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On Polynomial Interpolation on the Unit Ball

YUAN XU *

Dedicated to Borislav Bojanov on the occasion of his 60th birthday

Polynomial interpolation on the unit ball of $\mathbb{R}^d$ has a unique solution if the points are located on several spheres inside the ball and the points on each sphere solves the corresponding interpolation problem on the sphere. Furthermore, the solution can be computed in a recursive way.

1. Introduction

Let Π_n^d denote the space of polynomials of degree at most n in d variables and let $\mathcal{P}_n^d$ denote the space of homogeneous polynomials of degree n in d variables. It is known that

$$\dim \Pi_n^d = \binom{n+d}{n} \quad \text{and} \quad \dim \mathcal{P}_n^d = \binom{n+d-1}{n}.$$

Let $M_n^d = \dim \Pi_n^d$. The usual problem of polynomial interpolation is:

Problem 1. Let $X = \{\mathbf{a}_i : 1 \le i \le M_n^d\}$ be a set of distinct points. Find conditions on X such that there is a unique polynomial $P \in \Pi_n^d$ satisfying

$$P(\mathbf{a}_i) = f_i, \qquad \mathbf{a}_i \in X, \quad 1 \le i \le M_n^d,$$

for any given data $\{f_i\}$.

Let $B^d = \{x : \|x\| \le 1\}$ be the unit ball in $\mathbb{R}^d$, where $\|x\|$ is the Euclidean norm. We consider the problem of interpolation by polynomials on the points in B^d; that is, X is a subset of B^d. If there is a *unique* solution to the interpolation problem, we say that the problem is *poised* and that X solves Problem 1. Choosing a basis for Π_n^d, for example, the monomial basis $\{x^\alpha : |\alpha| \le n\}$, where $\alpha \in \mathbb{N}_0^d$ is a multiindex and $|\alpha| = \alpha_1 + \ldots + \alpha_d$. Then X solves Problem 1 if and

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only if the determinant $\det(x_i^\alpha)$ is nonzero. Hence, as it is well known, almost all choices of X solves Problem 1. In fact, those X that do not solve the problem lie on the hypersurface, in $x_1, \ldots, x_{M_n}$ variables, defined by $\det(x_i^\alpha) = 0$. However, testing whether a large determinant is zero can be difficult and costly. The hard part of Problem 1 is to determine if a *given* set X provides a unique solution. There are few general results that provide explicit X. We refer to the surveys [6, 7] and its references for results on interpolation of several variables in general. For the case on the unit ball, we refer to [2, 3, 9, 11, 16] and the references therein.

Naturally, instead of working with a randomly chosen set, we often require some structure of the points. In the present paper, we consider interpolation problems for which X consists of points located on several spheres inside B^d. Let $S^{d-1} = \{x : \|x\| = 1\}$ be the unit sphere. The problem of interpolation on S^{d-1} is similar to Problem 1. Let $\Pi_n(S^{d-1})$ denote the space of spherical polynomials of d variables, which are the restriction of polynomials in Π_n^d on S^{d-1}. It is known that $\dim \Pi_0(S^{d-1}) = 1$ and

$$\dim \Pi_n(S^{d-1}) = \binom{n+d-1}{d-1} + \binom{n+d-2}{d-1}, \qquad n \geq 1.$$

Let $N_n^d = \dim \Pi_n(S^{d-1})$. The interpolation problem by spherical polynomials is as follows:

Problem 2. Let $X = \{\mathbf{a}_i : 1 \leq i \leq N_n^d\}$ be a set of distinct points on S^{d-1}. Find conditions on X such that there is a unique polynomial $S \in \Pi_n(S^{d-1})$ satisfying

$$S(\mathbf{a}_i) = f_i, \qquad \mathbf{a}_i \in X, \quad 1 \leq i \leq N_n^d,$$

for any given data $\{f_i\}$.

Note that a dilation does not change the regularity of interpolation. That is, for $r > 0$ and $X \in S^{d-1}$, let $X(r)$ denote the set $\{rx : x \in X\}$ in S_r^{d-1}, the sphere of radius r; if $X \in S^{d-1}$ solves Problem 2, then $X(r)$ solves the interpolation problem in the space $\Pi_n(S_r^{d-1})$, the restriction of Π_n^d on S_r^{d-1}. In such a case, we also say that $X(r)$ solves Problem 2.

Just as in the case of Problem 1, there are few concrete examples of point sets that solve Problem 2. In fact, most examples are found only recently; see [5, 8, 13, 15, 16].

We will prove that if $X = \bigcup_{k=0}^n X(r_k)$, where r_k are distinct numbers between 0 and 1 and $X(r_k)$ solves Problem 2 in $\Pi_k(S^{d-1})$ with $\#X(r_k) = N_k$, then X solves Problem 1. Furthermore, we show that the interpolation polynomial so obtained can be computed by a recursive relation. These and other results on interpolation are studied in Section 2 and Section 3. Integrating an interpolation polynomial gives a numerical integration (cubature) formula. We discuss cubature formulas in Section 4.

2. Polynomial Interpolation on the Unit Ball

The proof of our main result on the polynomial interpolation on the unit ball uses an expression of the polynomials in d variables that appears to be of interest in its own. The expression is based on the spherical-polar coordinates,

$$x = rx', \qquad x' \in S^{d-1}, \quad r = \|x\| \geq 0, \quad x \in \mathbb{R}^d.$$

We will need the notion of spherical harmonics. A harmonic polynomial is a homogeneous polynomial Y that satisfies $\Delta Y(x) = 0$, where Δ is the usual Laplace operator,

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_d^2}.$$

Let $\mathcal{H}_n^d$ denote the space of harmonic polynomials of degree n. It is known that

$$L_n := \dim \mathcal{H}_n^d = \dim \mathcal{P}_n^d - \dim \mathcal{P}_{n-2}^d.$$

Since $Y \in \mathcal{H}_n^d$ is homogeneous, it can be written as $Y_n(x) = r^n Y_n(x')$ in the spherical-polar coordinates $x = rx'$. We denote by $\mathbb{Y}_k := \{Y_j^k : 1 \leq j \leq L_k\}$ a basis for $\mathcal{H}_k^d$.

Proposition 1. *In spherical-polar coordinates, a polynomial $P \in \Pi_n^d$ can be written as*

$$P(x) = \sum_{k=0}^{n} \sum_{j=0}^{L_k} a_{j,k}(\|x\|^2) Y_j^k(x), \qquad x \in \mathbb{R}^d, \tag{2.1}$$

where $a_{k,j}$ are polynomials in r with $\deg a_{k,j} = [(n-k)/2]$.

Proof. We use a short-hand notation of regarding $\mathbb{Y}$ as a column vector, so that every member $Y \in \mathcal{H}_k^d$ can be written as $\mathbf{a}\mathbb{Y}$ for some row vector $\mathbf{a}$. Since it is well-known that

$$\Pi_n^d = \bigcup_{k=0}^{n} \mathcal{P}_k^d \qquad \text{and} \qquad \mathcal{P}_k^d = \bigoplus_{j=0}^{[k/2]} r^{2j} \mathcal{H}_{k-2j}^d,$$

where $r = \|x\|$, we see that $P \in \Pi_n^d$ can be written as

$$P(x) = \sum_{k=0}^{n} \sum_{j=0}^{[k/2]} r^{2j} \mathbf{a}_{j,k-2j} \mathbb{Y}_{k-2j}.$$

The stated result comes from changing the order of the summation. Assume $n = 2m+1$, for example, then

$$
\begin{aligned}
P(x) &= \sum_{l=0}^{m}\sum_{j=0}^{l} r^{2j}\mathbf{a}_{j,2l-2j}\mathbb{Y}_{2l-2j}(x) + \sum_{l=0}^{m}\sum_{j=0}^{l} r^{2j}\mathbf{a}_{j,2l+1-2j}\mathbb{Y}_{2l+1-2j}(x) \\
&= \sum_{j=0}^{m}\sum_{l=j}^{m} r^{2l-2j}\mathbf{a}_{l-j,2j}\mathbb{Y}_{2j}(x) + \sum_{j=0}^{m}\sum_{l=j}^{m} r^{2l-2j}\mathbf{a}_{l-j,2j+1}\mathbb{Y}_{2j+1}(x) \\
&= \sum_{k=0}^{m} \mathbf{A}_k(r^2)\mathbb{Y}_k(x),
\end{aligned}
$$

where $\mathbf{A}_k(r^2)$ is a row vector $\sum_{l=0}^{m-j}\mathbf{a}_{l,k}r^{2l}$ for $k = 2j$ or $2j+1$, after a change of summation index, whose components are $a_{j,k}(r^2)$ in the statement of the theorem. The case of $n = 2m$ works similarly. □

With the above expression, the proof of the following result becomes easy.

Theorem 1. *Let m be a positive integer and $r_0, r_1, \ldots, r_{[n/2]}$ be distinct nonnegative real numbers. For each r_l, let*

$$X(r_l) = \left\{x_{j,l} : 1 \le j \le N^d_{n-2l},\ \|x_{j,l}\| = r_l\right\}$$

consist of distinct nodes on $S^{d-1}_{r_l}$ such that $X(r_l)$ solves Problem 2 in the space $\Pi_{n-2l}(S^{d-1}_{r_l})$. Then the point set $X := \bigcup_{l=0}^{[n/2]} X(r_l)$ solves Problem 1.

Proof. We again write $x = rx'$ with $r = \|x\|$ and $x' \in S^{d-1}$. By Proposition 1, we can write

$$P(x) = \sum_{k=0}^{n}\sum_{j=0}^{L_k} a_{j,k}(r^2)Y^k_j(x).$$

It suffices to prove that $P(x) \equiv 0$ if all interpolation conditions are zero. We start with r_0. The condition $P(x_{j,0}) = 0$ for $x_{j,0} \in X(r_0)$ gives

$$\sum_{k=0}^{n}\sum_{j=0}^{L_k} a_{j,k}(r_0^2)r_0^k Y^k_j(x'_{j,0}) = 0.$$

Since $X(r_0)$ solves Problem 2 in $\Pi_n(S^{d-1}_{r_0})$, it follows from the uniqueness of the interpolation polynomial that $r_0^k a_{j,k}(r_0^2) = 0$ for $k = 0, 1, \ldots, n$ and $0 \le j \le L_k$. As the degree of $a_{j,k}$ is $[(n-k)/2]$, this shows that $a_{j,n} = 0$, $0 \le j \le L_n$, and $a_{j,k}(r^2) = (r^2 - r_0^2)a^*_{j,k}(r^2)$, $1 \le j \le L_k$, $k \ge 1$, where $a^*_{j,k}$ has degree $[(n-k)/2] - 1 = [(n-2-k)/2]$. Consequently,

$$P(x) = (r^2 - r_0^2)\sum_{k=0}^{n-2}\sum_{j=0}^{L_k} a^*_{j,k}(r^2)Y^k_j(x).$$

Evidently, this allows us to repeat the same argument for $X(r_1), X(r_2), \ldots$ to complete the proof. □

This theorem is a special case of a more general result in [4], in which each $X(r_l)$ is allowed to be an algebraic variety on the surface $P_l(x) = 0$ and polynomials in $\{P_l\}$ are pairwise relatively prime.

The theorem states that the interpolation on the ball can be constructed from interpolation on the sphere. For $d = 2$, the spheres are just circles and interpolation on the circles are well understood. In fact, in this case, the space $\Pi_k(S^1)$ coincides with the space of trigonometric polynomials of degree n, which uniquely interpolates any distinct $2k+1$ points on the circle, as shown in [17]. Consequently, we can state the following corollary:

Corollary 1. *Let n be a positive integer and $r_0, r_1, \ldots, r_{[n/2]}$ be distinct positive real numbers. Let $(x_{j,l}, y_{j,l})$, $0 \le j \le 2(n-2l)$ be distinct points on the circle $S^1_{r_l}$, then the set $X = \{(x_{j,l}, y_{j,l}) : 0 \le j \le 2(n-2l),\ 0 \le l \le [n/2]\}$ solves Problem 1.*

In fact, this corollary is an easy consequence of the classical Bezout's theorem, and the circles can be replaced by other conics. Let us point it out that for $d = 2$, the expression in (2.1) takes the form of

$$P(x) = A_0(r^2) + \sum_{k=1}^{n} \left[r^k A_k(r^2) \cos k\theta + r^k B_k(r^2) \sin \theta \right],$$

where A_k and B_k are polynomials of degree $[(n-k)/2]$, which has been used in [2, 3] to study interpolation in two variables. In that case, we are able to prove much more than what is contained in the above theorem, by working with equidistant points on the circles. Furthermore, the result there has been extended in [10] to point sets on other conics.

Much more interesting applications of Theorem 1 lies in the case $d > 2$. There are, however, few explicit examples of point sets on S^{d-1} that will solve Problem 2 for $d \ge 2$. For $d = 2$, several families of points on S^2 are found recently in [15, 16] and [5]. For $a \in [-1,1]$, let $S^2(a) := \{(x,y,z) : (x,y,z) \in S^2,\ z = a\}$ denote the circle on S^2 resulted from the intersection of S^2 with the plane $z = a$ (called latitude at a). The main result of [15] is as follows:

Theorem 2. *Let n and σ be positive integers, such that $n + 1 - \sigma$ is an even integer and $\sigma \le n + 1$. Let $\lambda_1, \ldots, \lambda_\sigma$ be nonnegative integers such that*

$$\lambda_1 + \ldots + \lambda_\sigma = \frac{n+1-\sigma}{2}. \tag{2.2}$$

Define $n_k = n_{k-1} - (2\lambda_k + 1)$ for $1 \le k \le \sigma - 1$ with $n_0 = n$. Let $z_{0,k}, \ldots, z_{2\lambda_k,k}$ be distinct points in $(-1,1)$. If the set X consists of points located on σ groups of latitudes, $\{S^2(z_0,k), S^2(z_{1,k}), \ldots, S^2(z_{2\lambda_k,k})\}$, $1 \le k \le \sigma$, and on each latitude $S^2(z_{j,k})$ there are $2(n_{k-1} - \lambda_k) + 1$ equidistant points, with the same starting point for the latitudes in the same group, then X solves Problem 1 in $\Pi_n(S^2)$.

For a fixed n this theorem contains a number of distinct sets of interpolation points. In fact, for each σ satisfying the condition that $n+1-\sigma$ is a nonnegative even integer, every nonnegative integer solution of the equation (2.2) leads to a set of interpolation points that solves Problem 2; moreover, the order of $\lambda_1, \ldots, \lambda_\sigma$ matters. The number of solutions of such an equation grows exponentially as n goes to infinity. One simple case is when each group contains only one latitude, in which case $n = 2m$ and X contains $m + 1$ latitudes and each latitudes contains $2m + 1$ equidistant points.

The point sets in this theorem can then be used in Theorem 1 to generalize point sets on B^3 that will solve Problem 1. Let

$$\Theta_{\alpha,m} = \{\theta_j^\alpha : \theta_j^\alpha = (2j + \alpha)\pi/(2m + 1),\ j = 0, 1, \ldots, 2m,\ \alpha \in [0, 2)\}. \quad (2.3)$$

We state one result that comes from the simplest case of Theorem 2.

Corollary 2. *Let m be a positive integer and $1 = r_0 > r_1 > \ldots > r_m > 0$ be distinct real numbers. For each r_l, let $1 \geq t_{l,1} > t_{l,2} > \ldots > t_{l,m+1} > 0$ and let*

$$X(r_l) = \Big\{ r_l(x_{k,j}, y_{k,j}, z_k) : x_{k,j} = t_{l,k}\cos\theta_j,\ y_{k,j} = t_{l,k}\sin\theta_j,\ z_k = \pm\sqrt{1 - t_{l,k}^2},$$
$$\theta_j \in \Theta_{\alpha_k, m-l},\ 1 \leq k \leq m - l + 1 \Big\},$$

where α_k are real numbers in $[0, 2\pi)$. Then the point set $X := \bigcup_{l=0}^m X(r_l)$ is in B^3 and solves Problem 1 in Π_{2m}^3.

In fact, each $X(r_l)$ solves Problem 2 in $\Pi_{n-2l}(S_{r_l}^2)$ using Theorem 2.

Instead of constructing point sets that solve Problem 1 from those that solve Problem 2, we can also go to the other direction: constructing points sets on S^{d-1} that solve Problem 2 in $\Pi_n(S^d)$ from those on B^d that solve Problem 1 in Π_n^d. Such a result is proved in [16]. However, the condition it requires does not seem to apply to the point sets obtained in Corollary 2.

3. An Iterative Construction

According to Theorem 1, the set X that allows unique interpolation on Π_n^d consists of sets $X(r_l)$ that solve uniquely the interpolation problem on the spheres. In fact, starting from interpolation polynomials on the spheres, it is possible to give a recursive algorithm that allows one to construct the interpolation polynomial on the ball using interpolation polynomials on the spheres.

Let us denote by $\mathcal{L}_{n-2l}(f)$, $0 \leq l \leq [n/2]$, the polynomial in Π_n^d whose restriction on $S_{r_l}^{d-1}$ is the unique polynomial in $\Pi_{n-2l}^d(S_{r_l}^{d-1})$ that satisfies

$$\mathcal{L}_{n-2l}(f; x_{j,l}) = f(x_{j,l}), \qquad x_{j,l} \in X_{n-2l}, \quad 1 \leq j \leq N_{n-2l}^d, \qquad (3.1)$$

provided that $X(r_l)$ solves Problem 2 in $\Pi_{n-2l}^d(S_{r_l}^{d-1})$.

Theorem 3. *Let $X(r_l)$ be as in Theorem 1. Let Q^n_k, $0 \le k \le [n/2]$, be determined by the following algorithm: $Q^n_0(x) := \mathcal{L}_n(f;x)$ and, for $k = 1, 2, \ldots$,*

$$Q^n_k(x) = Q^n_{k-1}(x) + \prod_{j=0}^{k-1} \frac{\|x\|^2 - r_j^2}{r_k^2 - r_j^2}\, \mathcal{L}_{n-2k}(f - Q^n_{k-1}; x). \tag{3.2}$$

Then the polynomial $Q^n_{[n/2]}$ is the interpolation polynomial that solves Problem 1 with $X = \bigcup_{l=0}^{[n/2]} X(r_l)$.

Proof. Our construction follows the idea of the Newton interpolation in one variable. It is defined in such a way that Q^n_k, $0 \le k \le [n/2]$, interpolates f at points in $X(r_0), X(r_1), \ldots, X(r_k)$. Indeed, assume that the above assertion has been proved for Q^n_{k-1}. Then, since $g_k(x) := \prod_{j=0}^{k-1}(\|x\|^2 - r_j^2)$ vanishes on any point belonging to $X(r_0), X(r_1), \ldots, X(r_{k-1})$, the polynomial Q^n_k as defined will interpolate f at the points in these sets. Furthermore, the function $g_k(x)$ becomes 1 at points in $X(r_k)$, so that the definition of $\mathcal{L}_k(f)$ shows that Q^n_k also interpolates f at points in X_k. Consequently, by induction, we see that $Q_{[n/2]}$ interpolates all points in X and it is easy to see that $Q_{[n/2]}$ is a polynomial of degree n. □

Although the definition of Q^n_k is similar to that of Newton's form of interpolation polynomial, the formulation in several variable is not as powerful as that of Newton's form. Let us say a word about our notation here. The polynomial Q^n_0 is a polynomial of degree n which interpolates the functions at the points $X(r_0)$, the one that contains the most points among all $X(r_l)$. Since $\mathcal{L}_{n-2l}$ has degree $n - 2l$, all other polynomials Q^n_k also have degree n. In the case of one variable, $\#X(r_0) = \#X(r_1) = \ldots = 1$, so that adding one more point means adding one more term in Newton's form. This is no longer true in our definition for several variables, not even when we add another set of points on a separate sphere, since we start with $X(r_0)$, the one that has the most points, and move down from there. One can also use the Newton formula to derive the error term of an interpolation polynomial in one variable. Since each $X(r_k)$ for $k \ge 1$ contain more than one point in several variables, we do not get Newton type error formula automatically anymore.

To show how the algorithm works, let us consider the case $d = 2$; that is, consider polynomials of two variables that interpolates points on various circles in the unit disk. In this case, our results states that such a polynomial is built upon polynomials that interpolate points on the circles. It is more convenient to use the polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \qquad r \ge 0, \quad 0 \le \theta \le 2\pi.$$

Then the interpolation on the circle S^1 is the same as interpolation by trigonometric polynomials. A trigonometric polynomial of degree m is of the form

$$T_m(\theta) = a_0 + \sum_{k=1}^{m}(a_k\cos\theta + b_k\sin\theta), \qquad a_k, b_k \in \mathbb{R}.$$

Interpolation by trigonometric polynomials is studied extensively; see [17]. For our example, we shall assume that points on the circle are equally spaced. For a positive integer m, the equidistant points on the circle S^1 are $r(\cos\theta_j^\alpha, \sin\theta_j^\alpha)$ where $\theta_j^\alpha = (2j+\alpha)\pi/(2m+1)$ for $j = 0, 1, \ldots, 2m$. Recall that the collection of these θ_j^α is called $\Theta_{\alpha,m}$ as in (2.3).

We give an explicit formula for the interpolating polynomial for points on the circle. From [17] it follows that the trigonometric polynomial of degree m based on the $2m+1$ points $\Theta_{0,m}$ is given by the formula

$$T_m(f,\theta) = \sum_{j=0}^{m} f(\theta_j) D_m(\theta - \theta_j),$$

where the trigonometric polynomial D_m is given by

$$D_m(\theta) = \frac{2}{2m+1}\left[\frac{1}{2} + \sum_{k=1}^{m} \cos k\theta\right] = \frac{\sin(m+\frac{1}{2})\theta}{(2m+1)\sin\frac{1}{2}\theta}.$$

Let us define by $\widetilde{f}(r,\theta) = f(r\cos\theta, r\sin\theta)$. Then the polynomial $\mathcal{L}_{n-2k}(f)$ in Π_{n-2k}^2 whose restriction to the circle $S_{r_k}^1$ interpolates f at $(r_k\cos\theta_j^\alpha, r_k\sin\theta_j^\alpha)$, $\theta_j^\alpha \in \Theta_{\alpha,n-2k}$, is given by the formula

$$\widetilde{\mathcal{L}}_{n-2k}(f;r,\theta) = \sum_{j=0}^{2(n-2k)} \widetilde{f}(r_k,\theta_j^\alpha)\widetilde{D}_{n-2k}\left(\frac{r}{r_k}, \theta - \theta_j^\alpha\right), \tag{3.3}$$

where the polynomial $D_m(x,y)$ is defined by

$$\widetilde{D}_m(r,\theta) = \frac{1}{2m+1}\left(\frac{1}{2} + \sum_{j=1}^{m} r^j \cos j\theta\right).$$

The finite sum can be easily summed up, for example, as the real part of the sum $\sum_{k=1}^{n} r^k e^{ik\theta}$; the result is

$$\widetilde{D}_m(r,\theta) = \frac{1}{2m+1}\frac{1 - r^2 + 2r^{m+2}\cos m\theta - 2r^{m+1}\cos(m+1)\theta}{1 - 2r\cos\theta + r^2}.$$

Using the explicit formula (3.3), we can then use the algorithm in Theorem 3 to construct the interpolation polynomial on B^2. Let us give the explicit formula in the first two cases.

Example 3.1. The simplest case is 6 points with one in the center and 5 equally spaced points on the circle $S_{r_0}^1$; that is, the interpolation points are

$$X = \{(0,0),\ (r_0\cos\frac{2\pi j}{5}, r_0\sin\frac{2\pi j}{5}),\ 0 \le j \le 4\}.$$

Here $n = 2$ and using the notation of Theorem 3, we have $Q_0^2 = \mathcal{L}_2(f)$, where $\mathcal{L}_2$ is given in (3.3) with $n = 2$ and $k = 0$. Consequently, by (3.2), the interpolation polynomial Q_1^2 takes the form

$$Q_1^2(x,y) = Q_0^2(x,y) + \frac{1}{r_0^2}(r_0^2 - x^2 - y^2)\mathcal{L}_0(f - Q_{0,2}, x, y).$$

The polynomial $\mathcal{L}_0(f - Q_{0,2}, x, y)$ is a constant which is determined by the interpolation at $(0,0)$; its value is easily seen as $f(0,0) - Q_0^2(0,0)$. Consequently, we conclude that the interpolation polynomial based on X is given by

$$\begin{aligned} Q_1^2(x,y) =& (r_0^2 - x^2 - y^2)\left[f(0,0) - \frac{1}{5}\sum_{j=0}^{4}\widetilde{f}\left(r_0, \frac{2\pi j}{5}\right)\right] \\ &+ \frac{2}{5}\sum_{j=0}^{4}\widetilde{f}\left(r_0, \frac{2\pi j}{5}\right)\widetilde{D}_2\left(\frac{r}{r_0}, \theta - \frac{2\pi j}{5}\right). \end{aligned}$$

Using $\cos m\theta = T_m(x/r)$ and $\sin m\theta = (y/r)U_{m-1}(x/r)$, where T_m and U_m are Chebyshev polynomials of the first and the second kind, respectively, one can easily write the explicit formula of $D_m(x,y)$.

Example 3.2. In our next example, we have 15 points and interpolation by polynomials in Π_4^2. The point set is

$$X = \{(0,0)\} \cup \{r_1(\cos\theta, \sin\theta), \theta \in \Theta_{0,2}\} \cup \{r_0(\cos\theta, \sin\theta), \theta \in \Theta_{\alpha,4})\}.$$

Here $n = 4$. Following the notation of Theorem 3, the interpolation polynomial based on X can be written in the form of

$$Q_2^4(x,y) = Q_0^4(x,y) + \frac{r_0^2 - r^2}{r_1^2 - r_0^2}q_1(x,y) + \frac{(r_1^2 - r^2)(r_0^2 - r^2)}{r_1^2 r_0^2}q_2(x,y)$$

where $r^2 = x^2 + y^2$, $Q_0^4 = \mathcal{L}_4(f)$ in which $\mathcal{L}_4$ is given in (3.3) with $\alpha = 0$ and $k = 0$, $q_1(x,y) = \mathcal{L}_2(f - Q_{0,4}; x, y)$ with $\mathcal{L}_2$ given in (3.3) with $k = 1$, and $q_2(x,y)$ is a constant whose value is determined by

$$q_2(x,y) = f(0,0) - Q_0^4(0,0) - \frac{r_0^2}{r_1^2 - r_0^2}q_1(0,0). \tag{3.4}$$

By (3.3), it is then easy to write down the explicit formula of the interpolation polynomial.

The interpolation polynomial in Corollary 2 can also be constructed this way, provided the interpolation polynomials on the spheres have been worked out.

4. Cubature Formulas

Next we turn our attention to cubature formulas. Evidently, integrating an interpolating polynomial gives a cubature formula. We consider the formula obtained from integrating the polynomial that interpolates the points in the set $X = \bigcup_{l=0}^{[n/2]} X(r_l)$ as in Theorem 1. Such a cubature formula has the form

$$\int_{B^d} f(x)\,dx = \sum_{l=0}^{[n/2]} \sum_{j=1}^{N_{n-2l}^d} \lambda_{j,l} f(r_l x'_{j,n-2l}), \qquad x'_{j,n-2l} \in S^{d-1}, \quad f \in \Pi_n^d, \quad (4.1)$$

and it is exact at least for $f \in \Pi_n^d$. We note that the formula (4.1) is different from the classical cubature formulas on the ball. One family of the classical formulas is the product type ([14]), which is obtained from the one-dimensional quadrature formulas and the fact that $B^d = [-1,1] \times B_r^{d-1}$ with $r = \sqrt{1-x_d^2}$. Another family [12] is derived from the cubature formulas on the sphere in the form

$$\int_{B^d} f(x)\,dx \approx \sum_{k=1}^{m} B_k \int_{S_{r_k}^{d-1}} f(y)\,d\omega(y) = \sum_{k=1}^{m} B_k \sum_{j=1}^{N} \lambda_j f(r_k x_{j,k}),$$

in which all integrals over the spheres are approximated by cubature formulas of the same degree (N is independent of k). The first approximation in the above equation has also been studied in [1], where the best choices of B_k for the approximation to be exact for polynomials of the highest degree are found.

One can ask several questions about the formula in (4.1). For example, by choosing the radius of the spheres $r_0, r_1, \ldots, r_{[n/2]}$ and further specifying the points in X_{n-2l}, we should be able to obtain formulas that are exact for polynomials of degree higher than n. What will be a best choice in this regard? For example, it is well-known that

$$\int_{B^2} f(x)\,dx \approx \frac{\pi}{4} f(0,0) + \frac{3\pi}{20} \sum_{j=0}^{4} f\left(\frac{\sqrt{3}}{2}\cos\frac{2\pi j}{5}, \frac{\sqrt{3}}{2}\sin\frac{2\pi j}{5}\right)$$

is a cubature formula of degree 4 and the degree 4 is the highest possible for a formula with 6 points ([12, 14]). This formula can be obtained by integrating the explicit formula of the interpolation polynomial of degree 2 in Example 3.1 with $r_0 = \sqrt{3}/2$.

Let us consider the cubature formula obtained upon integrating the interpolation polynomial in Example 3.2. Define

$$I_2(f) = \frac{1}{5} \sum_{j=0}^{4} f\left(r_1 \cos\frac{2\pi j}{5}, r_1 \sin\frac{2\pi j}{5}\right),$$

and

$$I_4(f) = \frac{1}{9} \sum_{j=0}^{8} f\left(r_0 \cos\frac{2\pi j}{9}, r_0 \sin\frac{2\pi j}{9}\right).$$

Proposition 2. *The following cubature is exact for polynomials of degree* 4,

$$\frac{1}{\pi}\int_{B^2} f(x,y)\,dxdy \approx \lambda_0 f(0,0) + \lambda_1 I_2(f) + \lambda_2 I_4(f), \tag{4.2}$$

where the constants λ_i *is defined by*

$$\lambda_j = 2\int_0^1 r\ell_j(r^2)\,dr, \qquad \ell_j(t) = \prod_{i=0,i\neq j}^{2} \frac{t-r_i^2}{r_j^2-r_i^2}.$$

Furthermore, this formula is not exact for all polynomials of degree 5, *no matter how we choose* $\lambda_0, \lambda_1, \lambda_2$.

Proof. From the definition of $\widetilde{D}_m(r,\theta)$, it follows that for any polynomial p,

$$\int_{B^2} D_m(x,y)p(\|x\|)\,dxdy = \int_0^1 rp(r)\int_0^{2\pi}\widetilde{D}_m(r,\theta)\,d\theta dr = \frac{2\pi}{2m+1}\int_0^1 rp(r)\,dr.$$

Using the definition of λ_0, the cubature formula takes the form

$$\frac{1}{\pi}\int_{B^2} f(x,y)\,dxdy = I_4(f) + 2\int_0^1 r\,\frac{r_0^2-r^2}{r_1^2-r_0^2}\,dr\; I_2(f-Q_0^4) + \lambda_0\, q_2(0,0),$$

in which Q_0^4 and q_2 are as defined in Example 3.1. Since $Q_0^4(0,0) = I_4(f)$ and $q_1(0,0) = I_2(f-Q_0^4)$, by the definition of $q_2(x,y)$ in (3.4) and the explicit formula of ℓ_k, we can write the above formula as

$$\int_{B^2} f(x,y)\,dxdy = \lambda_0\, f(0,0) + \lambda_1(f)I_2(f) + \lambda_2 I_4(f) + \lambda_1\left[I_4(f) - I_2(Q_0^4)\right].$$

Using the fact that for $k = 1,2,\ldots,2m+1$,

$$\sum_{j=0}^{2m}\sin k\frac{2\pi j}{2m+1} = 0 \quad \text{and} \quad \sum_{j=0}^{2m}\cos k\frac{2\pi j}{2m+1} = (2m+1)\delta_{k,2m+1}, \tag{4.3}$$

a tedious computation shows that

$$\frac{1}{5}\sum_{j=0}^{4}\widetilde{D}_4\left(r_1, \frac{2\pi j}{5} - \frac{2\pi l}{9}\right) = \frac{1}{9},$$

so that, by the fact that $Q_0^4 = L_4 f$ with $L_4 f$ given in (2.3), $I_2(Q_0^4) = I_4(f)$. From which the cubature formula (4.2) follows by a straightforward computation.

Finally, to show that the cubature formula is not exact for all polynomials of degree 5, we choose the polynomial $P_5(x,y)$ given by $\widetilde{P}_5(r,\theta) = r^5\cos 5\theta$. This is a polynomial of degree 5 in x,y. Using (4.3), it is easy to see that $I_4(P_5) = 0$

but $I_2(P) = 1$. Since the integral of P_5 over B^2 is 0 and $P_5(0,0) = 0$, the cubature is not exact for this polynomial. □

The formula (4.2) looks simple in the sense that the same cubature weight applies to all points on the same circle. This turns out, however, to be a special feature of degree 4. In fact, the computation in (4.3) shows that the cubature formula (4.1) with equally spaced points on the circles will have different weights for points on the same circle for $n = 3$ and apprently also for $n > 4$.

We note that the cubature formula in Proposition 2 uses equally spaced points on the circles. Hence, it does not settle the question if one can obtain cubature formulas of the form (4.1) that are exact for polynomials of order higher than n. It is likely that with different choice of points, one can obtain cubature formulas with higher exactness.

Another question one can ask is when is the cubature formula (4.1) is positive; that is, with all $\lambda_{j,l}$ positive. The following proposition gives a necessary condition.

Proposition 3. *For the cubature formula* (4.1) *to be positive, it is necessary that* $r_0, \ldots, r_{n/2}$ *satisfies that*

$$\int_0^1 r^{d-1}\ell_k(r^2)\,dr > 0, \quad \textit{where} \quad \ell_k(t) = \prod_{j=0, j\neq k}^{[n/2]} \frac{t - r_j^2}{r_k^2 - r_j^2}, \quad 1 \le k \le [n/2].$$

Proof. With $r^2 = \|x\|^2$, the polynomial $f_k(x) = \ell_k(r^2)$ is a polynomial of degree $2[n/2] \le n$. Hence, the cubature formula (4.1) is exact for f_k. Since $\ell_k(r_j^2) = \delta_{k,j}$, we get

$$\sum_{j=1}^{N_{n-2k}^d} \lambda_{j,n-2k} = \int_{B^d} \ell_k(\|x\|^2)\,dx = \int_0^1 r^{d-1}\ell_k(r^2)\,dr.$$

If (4.1) is positive, then the right hand side is positive. □

The polynomial $\ell_k(t)$ is the k-th fundamental interpolation polynomial for one-dimensional interpolation based on the points $\{r_0^2, r_1^2, \ldots, r_{[n/2]}^2\}$. One can choose r_k as zeros of orthogonal polynomials, as in Gaussian quadrature formula, to ensure that the necessary conditions in the proposition are satisfied. However, the necessary conditions are not sufficient, since the weights in general are different for nodes on the same circle.

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Yuan Xu

Department of Mathematics
University of Oregon
Eugene, OR 97403-1222
USA
E-mail: yuan@math.uoregon.edu