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ORTHOGONAL POLYNOMIALS FOR A FAMILY OF PRODUCT WEIGHT FUNCTIONS ON THE SPHERES

YUAN XU

ABSTRACT. Based on the theory of spherical harmonics for measures invariant under a finite reflection group developed by Dunkl recently, we study orthogonal polynomials with respect to the weight functions $|x_1|^{\alpha_1} \cdots |x_d|^{\alpha_d}$ on the unit sphere S^{d-1} in $\mathbb{R}^d$. The results include explicit formulae for orthonormal polynomials, reproducing and Poisson kernel, as well as intertwining operator.

1. **Introduction and Preliminaries.** Among their many distinct properties, the spherical harmonics can be viewed as orthogonal polynomials with respect to the Lebesgue measure on the unit sphere S^{d-1} of $\mathbb{R}^d$. For years they remained to be the only orthogonal polynomials on spheres that had been studied in detail (see, however, [8] and the references there). Recently Dunkl [3–5] developed a theory of spherical harmonics for measures invariant under the finite reflection groups. The theory has remarkable similarities to the theory of spherical harmonics. Among other things, it opens a way to study orthogonal polynomials on the sphere with respect to a large class of measures. In this paper we consider the case of $h_\alpha^2(\mathbf{x})\,d\mathbf{x}$, where $h_\alpha(\mathbf{x}) = |x_1|^{\alpha_1} \cdots |x_d|^{\alpha_d}$, $\alpha_i \geq 0$; note that this measure reduces to the Lebesgue measure when all $\alpha_i = 0$. Our purpose is to study the system of orthogonal polynomials for this class of weight functions in detail, deriving explicit formulae for orthonormal polynomials and reproducing kernels, as well as to provide a case study for Dunkl's general theory.

It turns out that explicit formulae for the orthogonal polynomials can be given in terms of orthogonal polynomials with respect to the weight function $(1-x^2)^{\lambda-1/2}|x|^{2\mu}$ on $[-1,1]$. These polynomials of one variable can be written explicitly in terms of Jacobi polynomials, but they seem to possess properties that make them closer to the Gegenbauer polynomials; we shall call them *generalized Gegenbauer polynomials*. A large part of the paper is devoted to study these polynomials. Using a product formula for the Jacobi polynomials due to Dijksma and Koornwinder [2] (different from the one that follows from the addition formula of Koornwinder), we are able to derive product and addition formulae for the generalized Gegenbauer polynomials, from which the explicit formulae for the n-th reproducing kernel and the Poisson kernel for $h_\alpha^2\,d\mathbf{x}$ will follow.

The paper is organized as follows. In the remainder of this section we state the background and results from Dunkl's theory, together with other preliminaries. The study of

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the generalized Gegenbauer polynomials is contained in Section 2. The explicit formulae for orthonormal polynomials and n-th reproducing kernel are given in Section 3. The Poisson kernel and the intertwining operator are discussed in Section 4.

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ we let $\mathbf{x} \cdot \mathbf{y}$ denote the usual inner product of $\mathbb{R}^d$ and $|\mathbf{x}| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$ the Euclidean norm. Let $S^{d-1} = \{\mathbf{x} : |\mathbf{x}| = 1\}$ be the unit sphere in $\mathbb{R}^d$. We denote by $\mathbf{e}_1, \ldots, \mathbf{e}_d$ the standard basis of $\mathbb{R}^d$.

We restrict the statement from Dunkl's theory to the special case considered in this paper. The reflection group G is generated by the reflections along $\mathbf{e}_1, \ldots, \mathbf{e}_d$, which we denote by $\mathbf{x}\sigma_j$; *i.e.*,

$$\mathbf{x}\sigma_j = \mathbf{x} - 2(\mathbf{x} \cdot \mathbf{e}_j)\mathbf{e}_j = (x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_d).$$

For $\alpha = (\alpha_1, \ldots, \alpha_d)$, where $\alpha_i \geq 0$, we define

$$h_\alpha(\mathbf{x}) = H_\alpha |x_1|^{\alpha_1} \cdots |x_d|^{\alpha_d}, \quad \alpha_i \geq 0, \quad \mathbf{x} \in S^{d-1} \tag{1.1}$$

where H_α is chosen so that the integral of h_α^2 on S^{d-1} is 1; we have

$$H_\alpha^2 = \frac{\omega_{d-1}}{2} \frac{\Gamma(|\alpha|_1 + \frac{d}{2})}{\Gamma(\alpha_1 + \frac{1}{2}) \cdots \Gamma(\alpha_d + \frac{1}{2})},$$

which can be verified easily using the spherical coordinates.

The key ingredient of the theory is a family of commuting first-order differential-difference operators, $\mathcal{D}_i$ (Dunkl's operators), which act very much like the partial derivatives ∂_i. In the present much restricted case, the operators take the form

$$\mathcal{D}_j f(\mathbf{x}) = \partial_j f(\mathbf{x}) + \alpha_j \frac{f(\mathbf{x}) - f(\mathbf{x}\sigma_j)}{x_j}, \quad 1 \leq j \leq d. \tag{1.2}$$

The h-Laplacian, which plays the role similar to that of the usual Laplacian, is defined by

$$\Delta_h = \mathcal{D}_1^2 + \cdots + \mathcal{D}_d^2. \tag{1.3}$$

Indeed, let $\mathcal{P}_n^d$ denote the space of homogeneous polynomials of degree n in $x_1, \ldots, x_d$. Then $\mathcal{D}_i \mathcal{P}_n^d \subset \mathcal{P}_{n-1}^d$, $\Delta_h \mathcal{P}_n^d \subset \mathcal{P}_{n-2}^d$; moreover, if $P \in \mathcal{P}_n^d$, then

$$\int_{S^{d-1}} P Q h_\alpha^2 \, d\omega = 0, \quad \forall Q \in \bigcup_{k=0}^{n-1} \mathcal{P}_k^d$$

if and only if $\Delta_h P = 0$. The space $\mathcal{H}_n^h = \mathcal{H}_n^{h,d} := \mathcal{P}_n^d \cap \ker \Delta_h$ is called the space of *h-harmonic* polynomials of degree n. The dimension of $\mathcal{H}_n^h$ is the same as that of the usual spherical harmonics.

The theory of h-harmonics is established for measures invariant under a general reflection group, the reader should consult Dunkl's papers (*cf.* [3–6]).

2. **Generalized Gegenbauer Polynomials.** Throughout this paper we use the standard notation $P_n^{(\alpha,\beta)}$ for the Jacobi polynomials and $C_n^{(\lambda)}$ for the Gegenbauer polynomials. For the properties of them we refer to [7, Ch. X] and [13, Ch. 4].

In this section we study the orthogonal polynomials with respect to the weight function

$$w^{(\lambda,\mu)}(x) = w_{\lambda,\mu}(1-x^2)^{\lambda-\frac{1}{2}}|x|^{2\mu}, \quad -1 \le x \le 1, \quad \lambda,\mu > -1/2,$$

where

$$w_{\lambda,\mu} = \frac{\Gamma(\lambda+\mu+1)}{\Gamma(\mu+\frac{1}{2})\Gamma(\lambda+\frac{1}{2})}$$

is chosen so that the integral of $w^{(\lambda,\mu)}$ over the integral $[-1,1]$ is 1. We denote the *orthonormal* polynomials with respect to the weight function $w^{(\lambda,\mu)}$ by $D_n^{(\lambda,\mu)}$; that is,

$$\int_{-1}^{1} D_n^{(\lambda,\mu)}(x)D_m^{(\lambda,\mu)}(x)w^{(\lambda,\mu)}(x)\,dx = \delta_{n,m}, \quad D_n^{(\lambda,\mu)} \in \Pi_n,$$

where Π_n denotes the space of polynomials of degree at most n in one variable. When $\mu = 0$, the polynomials $D_n^{(\lambda,0)}$ are the normalized Gegenbauer polynomials $\tilde{C}_n^{(\lambda)}$, which differs from $C_n^{(\lambda)}$ by a normalization constant. We call $D_n^{(\lambda,\mu)}$ the generalized Gegenbauer polynomials.

PROPOSITION 2.1. *The generalized Gegenbauer polynomials can be expressed in terms of Jacobi polynomials as follows:*

$$D_{2n}^{(\lambda,\mu)}(x) = c_n(\lambda,\mu)P_n^{(\lambda-\frac{1}{2},\mu-\frac{1}{2})}(2x^2-1), \tag{2.1a}$$

$$D_{2n+1}^{(\lambda,\mu)}(x) = c_n(\lambda,\mu+1)\sqrt{\frac{\lambda+\mu+1}{\mu+1/2}}\,xP_n^{(\lambda-\frac{1}{2},\mu+\frac{1}{2})}(2x^2-1), \tag{2.1b}$$

where

$$c_n(\lambda,\mu) = \sqrt{\frac{\Gamma(\mu+\frac{1}{2})\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+\mu+1)}}\sqrt{\frac{(2n+\lambda+\mu)\Gamma(n+\lambda+\mu)\Gamma(n+1)}{\Gamma(n+\mu+\frac{1}{2})\Gamma(n+\lambda+\frac{1}{2})}}.$$

PROOF. In order to establish the relation for the case $2n$, it suffices to prove that

$$\int_{-1}^{1} P_n^{(\lambda-\frac{1}{2},\mu-\frac{1}{2})}(2x^2-1)p(x)(1-x^2)^{\lambda-\frac{1}{2}}|x|^{2\mu}\,dx = 0, \quad p \in \Pi_{2n-1},$$

and then determine the normalization constant. This is trivial if p is an odd polynomial. Let p be even and write it as $p(x) = q(x^2)$, where $q \in \Pi_{n-1}$. Then, by the orthogonality of $P_n^{(\alpha,\beta)}$,

$$\begin{aligned}
&\int_{-1}^{1} P_n^{(\lambda-\frac{1}{2},\mu-\frac{1}{2})}(2x^2-1)q(x^2)(1-x^2)^{\lambda-\frac{1}{2}}|x|^{2\mu}\,dx \\
&\qquad = 2\int_{0}^{1} P_n^{(\lambda-\frac{1}{2},\mu-\frac{1}{2})}(2x^2-1)q(x^2)(1-x^2)^{\lambda-\frac{1}{2}}x^{2\mu}\,dx \\
&\qquad = \int_{0}^{1} P_n^{(\lambda-\frac{1}{2},\mu-\frac{1}{2})}(2t-1)q(t)(1-t)^{\lambda-\frac{1}{2}}t^{\mu-\frac{1}{2}}\,dt \\
&\qquad = 0.
\end{aligned}$$

To determine the constant, we use the fact that

$$\begin{aligned}
&\int_{-1}^{1}[P_n^{(\lambda-\frac{1}{2},\mu-\frac{1}{2})}(2x^2-1)]^2(1-x^2)^{\lambda-\frac{1}{2}}|x|^{2\mu}\,dx \\
&\quad = 2^{-\lambda-\mu}\int_{-1}^{1}[P_n^{(\lambda-\frac{1}{2},\mu-\frac{1}{2})}(t)]^2(1-t)^{\lambda-\frac{1}{2}}(1+t)^{\mu-\frac{1}{2}}\,dt \\
&\quad = \frac{1}{2n+\lambda+\mu}\cdot\frac{\Gamma(n+\mu+\frac{1}{2})\Gamma(n+\lambda+\frac{1}{2})}{\Gamma(n+\lambda+\mu)\Gamma(n+1)}
\end{aligned}$$

and take into account the constant $w_{\lambda,\mu}$. A similar argument is used to prove the relation for the case $2n+1$. ∎

We note that for $\mu = 0$, the relations in the proposition are the well-known formulae that connect the Gegenbauer polynomials and the Jacobi polynomials (*cf.* [13, p. 59, (4.1.5)]). From this basic relation, a number of properties of $D_n^{(\lambda,\mu)}$ will follow easily. We record one below.

COROLLARY 2.2. *For $\lambda, \mu \geq 0$,*

(2.2)
$$D_{2n+1}^{(\lambda,\mu)}(x) = \sqrt{\frac{\lambda+\mu+1}{\mu+\frac{1}{2}}}\,xD_{2n}^{(\lambda,\mu+1)}(x).$$

Our starting point is the following important relation first proved in [2, p. 192, (2.5)] by group theoretic method, an analytic proof appeared in [10, p. 133].

For $\alpha > -1/2$, $\beta > -1/2$,

(2.3)
$$\begin{aligned}
P_n^{(\alpha,\beta)}(\cos 2\theta)P_n^{(\alpha,\beta)}(\cos 2\phi) &= \frac{\Gamma(\alpha+\beta+1)\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\pi\Gamma(n+1)\Gamma(n+\alpha+\beta)\Gamma(\alpha+\frac{1}{2})\Gamma(\beta+\frac{1}{2})} \\
&\quad\times\int_{-1}^{1}\int_{-1}^{1}C_{2n}^{(\alpha+\beta+1)}(t\cos\theta\cos\phi+s\sin\theta\sin\phi)(1-t^2)^{\alpha-\frac{1}{2}}(1-s^2)^{\beta-\frac{1}{2}}\,ds\,dt.
\end{aligned}$$

Since $\cos 2\theta = 2\cos^2\theta - 1$, we can use the basic relation (2.1) with $\alpha = \lambda - 1/2$ and $\beta = \mu - 1/2$ to write the above formula as

For $\lambda > 0$, $\mu > 0$,

(2.4)
$$\begin{aligned}
D_{2n}^{(\lambda,\mu)}(\cos\theta)D_{2n}^{(\lambda,\mu)}(\cos\phi) &= \frac{2n+\lambda+\mu}{\lambda+\mu}c_\lambda c\mu \\
&\quad\times\int_{-1}^{1}\int_{-1}^{1}C_{2n}^{(\lambda+\mu)}(t\cos\theta\cos\phi+s\sin\theta\sin\phi)(1-t^2)^{\lambda-1}(1-s^2)^{\mu-1}\,ds\,dt,
\end{aligned}$$

where

$$c_\lambda^{-1} = \int_{-1}^{1}(1-t^2)^{\lambda-1}\,dt = \frac{\pi^{\frac{1}{2}}\Gamma(\lambda)}{\Gamma(\lambda+\frac{1}{2})}.$$

This product formula allows us to derive an additional formula for the Gegenbauer polynomial that involves generalized Gegenbauer polynomials.

THEOREM 2.3. *For $\lambda > 0$ and $\mu > 0$,*

$$C_n^{(\lambda+\mu)}(\cos\theta\cos\phi t + \sin\theta\sin\phi s) = \sum_{m=0}^{[\frac{n}{2}]} \sum_{k+j=n-2m} b_{k,j}^n (\cos\theta\cos\phi)^k \tag{2.5}$$
$$\times (\sin\theta\sin\phi)^j D_{n-k-j}^{(\lambda+j,\mu+k)}(\cos\theta) D_{n-k-j}^{(\lambda+j,\mu+k)}(\cos\phi) C_k^{(\mu-\frac{1}{2})}(t) C_j^{(\lambda-\frac{1}{2})}(s)$$

where

$$b_{k,j}^n = \frac{\Gamma(\mu-\frac{1}{2})\Gamma(\lambda-\frac{1}{2})}{\Gamma(\lambda+\mu)} \frac{\Gamma(\lambda+\mu+k+j+1)}{(n+\lambda+\mu)\Gamma(k+\mu-\frac{1}{2})\Gamma(j+\lambda-\frac{1}{2})}. \tag{2.6}$$

This formula has appeared in [9, p. 242, (4.7)], where a proof was given using group theoretic method, but the constants were not given explicitly. We give an analytic proof in the following.

PROOF OF THEOREM 2.3. Since the left hand side of (2.5) is a polynomial of degree n in variables t and s, it can be written in terms of the product orthogonal polynomials $\{C_k^{(\mu-\frac{1}{2})}(t)C_j^{(\lambda-\frac{1}{2})}(s)\}_{k,j}$ as

$$C_n^{(\lambda+\mu)}(\cos\theta\cos\phi t + \sin\theta\sin\phi s) = \sum_{0\le k+j\le n} F_{k,j}^n(\cos\theta,\cos\phi) C_k^{(\mu-\frac{1}{2})}(t) C_j^{(\lambda-\frac{1}{2})}(s),$$

where, by orthogonality of the product polynomials,

$$F_{k,j}^n(\cos\theta,\cos\phi) = \frac{1}{h_k^{(\mu)}h_j^{(\lambda)}} \int_{-1}^1\int_{-1}^1 C_n^{(\lambda+\mu)}(\cos\theta\cos\phi t + \sin\theta\sin\phi s)$$
$$\times C_k^{(\mu-\frac{1}{2})}(t) C_j^{(\lambda-\frac{1}{2})}(s)(1-t^2)^{\lambda-\frac{1}{2}}(1-s^2)^{\mu-\frac{1}{2}}\, ds\, dt,$$

in which we write (*cf.* [13, p. 80, (4.7.14)])

$$h_k^{(\mu)} = \int_{-1}^1 [C_k^{(\mu-\frac{1}{2})}(t)]^2(1-t^2)^{\mu-1}\, dt = 2^{2-3\mu}\pi \frac{\Gamma(k+2\mu-1)}{(k+\mu-\frac{1}{2})k![\Gamma(\mu-\frac{1}{2})]^2}.$$

We first make the observation that changing variables $s \longmapsto -s$ and $t \longmapsto -t$ and using the fact that $C_m^{(\gamma)}(-t) = (-1)^m C_m^{(\gamma)}(t)$ leads to

$$F_{k,j}^n(\cos\theta,\cos\phi) = 0, \quad \text{if } n-k-j \text{ is an odd integer .}$$

Therefore, we may assume that $n-k-j$ is even in the following. We use Rodrigues' formula for the Gegenbauer polynomial which states [13, p. 81, (4.7.12)],

$$(1-t^2)^{\mu-1} C_k^{(\mu-\frac{1}{2})}(t) = B_k^{(\mu)}\left(\frac{d}{dt}\right)^k (1-t^2)^{k+\mu-1},$$

where

$$B_k^{(\mu)} = \frac{(-2)^k}{k!} \frac{\Gamma(k+\mu-\frac{1}{2})\Gamma(k+2\mu-1)}{\Gamma(\mu-\frac{1}{2})\Gamma(2k+2\mu-1)}.$$

For $n-k-j$ even, we use this formula and integration by parts to conclude that

$$\begin{aligned}
&F_{k,j}^n(\cos\theta,\cos\phi)\\
&\quad=\frac{B_k^{(\mu)}B_j^{(\lambda)}}{h_k^{(\mu)}h_j^{(\lambda)}}\int_{-1}^1\int_{-1}^1 C_n^{(\lambda+\mu)}(\cos\theta\cos\phi t+\sin\theta\sin\phi s)\\
&\qquad\times\left(\frac{d}{dt}\right)^k(1-t^2)^{k+\mu-1}\left(\frac{d}{ds}\right)^j(1-s^2)^{j+\lambda-1}\,dt\,ds\\
&\quad=(-1)^{k+j}\frac{B_k^{(\mu)}B_j^{(\lambda)}}{h_k^{(\mu)}h_j^{(\lambda)}}\int_{-1}^1\int_{-1}^1\left(\frac{d}{dt}\right)^k\left(\frac{d}{ds}\right)^j C_n^{(\lambda+\mu)}(\cos\theta\cos\phi t+\sin\theta\sin\phi s)\\
&\qquad\times(1-t^2)^{k+\mu-1}(1-s^2)^{j+\lambda-1}\,dt\,ds.
\end{aligned}$$

Using the derivative formula of the Gegenbauer polynomials ([13, p. 81, (4.7.14)]) repeatedly leads to

$$\left(\frac{d}{dt}\right)^m C_n^{(\gamma)}(t)=2^m\frac{\Gamma(\gamma+m)}{\Gamma(\gamma)}C_{n-m}^{(\gamma+m)}(t),$$

from which it follows readily that

$$\begin{aligned}
&F_{k,j}^n(\cos\theta,\cos\phi)\\
&\quad=(-1)^{k+j}\frac{B_k^{(\mu)}B_j^{(\lambda)}}{h_k^{(\mu)}h_j^{(\lambda)}}2^{k+j}\frac{\Gamma(\lambda+\mu+k+j)}{\Gamma(\lambda+\mu)}(\cos\theta\cos\phi)^k(\sin\theta\sin\phi)^j\\
&\qquad\times\int_{-1}^1\int_{-1}^1 C_{n-k-j}^{(\lambda+\mu+k+j)}(\cos\theta\cos\phi t+\sin\theta\sin\phi s)(1-t^2)^{k+\mu-1}(1-s^2)^{j+\lambda-1}\,dt\,ds\\
&\quad=b_{k,j}^n(\cos\theta\cos\phi)^k(\sin\theta\sin\phi)^j D_{n-k-j}^{(\lambda+j,\mu+k)}(\cos\theta)D_{n-k-j}^{(\lambda+j,\mu+k)}(\cos\phi),
\end{aligned}$$

where we have used (2.4) since $n-k-j$ is even and the constant $b_{k,j}^n$ is given by

$$b_{k,j}^n=(-1)^{k+j}\frac{B_k^{(\mu)}B_j^{(\lambda)}}{h_k^{(\mu)}h_j^{(\lambda)}}2^{k+j}\frac{\Gamma(\lambda+\mu+k+j)}{\Gamma(\lambda+\mu)}\frac{\lambda+\mu+k+j}{n+\lambda+\mu}\frac{1}{c_{\lambda+j}c_{\mu+k}}.$$

Using the formulae for various constants in $b_{k,j}^n$ and making use of the formula

$$\Gamma\left(\gamma-\frac{1}{2}\right)\Gamma(\gamma)=\sqrt{\pi}2^{2-2\gamma}\Gamma(2\gamma-1),$$

we can simplify the formula for $b_{k,j}^n$ to the desired form. This completes the proof. ∎

For our purpose, the following corollaries of the theorem are of most interest.

THEOREM 2.4. *For $\lambda>0$,*

$$\begin{aligned}
&c_\mu\int_{-1}^1 C_n^{(\lambda+\mu)}(\cos\theta\cos\phi t+\sin\theta\sin\phi s)(1+t)(1-t^2)^{\mu-1}\,dt\\
&\quad=\sum_{k=0}^n a_{k,n}^{(\lambda,\mu)}\sin^k\theta D_{n-k}^{(\lambda+k,\mu)}(\cos\theta)\sin^k\phi D_{n-k}^{(\lambda+k,\mu)}(\cos\phi)C_k^{(\lambda-\frac{1}{2})}(s)
\end{aligned}\tag{2.7}$$

where

$$a_{k,j}^{(\lambda,\mu)} = \frac{\Gamma(\lambda - \frac{1}{2})}{\Gamma(\lambda + \mu)} \frac{\Gamma(\lambda + \mu + k + 1)}{(n + \lambda + \mu)\Gamma(k + \lambda - \frac{1}{2})}. \tag{2.8}$$

PROOF. Taking integral of (2.5) in the previous theorem with respect to $(1 - t^2)^{\mu-1/2}\, dt$ and $C_1^{(\mu-1/2)}(t)(1 - t^2)^{\mu-1/2}\, dt$, respectively, leads to

$$\begin{aligned} c_\mu \int_{-1}^{1} & C_n^{(\lambda+\mu)}(\cos\theta\cos\phi t + \sin\theta\sin\phi s)(1 - t^2)^{\mu-1}\, dt \\ &= \sum_{m=0}^{[\frac{n}{2}]} b_{0,n-2m}^n (\sin\theta\sin\phi)^{n-2m} D_{2m}^{(\lambda+n-2m,\mu)}(\cos\theta) D_{2m}^{(\lambda+n-2m,\mu)}(\cos\phi) C_{n-2m}^{(\lambda-\frac{1}{2})}(s) \end{aligned}$$

and

$$\begin{aligned} c_\mu \int_{-1}^{1} & C_n^{(\lambda+\mu)}(\cos\theta\cos\phi t + \sin\theta\sin\phi s)(1 - t^2)^{\mu-1}\, dt \\ &= \sum_{m=0}^{[\frac{n}{2}]} b_{1,n-2m-1}^n \frac{\mu - \frac{1}{2}}{n - 2m + \lambda + \mu} \\ &\qquad \times (\sin\theta\sin\phi)^{n-2m-1} D_{2m+1}^{(\lambda+n-2m-1,\mu)}(\cos\theta) D_{2m+1}^{(\lambda+n-2m-1,\mu)}(\cos\phi) C_{n-2m-1}^{(\lambda-\frac{1}{2})}(s) \end{aligned}$$

where in deriving the second equation we have used (2.2) and the fact that

$$C_1^{(\frac{\mu-1}{2})}(t) = (2\mu - 1)t \text{ and } \int_{-1}^{1} [C_1^{(\mu-\frac{1}{2})}(t)]^2 (1 - t^2)^{\mu-1}\, dt = c_\mu^{-1} \frac{\mu - \frac{1}{2}}{\mu + \frac{1}{2}}.$$

From (2.5) it is readily verified that

$$\begin{aligned} \frac{\mu - \frac{1}{2}}{n - 2m + \lambda + \mu} b_{1,n-2m-1}^n &= \frac{\Gamma(n - 2m + \lambda + \mu + 1)}{(n + \lambda + \mu)\Gamma(n - 2m + \lambda - \frac{3}{2})} \frac{\Gamma(\lambda - \frac{1}{2})}{\Gamma(\lambda + \mu)} \\ &= b_{0,n-2m-1}^n. \end{aligned}$$

Therefore, adding the two formulae together and changing the summation index, we have the desired equation. ∎

We can take the limit $\mu \longrightarrow 0$ in the formula by using the relation

$$\lim_{\mu\to 0} c_\mu \int_{-1}^{1} f(t)(1 - t^2)^{\mu-1}\, dt = \frac{f(1) + f(-1)}{2}, \tag{2.9}$$

then the formula (2.6) becomes

$$\begin{aligned} C_n^{(\lambda)}(\cos\theta\cos\phi + \sin\theta\sin\phi s) &= \sum_{k=0}^{n} \frac{\Gamma(\lambda + k + 1)\Gamma(\lambda - \frac{1}{2})}{2(n + \lambda)\Gamma(k + \lambda - \frac{1}{2})\Gamma(\lambda)} \\ &\qquad \times (\sin\theta\sin\phi)^k \tilde{C}_{n-k}^{(\lambda+k)}(\cos\theta) \tilde{C}_{n-k}^{(\lambda+k)}(\cos\phi) C_k^{(\lambda-\frac{1}{2})}(s), \end{aligned}$$

which is the addition formula for the Gegenbauer polynomials (*cf.* [7, Vol. I, Sec. 3.15.1, (19)] and taking into account the normalization constant of $\tilde{C}_n^{(\mu)}$).

THEOREM 2.5. *For $\lambda, \mu > 0$,*

(2.10)
$$D_n^{(\lambda,\mu)}(\cos\theta) D_n^{(\lambda,\mu)}(\cos\phi) = \frac{n+\lambda+\mu}{\lambda+\mu} c_\lambda c_\mu \times \int_{-1}^1 \int_{-1}^1 C_n^{(\lambda+\mu)}(t\cos\theta\cos\phi + s\sin\theta\sin\phi)(1+t)(1-t^2)^{\mu-1}(1-s^2)^{\lambda-1}\, dt\, ds.$$

PROOF. Because of (2.1) and the fact that $t(1-t^2)^{\mu-1/2}$ is odd, it follows from change of variables $t \longmapsto -t$ and $s \longmapsto -s$ that the factor $(1+t)(1-t^2)^{\mu-1/2}$ in the integral can be replaced by $(1-t^2)^{\mu-1/2}$ when n is even and by $t(1-t^2)^{\mu-1/2}$ when n is odd. Thus, for $2n$, the formula (2.10) is the same as the formula (2.4). For $2n+1$ we multiply the formula (2.5) by $C_1^{(\mu-1/2)}(t) = (2\mu-1)t$ and integrate with respect to $(1-t^2)^{\mu-1}(1-s^2)^{\lambda-1}\, ds\, dt$ to conclude that

$$c_\lambda \int_{-1}^1 \int_{-1}^1 C_{2n+1}^{(\lambda+\mu)}(\cos\theta\cos\phi t + \sin\theta\sin\phi s) t(1-t^2)^{\mu-1}(1-s^2)^{\lambda-1}\, dt\, ds$$
$$= \frac{b_{1,0}^{2n+1}}{2\mu-1} \int_{-1}^1 [C_1^{\mu-\frac{1}{2}}(t)]^2 (1-t^2)^{\mu-1}\, dt \cos\theta\cos\phi D_{2n}^{(\lambda,\mu+1)}(\cos\theta) D_{2n}^{(\lambda,\mu+1)}(\cos\phi),$$

from which the desired result follows from (2.2), while the constant can be easily verified. ∎

As $\mu \longrightarrow 0$, the above formula reduces to the product formula of the Gegenbauer polynomials (*cf.* [7, Vol. I. Sec. 3.15.1, (20)]). One immediate consequence of the above theorem is the following interesting representation of the generalized Gegenbauer polynomial.

COROLLARY 2.6. *For $\lambda, \mu > 0$, $n \geq 0$,*

$$D_n^{(\lambda,\mu)}(x) D_n^{(\lambda,\mu)}(1) = \frac{n+\lambda+\mu}{\lambda+\mu} c_\mu \int_{-1}^1 C_n^{(\lambda+\mu)}(xt)(1+t)(1-t^2)^{\mu-1}\, dt. \tag{2.11}$$

We will not use this particular representation. However, the following remark seems to be worthwhile. The normalization of the usual Gegenbauer polynomials $C_n^{(\lambda)}$ comes more or less from the simple generating function

$$\sum_{n=0}^{\infty} C_n^{(\lambda)}(x) r^n = \frac{1}{(1-2rx+r^2)^\lambda}.$$

In our definition of $D_n^{(\lambda,\mu)}$ we have chosen the constant so that the polynomial is orthonormal since there is no obvious reason to choose any other one at the time. In this respect, the formula (2.11) suggests that a reasonable choice would be

$$C_n^{(\lambda,\mu)}(x) = \frac{\lambda+\mu}{n+\lambda+\mu} D_n^{(\lambda,\mu)}(1) D_n^{(\lambda,\mu)}(x),$$

since these polynomials will have a generating function

$$\sum_{n=0}^{\infty} C_n^{(\lambda,\mu)}(x) r^n = c_\mu \int_{-1}^1 \frac{1}{(1-2rtx+r^2)^{\lambda+\mu}} (1+t)(1-t^2)^{\mu-1}\, dt, \tag{2.12}$$

which can be taken as the definition of $C_n^{(\lambda,\mu)}$. In particular, if $\mu \longrightarrow 0$, then (2.12) reduces to the generating function of $C_n^{(\lambda)}$.

THEOREM 2.7. *For $\lambda, \mu > 0$,*

(2.13)
$$\frac{\lambda+\mu+1}{\lambda+\frac{1}{2}}\sin\theta\sin\phi D_{n-1}^{(\lambda+1,\mu)}(\cos\theta)D_{n-1}^{(\lambda+1,\mu)}(\cos\phi) = \frac{n+\lambda+\mu}{\lambda+\mu}c_\lambda c_\mu \times \int_{-1}^{1}\int_{-1}^{1} C_n^{(\lambda+\mu)}(\cos\theta\cos\phi t + \sin\theta\sin\phi s)(1+t)(1-t^2)^{\mu-1}s(1-s^2)^{\lambda-1}\,dt\,ds.$$

PROOF. The proof is very similar to that of Theorem 2.5. For the case of odd integer $2n+1$, we multiply (2.7) by $C_1^{(\lambda-1/2)}(s)$ and then integrate against $(1-t^2)^{\mu-1}(1-s^2)^{\lambda-1}\,dt\,ds$. For the case of even integer $2n$, we multiply (2.7) by $C_1^{(\lambda-1/2)}(s)C_1^{(\mu-1/2)}(t)$ and then integrate against the same measure; here we need to use the formula (2.2). We omit the details. ∎

3. **Orthogonal polynomials on spheres.** We study orthogonal polynomials on S^{d-1} with respect to the weight function $h_\alpha^2\,d\omega$, where the function h_α is defined in (1.1). For $d = 2$, one family of h-spherical harmonics is explicitly given in terms of Jacobi polynomials in [4]. We state it below using the notation of the generalized Gegenbauer polynomials.

THEOREM 3.1. *Let $d = 2$ and $h_\alpha = H_\alpha|x_1|^{\alpha_1}|x_2|^{\alpha_2}$. An orthonormal basis for $\mathcal{H}_n^h$ is given by*

$$Y_1^n(\mathbf{x}) = r^n D_n^{(\alpha_1,\alpha_2)}(\cos\theta), \quad Y_2^n(\mathbf{x}) = r^n\sqrt{\frac{\alpha_1+\alpha_2+1}{\alpha_1+\frac{1}{2}}}\sin\theta D_{n-1}^{(\alpha_1+1,\alpha_2)}(\cos\theta), \tag{3.1}$$

where we use the polar coordinates $\mathbf{x} = (r\sin\theta, r\cos\theta)$ and we take $Y_2^0(\mathbf{x}) = 0$.

In particular, the restriction of Y_i^n on S^1 are orthonormal polynomials of degree n with respect to the weight function h_α^2 on S^1. The theorem can be easily proved by verifying the orthonormal relation directly.

To describe the result for $d \ge 3$, we need the definition of the spherical coordinates. For $\mathbf{x} \in \mathbb{R}^d$, these coordinates are defined by

$$\begin{aligned} x_1 &= r\sin\theta_{d-1}\cdots\sin\theta_2\sin\theta_1 \\ x_2 &= r\sin\theta_{d-1}\cdots\sin\theta_2\cos\theta_1 \\ &\vdots \\ x_{d-1} &= r\sin\theta_{d-1}\cos\theta_{d-2} \\ x_d &= r\cos\theta_{d-1}, \end{aligned}$$

where $r \ge 0$, $0 \le \theta_1 \le 2\pi$, $0 \le \theta_k \le \pi$, $k \ne 2$. We also introduce the following notations. For each $n \in \mathbb{N}_0$, let

$$n = k_0 \ge k_1 \ge \cdots \ge k_{d-2} \ge 0, \quad \mathbf{k} = (k_1, \ldots, k_{d-2}).$$

For $\alpha = (\alpha_1, \ldots, \alpha_d)$, we define

$$\alpha^j = (\alpha_1, \ldots, \alpha_j), \quad 1 \le j \le d.$$

Since α^1 consists of only the first element of α, we write $\alpha^1 = \alpha_1$.

THEOREM 3.2. *Let $d \geq 2$ and $n = k_0 \geq k_1 \geq \cdots \geq k_{d-2} \geq 0$. In spherical coordinates an orthonormal basis of $\mathcal{H}_n^h$ is given by*

$$
\begin{aligned}
Y_{\mathbf{k}}^{n,i}(\mathbf{x}) = A_{\mathbf{k}}^n r^n \prod_{j=1}^{d-2} D_{k_{j-1}-k_j}^{(k_j+|\alpha^{d-j}|_1+\frac{d-j-1}{2},\alpha_{d-j+1})}(\cos\theta_{d-j}) \\
\times (\sin\theta_{d-j})^{k_j} Y_i^{k_{d-2}}(\sin\theta_1,\cos\theta_1),
\end{aligned} \tag{3.2}
$$

where $Y_i^{k_{d-2}}$, $i = 1, 2$, are the h-harmonics in (3.1), and

$$
[A_{\mathbf{k}}^n]^2 = \frac{\Gamma(\alpha_1+\alpha_2+1)}{\Gamma(|\alpha|_1+\frac{d}{2})} \prod_{j=1}^{d-2} \frac{\Gamma(k_j+|\alpha^{d-j+1}|_1+\frac{d-j+1}{2})}{\Gamma(k_j+|\alpha^{d-j}|_1+\frac{d-j}{2})}. \tag{3.3}
$$

We note that if $k_{d-2} = 0$ then $Y_{\mathbf{k}}^{n,2} = 0$ by the convention we adopted in Theorem 3.1.

These explicit formulae are known to Dunkl (personal communication) in terms of Jacobi polynomials. The orthonormal relation as stated can be verified by computing the relevant integrals. However, to stress the analog of h-harmonics and the usual spherical harmonics, we shall derive the h-harmonics from $\Delta_h p = 0$. Since the development is similar to that of usual spherical harmonics (*cf.* [14, Ch. IX]), we shall give only an outline of the proof.

PROOF OF THEOREM 3.2. We start with the following decomposition of $\mathcal{P}_n^d$

$$
\mathcal{P}_n^d = \sum_{k=0}^{n} x_d^{n-k} \mathcal{H}_k^{d-1,h} + r^2 \mathcal{P}_{n-2}^d. \tag{3.4}
$$

It follows from the fact that $f \in \mathcal{P}_n^d$ can be written as

$$
f(\mathbf{x}) = r^2 F(\mathbf{x}) + x_d \phi_1(\mathbf{x}') + \phi_2(\mathbf{x}'), \quad \mathbf{x} = (\mathbf{x}', x_d),
$$

where $F \in \mathcal{P}_{n-2}^d$, $\phi_1 \in \mathcal{P}_{n-1}^d$ and $\phi_2 \in \mathcal{P}_n^d$, then use the canonical decomposition of [3, Theorem 1.7]

$$
\mathcal{P}_n^d = \sum_{j=0}^{[\frac{n}{2}]} \oplus |\mathbf{x}|^{2j} \mathcal{H}_{n-2j}^{d,h}
$$

to expand ϕ_i and collect terms according to the power of x_d after replacing $|\mathbf{x}'|^2$ by $|\mathbf{x}|^2 - x_d^2$. For any $P \in \mathcal{H}_n^{d,h}$, the decomposition (3.4) allows us to write

$$
P(\mathbf{x}) = \sum_{k=0}^{n} x_d^{n-k} P_k(\mathbf{x}') + r^2 Q(\mathbf{x}), \quad Q \in \mathcal{P}_{n-2}^d.
$$

Therefore, using the harmonic projection operator $\operatorname{proj}_{\mathcal{H}_n}$ given in [3, Theorem 1.11] it follows that

$$
\begin{aligned}
P(\mathbf{x}) &= \sum_{k=0}^{n} \operatorname{proj}_{\mathcal{H}_n}\left(x_d^{n-k} P_k(\mathbf{x}')\right) \\
&= \sum_{k=0}^{n} \sum_{j=0}^{[\frac{n}{2}]} \frac{(-1)^j |\mathbf{x}|^{2j} \Delta_h^j \left(x_d^{n-k} P_k(\mathbf{x}')\right)}{4^j j! (-\frac{d}{2} - |\alpha|_1 - n + 2)_j}.
\end{aligned}
$$

We write $\Delta_{h,d}$ for Δ_h to indicate the dependence on d. Since our h_α is separable in variables, it follows from the definition of Δ_h and Dunkl's operators (1.3) that

$$\Delta_{h,d} = \Delta_{h,d-1} + \mathcal{D}_d^2,$$

where $\Delta_{h,d-1}$ is with respect to $\mathbf{x}' = (x_1, \ldots, x_d)$. Therefore, since $P_k \in \mathcal{H}_k^{d-1,h}$, it follows that

$$\Delta_{h,d}\big(x_d^{n-k} P_k(\mathbf{x}')\big) = \mathcal{D}_d^2(x_d^{n-k}) P_k(\mathbf{x}').$$

By the definition of T_d, it follows easily that

$$\mathcal{D}_d^2(x_d^{n-k}) = \big(n-k+[1-(-1)^{n-k}]\alpha_d\big)\big(n-k-1+[1-(-1)^{n-k-1}]\alpha_d\big)x_d^{n-k-2}.$$

Using the formula repeatedly, we end up with that for $n-k$ being an even integer,

$$\Delta_{h,d}^j\big(x_d^{n-k}P_k(\mathbf{x}')\big) = 2^{2j}\Big(-\frac{n-k}{2}\Big)_j\Big(-\frac{n-k-1}{2}-\alpha_d\Big)_j x_d^{n-k-2j}P_k(\mathbf{x}'),$$

where $(a)_j = a(a+1)\cdots(a+j-1)$. Therefore, for $n-k$ even, we get

$$\begin{aligned}
&\operatorname{proj}_{\mathcal{H}_n}\big(x_d^{n-k}P_k(\mathbf{x}')\big)\\
&\quad= \sum_{j=0}^{[\frac{n-k}{2}]} \frac{(-1)^j(-\frac{n-k}{2})_j(-\frac{n-k-1}{2}-\alpha_d)_j}{(-n-|\alpha|_1-\frac{d}{2}+2)_j j!} r^{2j} x_d^{n-k-2j} P_k(\mathbf{x}')\\
&\quad= P_k(\mathbf{x}')x_d^{n-k}{}_2F_1\Big(-\frac{n-k}{2}, -\frac{n-k-1}{2}-\alpha_d; -n-|\alpha|_1-\frac{d}{2}+2; \frac{r^2}{x_d^2}\Big)\\
&\quad= \mathrm{const}\, P_k(\mathbf{x}')r^{n-k}P_{\frac{n-k}{2}}^{(k+|\alpha|_1-\alpha_d+\frac{d-3}{2},\alpha_d-\frac{1}{2})}\Big(2\frac{x_d^2}{r^2}-1\Big),
\end{aligned}$$

where in the last step we have used the formula on [13, p. 64, (4.22.)]. A similar equation holds for $n-k$ odd. By the definition of the generalized Gengenbauer polynomials, we then have

$$\operatorname{proj}_{\mathcal{H}_n}\big(x_d^{n-k}P_k(\mathbf{x}')\big) = \mathrm{const}\, P_k(\mathbf{x}')r^{n-k}C_{n-k}^{(k+|\alpha|_1-\alpha_d+\frac{d-2}{2},\alpha_d)}(\cos\theta_d).$$

Since $P_k \in \mathcal{H}_k^{h,d-1}$ they admit a similar decomposition. We can continue this process until we get to the case of h-harmonic polynomials of two variables x_1 and x_2, which can be written as linear combinations of the spherical h-harmonics in Theorem 3.1. Therefore, taking into account that in spherical coordinates $x_j/r_j = \cos\theta_{j-1}$ and $r_{j-1}/r_j = \sin\theta_{j-1}$, where $r_j = x_1^2 + \cdots + x_j^2$, we conclude that any polynomials in $\mathcal{H}_n^{h,d}$ can be uniquely presented as a linear combination of functions of the form $r^n Y_{\mathbf{k}}^{n,i}$. The value of $A_{\mathbf{k}}^n$ is determined by

$$\int_{S^{d-1}} [Y_{\mathbf{k}}^{n,i}]^2 h_\alpha^2(\mathbf{x})\, d\omega = 1,$$

where the integral can be evaluated by using the spherical coordinates. We omit the details. ∎

For each $n \in \mathbb{N}_0$, the reproducing kernel function, P_n^h, for $\mathcal{H}_n^h$ is defined by the property that

$$\int_{S^{d-1}} Q(\mathbf{y}) P_n^h(\mathbf{x},\mathbf{y}) h_\alpha^2 \, d\omega(\mathbf{y}) = Q(\mathbf{x}), \quad Q \in \mathcal{H}_n^h.$$

Let $Y_{\mathbf{k}}^{n,i}$ be the basis of $\mathcal{H}_n^h$ in Theorem 3.2. It follows readily that

$$P_n^h(\mathbf{x},\mathbf{y}) = \sum_{k_1=0}^{n} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{d-2}=0}^{k_{d-3}} [Y_{\mathbf{k}}^{n,1}(\mathbf{x}) Y_{\mathbf{k}}^{n,1}(\mathbf{y}) + Y_{\mathbf{k}}^{n,2}(\mathbf{x}) Y_{\mathbf{k}}^{n,2}(\mathbf{y})];$$

in other words, P_n^h is equal to the sum of the product spherical h-harmonics of degree n, where the sum is over all h-harmonics. In fact, since any two orthonormal basis of $\mathcal{H}_n^h$ differ only by an orthonormal transform, it's easy to see that P_n^h is independent of the choice of the orthonormal bases—a fact that is of interest in the study of orthogonal polynomials in several variables in general (*cf.* [15]).

THEOREM 3.3. *For $h_\alpha^2 \, d\omega$ on S^{d-1},*

$$\begin{aligned} P_n^h(\mathbf{x},\mathbf{y}) = {} & \frac{n + |\alpha|_1 + \frac{d-2}{2}}{|\alpha|_1 + \frac{d-2}{2}} \int_{[-1,1]^d} C_n^{(|\alpha|_1 + \frac{d-2}{2})}(x_1 y_1 t_1 + \cdots + x_d y_d t_d) \\ & \times \prod_{i=1}^{d} (1 + t_i) \prod_{i=1}^{d} c_{\alpha_i} (1 - t_i^2)^{\alpha_i - 1} \, d\mathbf{t}. \end{aligned} \tag{3.5}$$

PROOF. First we derive the compact formula for the case $d = 2$, which will be used in deriving the formula for $d > 2$. Let $\mathbf{x} = (\sin\theta, \cos\theta)$ and $\mathbf{y} = (\sin\phi, \cos\phi)$. From Theorem 3.1 we need to derive a formula for

$$\begin{aligned} P_{n,2}^{(\lambda,\mu)}(\mathbf{x},\mathbf{y}) &= P_{n,2}^h(\mathbf{x},\mathbf{y}) \\ &= D_n^{(\lambda,\mu)}(\cos\theta) D_n^{(\lambda,\mu)}(\cos\phi) \\ &\quad + \frac{\lambda + \mu + 1}{\lambda + \frac{1}{2}} \sin\theta \sin\phi D_{n-1}^{(\lambda+1,\mu)}(\cos\theta) D_{n-1}^{(\lambda+1,\mu)}(\cos\phi). \end{aligned}$$

From Theorem 2.5 and Theorem 2.7 we have

$$\begin{aligned} P_{n,2}^{(\lambda,\mu)}(\mathbf{x},\mathbf{y}) = {} & \frac{n + \lambda + \mu}{\lambda + \mu} c_\lambda c_\mu \int_{-1}^{1} \int_{-1}^{1} C_n^{(\lambda+\mu)}(\cos\theta\cos\phi t + \sin\theta\sin\phi s) \\ & \times (1 + t)(1 - t^2)^{\mu-1} (1 + s)(1 - s^2)^{\lambda-1} \, dt \, ds. \end{aligned}$$

For the general case $d \ge 3$, we use the spherical coordinates. Let $\mathbf{x}$ be associated with $(\theta_1, \ldots, \theta_{d-1})$ and $\mathbf{y}$ be associated with $(\phi_1, \ldots, \phi_{d-1})$. For $\mathbf{x} \in S^{d-1}$ we write $\mathbf{x} = (\sin\theta_{d-1} \mathbf{x}_{d-1}, \cos\theta_{d-1})$ where $\mathbf{x}_{d-1} \in S^{d-2}$. Since

$$x_1 y_1 t_1 + \cdots + x_d y_d t_d = \cos\theta_{d-1} \cos\phi_{d-1} t_d + (x_1 y_1 t_1 + \cdots + x_{d-1} y_{d-1} t_{d-1}),$$

we can apply the formula (2.7) to conclude that

$$\int_{[-1,1]^d} C_n^{(|\alpha|_1+\frac{d-2}{2})}(x_1y_1t_1+\cdots+x_dy_dt_d)\prod_{i=1}^d(1+t_i)\prod_{i=1}^d c_{\alpha_i}(1-t_i^2)^{\alpha_i-1}\,d\mathbf{t}$$
$$=\sum_{k_1=0}^n a_{k_1,n}^{(|\alpha^{d-1}|_1+\frac{d-2}{2},\alpha_d)}(\sin\theta_{d-1}\sin\phi_{d-1})^{k_1}D_{n-k_1}^{(|\alpha^{d-1}|_1+\frac{d-2}{2}+k_1,\alpha_d)}(\cos\theta_{d-1})$$
$$\times D_{n-k_1}^{(|\alpha^{d-1}|_1+\frac{d-2}{2}+k_1,\alpha_d)}(\cos\phi_{d-1})$$
$$\times\int_{[-1,1]^{d-1}} C_{k_1}^{(|\alpha^{d-1}|_1+\frac{d-3}{2})}(x_1y_1t_1+\cdots+x_{d-1}y_{d-1}t_{d-1})\prod_{i=1}^{d-1}(1+t_i)\prod_{i=1}^{d-1}c_{\alpha_i}(1-t_i^2)^{\alpha_i-1}\,d\mathbf{t}.$$

Clearly, we can repeat the above process and reduce the integral in the right hand side one at a time until we are down to the integral

$$\int_{[-1,1]^2} C_{k_{d-2}}^{(|\alpha^2|_1)}(x_1y_1t_1+x_2y_2t_2)(1+t_1)(1+t_2)c_{\alpha_1}c_{\alpha_2}(1-t_1^2)^{\alpha_1-1}(1-t_2^2)^{\alpha_2-1}\,dt_1\,dt_2,$$

which we use the formula we derived for $d=2$. This way, we conclude that

$$\int_{[-1,1]^d} C_n^{(|\alpha|_1+\frac{d-2}{2})}(x_1y_1t_1+\cdots+x_dy_dt_d)\prod_{i=1}^d(1+t_i)\prod_{i=1}^d c_{\alpha_i}(1-t_i^2)^{\alpha_i-1}\,d\mathbf{t}$$
$$=\sum_{k_1=0}^n\sum_{k_2=0}^{k_1}\cdots\sum_{k_{d-2}=0}^{k_{d-3}}\left(\prod_{j=1}^{d-2}a_{k_j,k_{j-1}}^{(|\alpha^{d-j}|_1+\frac{d-j-1}{2},\alpha_{d-j+1})}\right)\frac{\alpha_1+\alpha_2}{k_{d-2}+\alpha_1+\alpha_2}\prod_{j=1}^{d-2}(\sin\theta_{d-j}\sin\phi_{d-j})^{k_j}$$
$$\times D_{k_{j-1}-k_j}^{(|\alpha^{d-j}|_1+\frac{d-j-1}{2}+k_j,\alpha_{d-j+1})}(\cos\theta_{d-j})D_{k_{j-1}-k_j}^{(|\alpha^{d-j}|_1+\frac{d-j-1}{2}+k_j,\alpha_{d-j+1})}(\cos\phi_{d-j})$$
$$\times P_{k_{d-2},2}^{(\alpha_1,\alpha_2)}\big((\sin\theta_1,\cos\theta_1),(\sin\phi_1,\cos\phi_1)\big).$$

From the definition of $a_{k,n}^{(\lambda,\mu)}$ and $A_{\mathbf{k}}$, it's not hard to see that

$$\prod_{j=1}^{d-2}a_{k_j,k_{j-1}}^{(|\alpha^{d-j}|_1+\frac{d-j-1}{2},\alpha_{d-j+1})}\frac{\alpha_1+\alpha_2}{k_{d-2}+\alpha_1+\alpha_2}=\frac{|\alpha|_1+\frac{d-2}{2}}{n+|\alpha|_1+\frac{d-2}{2}}A_{\mathbf{k}}^2,$$

from which the desired result follows from the definition of $Y_{\mathbf{k}}^{n,i}$ in Theorem 3.2. ■

REMARK. It's worthwhile to mention that the formula in the theorem reduces to the classical addition formula for the spherical harmonics (*cf.* [11])

$$P_n(\mathbf{x},\mathbf{y})=\frac{n+\frac{d-2}{2}}{\frac{d-2}{2}}C_n^{(\frac{d-2}{2})}(\mathbf{x}\cdot\mathbf{y})$$

when $\alpha=0$. This follows easily from the limit relation (2.9).

Since $|C_n^{(\lambda)}(t)|\le|C_n^{(\lambda)}(1)|$ for all $|t|\le 1$, it follows readily that

$$|P_n^h(\mathbf{x},\mathbf{y})|\le\frac{n+|\alpha|_1+\frac{d-2}{2}}{|\alpha|_1+\frac{d-2}{2}}C_n^{(|\alpha|_1+\frac{d-2}{2})}(1).$$

uniformly in $\mathbf{x}$ and $\mathbf{y}$.

4. The Poisson kernel and the intertwining operator. The Poisson kernel $P^h(\mathbf{x},\mathbf{y})$ is a function that satisfies the following properties: for each $f \in \mathcal{P}_n$,

$$\int_{S^{d-1}} P^h(\mathbf{x},\mathbf{y}) f(\mathbf{y}) h^2(\mathbf{y})\, d\omega(\mathbf{y}) = f(\mathbf{x}),$$

where $|\mathbf{x}| < |\mathbf{y}| = 1$ and $n \geq 0$. The following theorem gives an explicit formula of this kernel.

THEOREM 4.1. *Let h_α be defined as in (1.1). For* $|\mathbf{x}| < |\mathbf{y}| = 1$,

$$\begin{aligned} P^h(\mathbf{x},\mathbf{y}) = \int_{[-1,1]^d} & \frac{1-|\mathbf{x}|^2}{\big(1-2(x_1y_1t_1+\cdots+x_dy_dt_d)+|\mathbf{x}|^2\big)^{|\alpha|_1+\frac{d}{2}}} \\ & \times \prod_{i=1}^d (1+t_i) \prod_{i=1}^d c_{\alpha_i}(1-t_i^2)^{\alpha_i-1} d\mathbf{t}. \end{aligned} \tag{4.1}$$

PROOF. It is easy to see that the Poisson kernel is given by

$$P^h(\mathbf{x},\mathbf{y}) = \sum_{n=0}^{\infty} P_n^h(\mathbf{x},\mathbf{y}).$$

Therefore, by the compact formula of P_n^h, we only need to sum up a series of Gegenbauer polynomials. From the generating function of the Gegenbauer polynomials, we conclude that

$$\sum_{n=0}^{\infty} \frac{n+\lambda}{\lambda} C_n^{(\lambda)}(t) r^n = \frac{1-r^2}{(1-2tr+r^2)^{\lambda+1}},$$

from which the desired result follows readily from that of Theorem 3.3. ∎

REMARK. Again let us mention that as $\alpha \longrightarrow 0$, we end up with the classical Poisson kernel for the ball (*cf.* [12]),

$$P(\mathbf{x},\mathbf{y}) = \frac{1}{\omega_{d-1}} \frac{1-|\mathbf{x}|^2}{(1-2\mathbf{x}\cdot\mathbf{y}+|\mathbf{x}|^2)^{\frac{d}{2}}}.$$

For $d = 2$, this formula has appeared in [5].

From the formula (4.1), we obtain the following properties of the Poisson kernel,

$$0 \leq P^h(\mathbf{x},\mathbf{y}) \leq 1, \quad |\mathbf{x}| < |\mathbf{y}| = 1, \tag{i}$$

$$\int_{S^{d-1}} P^h(\mathbf{x},\mathbf{y}) h^2(\mathbf{y})\, d\omega(y) = 1, \quad |\mathbf{x}| < 1. \tag{ii}$$

Moreover, for each integrable function f on S^{d-1}, we define the Abel means $S_r(f)$ by

$$S_r(f,\mathbf{x}) = \int_{S^{d-1}} f(y) P^h(r\mathbf{x}',\mathbf{y}) h_\alpha^2(\mathbf{y})\, d\omega(\mathbf{y}), \quad \mathbf{x} = r\mathbf{x}', \quad \mathbf{x}' \in S^{d-1}, \quad r < 1.$$

We have the following theorem.

THEOREM 4.2. *If f is continuous on S^{d-1}, then*

$$\lim_{r\to 0} S_r(f,\mathbf{x}) = f(\mathbf{x}'), \quad \mathbf{x}' \in S^{d-1}.$$

PROOF. Let $A_\delta = \{\mathbf{y} \in S^{d-1} : |\mathbf{x}' - \mathbf{y}| < \delta\}$. Then, by the properties (i) and (ii) of P^h, we have

$$\begin{aligned} |S_r(f,\mathbf{x}) - f(\mathbf{x}')| &\le \Big(\int_{A_\delta} + \int_{S^{d-1}\setminus A_\delta}\Big) f(\mathbf{y}) P^h(r\mathbf{x}',\mathbf{y}) h_\alpha^2(\mathbf{y})\, d\omega(\mathbf{y}) \\ &\le \sup_{|\mathbf{x}'-\mathbf{y}|\le\delta} +2\|f\|_\infty \int_{S^{d-1}\setminus A_\delta} P^h(r\mathbf{x}',\mathbf{y}) h_\alpha^2(\mathbf{y})\, d\omega(\mathbf{y}), \end{aligned}$$

where $\|f\|_\infty$ is the maximum of f over S^{d-1}. Since f is continuous over S^{d-1}, we only need to prove that the last integral converges to zero when $r \longrightarrow 1$. By the definition of P^h, it suffices to show that

$$\begin{aligned} \limsup_{r\to 1} \int_{S^{d-1}} \int_{[0,1]^d} & \frac{1}{\big(1 - 2r(x_1'y_1t_1 + \cdots + x_d'y_dt_d) + r^2\big)^{|\alpha|_1 + \frac{d}{2}}} \\ & \times \prod_{i=1}^d c_{\alpha_i}(1-t_i^2)^{\alpha_i - 1}\, d\mathbf{t} h_\alpha^2(\mathbf{y})\, d\mathbf{y} \end{aligned} \tag{4.2}$$

is finite for every $\mathbf{x}' \in S^{d-1}$. Let $B_\sigma = \{\mathbf{t} = (t_1,\ldots,t_d) : t_i > 1-\sigma, 1 \le i \le d\}$ with $\sigma = \delta^2/4$. For $\mathbf{y} \in S^{d-1} \setminus A_\delta$ and $\mathbf{t} \in B_\sigma$ we have that

$$\begin{aligned} |x_1'y_1t_1 + \cdots x_d'y_dt_d| &= |\mathbf{x}'\cdot\mathbf{y} - x_1'y_1(1-t_1) + \cdots + x_d'y_d(1-t_d)| \\ &\le 1 - \delta^2/2 + \max_i(1-t_i) \le 1 - \delta^2/4 \end{aligned}$$

from which it follows readily that

$$1 - 2r(x_1'y_1t_1 + \cdots x_d'y_dt_d) + r^2 \ge 1 + r^2 - 2r(1-\delta^2/4) = (1-r)^2 + r\delta^2/2 \ge \delta^2/4.$$

For $\mathbf{t} \in [0,1]^d \setminus B_\sigma$, we have that $t_i \le 1-\sigma$ for at least one i. Let us assume that $t_1 \le 1-\sigma$. We can assume that $x_1' \ne 0$, since otherwise, t_1 does not appear in the integral (4.2), and we can repeat the above argument for $(t_2,\ldots,t_d) \in [0,1]^{d-1}$. It follows then that

$$\begin{aligned} 1 - 2r(x_1'y_1t_1 + \cdots + x_d'y_dt_d) + r^2 &= r^2 x_1'^2(1-t_1^2) \\ &\quad + \sum_{i=1}^d (y_i - rx_i't_i)^2 + r^2\Big(1 - x_1'^2 - \sum_{i=2}^d x_i'^2 t_i^2\Big) \\ &\ge r^2 x_1'^2(1-t_1^2) \\ &\ge r^2\sigma x_1'^2 > 0. \end{aligned}$$

Therefore, for each $\mathbf{x} = r\mathbf{x}'$, the denominator of the integrand in (4.2) is never zero and the expression is finite as $r \longrightarrow 1$. ∎

In [6], it is proved that the Poisson kernel for measures invariant under a reflection group can be given by the intertwining operator, V, uniquely determined by

$$V\mathcal{P}_n \subset \mathcal{P}_n, \quad V1 = 1, \quad \mathcal{D}_i V = V\partial_i, \quad 1 \le i \le d;$$

for the h_α defined in (1.1), the formula in [6, Theorem 4.2] states that

$$P^h(\mathbf{x},\mathbf{y}) = V_{\mathbf{y}}\Big((1-|\mathbf{x}|^2)(1-2\mathbf{x}\cdot\mathbf{y}+|\mathbf{x}|^2)^{-|\alpha|_1-\frac{d}{2}}\Big) \tag{4.3}$$

where $V_{\mathbf{y}}$ means that the operator acts on the variable $\mathbf{y}$. For the special weight function that we are interested in, the Theorem 4.1 suggests the following explicit formula of the intertwining operator.

THEOREM 4.3. *For the h_α defined in (1.1),*

$$Vf(\mathbf{x}) = \int_{[-1,1]^d} f(x_1t_1,\ldots,x_dt_d)\prod_{i=1}^d(1+t_i)\prod_{i=1}^d c_{\alpha_i}(1-t_i^2)^{\alpha_i-1}\,d\mathbf{t}. \tag{4.4}$$

PROOF. The fact that $V1 = 1$ and $V\mathcal{P}_n \subset \mathcal{P}_n$ are obvious from the definition. Thus, we only need to verify that $\mathcal{D}_i V = V\partial_i$. From the definition of $\mathcal{D}_i$, we write

$$\mathcal{D}_i f = \partial_i f + \tilde{\mathcal{D}}_i f, \quad \tilde{\mathcal{D}}_i f(\mathbf{x}) = \alpha_i \frac{f(\mathbf{x}) - f(\mathbf{x}-2x_i\mathbf{e}_i)}{x_i}.$$

Take, for example, $i = 1$, we consider

$$\begin{aligned}\tilde{\mathcal{D}}_1 Vf(\mathbf{x}) = \alpha_1 \int_{[-1,1]^d} &\frac{f(x_1t_1,\ldots,x_dt_d) - f(-x_1t_1,x_2t_2,\ldots,x_dt_d)}{x_1}\\ &\times \prod_{i=1}^d(1+t_i)\prod_{i=1}^d c_{\alpha_i}(1-t_i^2)^{\alpha_i-1}\,d\mathbf{t}.\end{aligned}$$

Since the difference in the integral is an odd function of t_1, it follows that

$$\begin{aligned}\tilde{\mathcal{D}}_1 Vf(\mathbf{x}) &= \frac{2\alpha_1}{x_1}\int_{[-1,1]^d} f(x_1t_1,\ldots,x_dt_d)t_1\prod_{i=2}^d(1+t_i)\prod_{i=1}^d c_{\alpha_i}(1-t_i^2)^{\alpha_i-1}\,d\mathbf{t}\\ &= \int_{[-1,1]^d} \partial_1 f(x_1t_1,\ldots,x_dt_d)(1-t_1)\prod_{i=1}^d(1+t_i)\prod_{i=1}^d c_{\alpha_i}(1-t_i^2)^{\alpha_i-1}\,d\mathbf{t}\end{aligned}$$

where the second integral follows from integral by parts. Since

$$\partial_1 Vf(\mathbf{x}) = \int_{[-1,1]^d} \partial_1 f(x_1t_1,\ldots,x_dt_d)t_1\prod_{i=1}^d(1+t_i)\prod_{i=1}^d c_{\alpha_i}(1-t_i^2)^{\alpha_i-1}\,d\mathbf{t},$$

the desired result follows easily from the last two equations. ■

For $d = 1$ and $h(x) = |x|^\alpha$ the formula (4.4) has been obtained in [6]. Although the form of Vf is suggested by the formula of Poisson kernel in Theorem 4.2, its proof is only a simple verification. Thus, one can start with Theorem 4.3 and use (4.3) to derive the formula (4.2) for the Poisson kernel.

The intertwining operator for the general reflection group has been discussed in [6]. It may allow one to transform the results from the ordinary spherical harmonics to the h-harmonics. In particular, in the case of $d = 2$, we know that the spherical harmonics are given by $\cos n\theta$ and $\sin n\theta$. Thus, the intertwining operator shows that the h-harmonics can be written as integrals against trigonometric functions. In fact, the analytic proof of the formula (2.3) is based on the following formula in [1],

$$P_n^{(\lambda-\frac{1}{2},\mu-\frac{1}{2})}(\cos 2\theta) = \frac{\Gamma(n+\lambda+\frac{1}{2})\Gamma(n+\mu+\frac{1}{2})}{\Gamma(\lambda+\frac{1}{2})\Gamma(\mu+\frac{1}{2})} \times \int_{-1}^{1}\int_{-1}^{1}(t\cos\theta + is\sin\theta)^{2n}(1-t^2)^{\mu-1}(1-s^2)^{\lambda-1}\,dt\,ds.$$

Integrating by parts and using the definition of $D_n^{(\lambda,\beta)}$, it is not hard to see that the following formula holds

$$D_n^{(\lambda,\mu)}(\cos\theta) = B_n^{(\lambda,\mu)} c_\lambda c_\mu \int_{-1}^{1}\int_{-1}^{1}(t\cos\theta + is\sin\theta)^n(1+t)(1-t^2)^{\mu-1}(1-s^2)^{\lambda-1}\,dt\,ds,$$

where $B_n^{(\lambda,\mu)}$ is a constant, from which we can perform integration by parts again to derive the formula

$$i\sin\theta D_{n-1}^{(\lambda+1,\mu)}(\cos\theta) = B_{n-1}^{(\lambda+1,\mu)}\frac{2\lambda}{n}c_\lambda c_\mu \int_{-1}^{1}\int_{-1}^{1}(t\cos\theta + is\sin\theta)^n \times (1+t)s(1-t^2)^{\mu-1}(1-s^2)^{\lambda-1}\,dt\,ds.$$

Therefore, we have the following formula which gives explicitly the action of intertwining operator,

$$A_n^{(\lambda,\mu)}D_n^{(\lambda,\mu)}(\cos\theta) + i\frac{n}{2\lambda}A_{n-1}^{(\lambda+1,\mu)}\sin\theta D_{n-1}^{(\lambda+1,\mu)}(\cos\theta) = c_\lambda c_\mu \int_{-1}^{1}\int_{-1}^{1}(t\cos\theta + is\sin\theta)^n(1+t)(1+s)(1-t^2)^{\mu-1}(1-s^2)^{\lambda-1}\,dt\,ds,$$

the constant $A_n^{(\lambda,\mu)}$ can be written as an integral by setting $\theta = 0$ in the above formula. This integral generalizes the Dirichlet type integral for the Gegenbauer polynomials (see [6, p. 1226]).

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Department of Mathematics
University of Oregon,
Eugene, Oregon
USA 97403-1222
e-mail: yuan@math.uoregon.edu