Numer. Math. 85 (2000), 155-173.

CONSTRUCTING CUBATURE FORMULAE BY THE METHOD OF REPRODUCING KERNEL

Yuan Xu

March 10, 1998, Revised Nov. 11, 1998

Abstract. We examine the method of reproducing kernel for constructing cubature formulae on the unit ball and on the triangle in light of the compact formulae of the reproducing kernels that are discovered recently. Several new cubature formulae are derived.

1. Introduction

Among various methods of constructing cubature formulae, the method of reproducing kernel stands out with its simplicity. The purpose of this paper is to have a close look of this method for the unit ball and for the standard simplex, making use of the compact formulae of the reproducing kernels that are discovered recently. We first describe the method of reproducing kernel; the method was first considered by Mysovskikh and later studied by Möller ([8, 10, 11]).

For a given weight function W on a subset Ω of $\mathbb{R}^d$, a cubature formula is a finite linear combination of point evaluations that approximates $\int_\Omega fW d\mathbf{x}$. The cubature formula is said to have degree m if it yields the exact value of the integrals whenever f are polynomials of degree at most m. We denote the space of polynomials of degree at most n by Π_n^d. Let $\{P_k^n : 1 \le k \le r_n^d, 0 \le n < \infty\}$, where $r_n^d = \binom{n+d-1}{d}$, denote a sequence of orthonormal polynomials of degree n with respect to the inner product $(f,g) = \int_\Omega fgW d\mathbf{x}$, where the superscript n means that $P_k^n \in \Pi_n^d$. Throughout this paper, we will use the notation $\mathbb{P}_n = (P_1^n, \dots, P_{r_n^d}^n)$ either as a subset of Π_n^d or as a column vector. The n-th reproducing kernel $K_n(\mathbf{x},\mathbf{y})$ of the space Π_n^d is defined by

$$K_n(\mathbf{x},\mathbf{y}) = \sum_{k=0}^{n} \mathbb{P}_k^T(\mathbf{x})\mathbb{P}_k(\mathbf{y}) = \sum_{k=0}^{n}\sum_{j=1}^{r_k^d} P_j^k(\mathbf{x})P_j^k(\mathbf{y}), \qquad \mathbf{x},\mathbf{y} \in \mathbb{R}^d. \tag{1.1}$$

The method of reproducing kernel requires one to choose d points $\mathbf{a}_1, \dots, \mathbf{a}_d$ in $\mathbb{R}^d$ such that the hypersurfaces $H_1, \dots, H_d$, where H_i is the surface defined by $K_n(\mathbf{x}, \mathbf{a}_i) = 0$,

---

1991 *Mathematics Subject Classification.* 33C50, 42C05, 41A 63.

Key words and phrases. Reproducing kernel, cubature formula, orthogonal polynomials in several variables, on simplex, on ball.

Supported by the National Science Foundation under Grant DMS-9802265.

Typeset by $\mathcal{AMS}$-TeX

intersect at n^d points. The points $\mathbf{a}_1, \ldots, \mathbf{a}_d$ are chosen as follows. For $\mathbf{a}_1$ we choose any point that is not a common zero of $\mathbb{P}_n$. Suppose the points $\mathbf{a}_1, \ldots, \mathbf{a}_{r-1}$ have been chosen. We pick $\mathbf{a}_r$ as a point from the intersection $H_1 \cap H_2 \cap \ldots \cap H_{r-1}$ which is not a common zero of $\mathbb{P}_n$. We assume further that the infinity is not a common point of $H_1, \cdots, H_d$. Then the following result holds.

Method of reproducing kernel: *If $H_1, \ldots, H_d$ defined by $\mathbf{a}_1, \ldots, \mathbf{a}_d$ intersect at n^d distinct points, $\mathbf{x}_1, \ldots, \mathbf{x}_{n^d}$, then there is a cubature formula of degree $2n$,*

$$\mathcal{L}_n(f) = \sum_{i=1}^{d} \lambda_i f(\mathbf{a}_i) + \sum_{j=1}^{n^d} \mu_j f(\mathbf{x}_j), \quad f \in \Pi_{2n}^d, \tag{1.2}$$

where $\lambda_i = 1/K_n(\mathbf{a}_i, \mathbf{a}_i)$.

If the weight function W is of centrally symmetric, that is, W and its support set Ω satisfy

$$\mathbf{x} \in \Omega \Rightarrow -\mathbf{x} \in \Omega, \qquad W(\mathbf{x}) = W(-\mathbf{x}), \tag{1.3}$$

then there is a modified method of reproducing kernel due to Möller ([8]). Let $\tilde{K}_n$ denote

$$\tilde{K}_n(\mathbf{x}, \mathbf{y}) = {\sum_{k=0}^{n}}' {\sum_{j=1}^{r_k^d}}' P_j^k(\mathbf{x}) P_j^k(\mathbf{y}), \qquad \mathbf{x}, \mathbf{y} \in \mathbb{R}^d, \tag{1.4}$$

where $\sum'$ means that the summation is taken over those j so that the corresponding P_j^k has the same parity as n. We choose points $\mathbf{a}_i$ as before except that we replace H_i by the hypersurface $\tilde{H}_i$ defined by the equation $\tilde{K}_n(\mathbf{x}, \mathbf{a}_i) = 0$. We also assume that the infinity is not a common point of $\tilde{H}_1, \ldots, \tilde{H}_d$. Then the following holds,

Modified method of reproducing kernel: *Let W be centrally symmetric. If $\tilde{H}_1, \ldots, \tilde{H}_d$ defined by $\mathbf{a}_1, \ldots, \mathbf{a}_d$ intersect at n^d distinct points, $\mathbf{x}_1, \ldots, \mathbf{x}_{n^d}$, then there is a cubature formula of degree $2n+1$,*

$$\mathcal{L}_n(f) = \sum_{i=1}^{d} \lambda_i [f(\mathbf{a}_i) + f(-\mathbf{a}_i)]/2 + \sum_{j=1}^{n^d} \mu_j f(\mathbf{x}_j), \quad f \in \Pi_{2n+1}^d, \tag{1.5}$$

where $\lambda_i = 1/\tilde{K}_n(\mathbf{a}_i, \mathbf{a}_i)$.

In particular, if $d = 2$, then the method only requires two points $\mathbf{a}_1$ and $\mathbf{a}_2$ so that the polynomial surfaces $\tilde{K}_n(\mathbf{x}, \mathbf{a}_1)$ and $\tilde{K}_n(\mathbf{x}, \mathbf{a}_2)$ have n^2 common zeros. Because of its simplicity, this method has been used to generate lower degree cubature formulae on the classical domains such as square, disk and triangle; see, for example, [1, 2, 8, 11, 12, 14] and the references therein. There is also further modification of this method, see [14]. Evidently, if a compact formula of the reproducing kernel is known, it may help us to identify the points $\mathbf{a}_i$ in the method. There have been attempts to find compact formulae for the reproducing kernels for the unit weight function over the classical domains. For example, in the case of unit weight function on the unit ball, see [2, 12]. Recently, we found compact formulae of the reproducing kernel for the Jacobi type weight functions on the unit ball and on the standard simplex ([18, 19], see the following section). The compact formulae allow us to have a close look at the method of the reproducing kernel. It shows, in particular, that the method works for all n in these two cases, although some of the nodes may lie outside the domain. The method of reproducing kernel yields cubature formulae of degree $2n+1$ or $2n$ with $n^d + dn$ or $n^d + dn - 1$ nodes, which is much more than the theoretic lower bound for the nodes. Only for small n and d will the method yield formulae with nodes close to the lower bound. We will construct several cubature formulae of lower degrees making use of the compact formulae. For large n, the number of nodes in the method of reproducing kernel is just slightly less than $(n+1)^d$, the number of nodes of the product type formula on these domains. In fact, we will show in the case of $d=2$ that if one of the point, say, $\mathbf{a}_1$, goes to infinity in a proper setting, then the method will yield a formula that coincides with a special type of product formula.

For references on cubature formulae we refer to [3, 5, 7, 11, 15] and the references therein. We discuss cubature formulae on the ball in Section 2, and cubature formulae on the simplex in Section 3.

2. Cubature formulae on the unit ball

For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, we use the notation $\mathbf{x}\cdot\mathbf{y}$ for the usual Euclidean inner product, $\mathbf{x}\cdot\mathbf{y} = x_1y_1 + \ldots + x_dy_d$, and $|\mathbf{x}|$ for the usual Euclidean norm, $|\mathbf{x}|^2 = \mathbf{x}\cdot\mathbf{x}$. We consider cubature formulae on the unit ball $B^d = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| \le 1\}$ with respect to the normalized weight function

$$W_\mu(\mathbf{x}) = W_{\mu,d}(\mathbf{x}) = w_\mu (1 - |\mathbf{x}|^2)^{\mu - \frac{1}{2}}, \quad \mu \ge 0, \qquad \mathbf{x} \in B^d, \tag{2.1}$$

where w_μ is a constant chosen so that the integral of W_μ is 1,

$$w_\mu = w_{\mu,d} = \frac{2}{\omega_{d-1}} \frac{\Gamma(\mu + \frac{d+1}{2})}{\Gamma(\mu+\frac{1}{2})\Gamma(\frac{d}{2})} = \frac{\Gamma(\mu + \frac{d+1}{2})}{\pi^{d/2}\Gamma(\mu+\frac{1}{2})}, \tag{2.2}$$

where $\omega_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$ is the surface area of the unit sphere in $\mathbb{R}^d$. Clearly, the function W_μ is centrally symmetric, we will use the modified method of reproducing kernel to generate cubature formulae for this weight function.

2.1 Reproducing kernel. We denote the n-th reproducing kernel with respect to W_μ by $\mathbf{K}_n^{(\mu)}(\cdot,\cdot)$. It has the following closed formula

(2.3)
$$\begin{aligned} K_n(W_\mu; \mathbf{x}, \mathbf{y}) = c_\mu \int_{-1}^1 \Big[& C_n^{(\mu + \frac{d+1}{2})}(\mathbf{x}\cdot\mathbf{y} + \sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}\, t) \\ & + C_{n-1}^{(\mu + \frac{d+1}{2})}(\mathbf{x}\cdot\mathbf{y} + \sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}\, t) \Big] (1-t^2)^{\mu-1} dt, \end{aligned}$$

where $c_\mu = 1/\int_{-1}^1 (1-t^2)^{\mu-1}dt$, and $C_n^{(\lambda)}$ is the Gegenbauer polynomial of degree n, defined by the generating function

$$\frac{1}{(1-2tr+r^2)^\lambda} = \sum_{n=0}^\infty C_n^{(\lambda)}(t) r^n$$

(see [16, p. 80]). This formula is derived recently in [18] and has been used in [20]. Mysovskikh has derived a formula for $K_n(W_{1/2};\cdot,\cdot)$ in [12], which is a one fold sum of Jacobi polynomials, not as compact as the above one. Since $C_n^{(\lambda)}(-t) = (-1)^n C_n^{(\lambda)}(t)$, by considering $K_n(W_\mu;\mathbf{x},\mathbf{y}) \pm K_n(W_\mu;\mathbf{x},-\mathbf{y})$ for n being even and odd, respectively, it follows from the formula (2.3) that the modified reproducing kernel function $\tilde{K}_n(W_\mu;\cdot,\cdot)$ is given by the formula

$$\tilde{K}_n(W_\mu;\mathbf{x},\mathbf{y}) = c_\mu \int_{-1}^1 C_n^{(\mu+\frac{d+1}{2})}(\mathbf{x}\cdot\mathbf{y} + \sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}\, t)(1-t^2)^{\mu-1}dt. \tag{2.4}$$

For $\mu \to 0$, both of these formulae hold upon using the limit

$$\lim_{\mu\to 0} c_\mu \int_{-1}^1 f(t)(1-t^2)^{\mu-1}dt = \frac{f(1)+f(-1)}{2}. \tag{2.5}$$

In the case $\mu = 1/2$, we have $W_{1/2}(\mathbf{x}) = d/\omega_{d-1}$, a multiple of the unit weight function. Another interesting case is $\mu = 0$, that is, $W_0(\mathbf{x}) = w_0(1-|\mathbf{x}|)^{-1/2}$. In [21], it is shown that a cubature formula for W_0 corresponds to a formula for the surface measure $d\omega$ on the unit sphere S^d. The formulae for $K_n(W_0;\cdot,\cdot)$ and $\tilde{K}_n(W_0;\cdot,\cdot)$ take a simple form, for example,

$$\tilde{K}_n(W_0;\cdot,\cdot) = \frac{1}{2}\Big[C_n^{(\frac{3}{2})}(\mathbf{x}\cdot\mathbf{y} + \sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}) + C_n^{(\frac{3}{2})}(\mathbf{x}\cdot\mathbf{y} - \sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2})\Big],$$

which follows from (2.4) by using the limit (2.5) repeatly.

The compact formula takes a particular simple form when one of its argument has norm one; we have

$$\tilde{K}_n(W_\mu;\mathbf{x},\mathbf{a}) = C_n^{(\mu+(d+1)/2)}(\mathbf{x}\cdot\mathbf{a}), \qquad |\mathbf{a}| = 1. \tag{2.6}$$

It should be pointed out that if $|\mathbf{a}| = 1$ then $\mathbf{a}$ is not a common zero of $\mathbb{P}_n$. This follows from the fact that $\mathbb{P}_n$ has no common zero if n is even, and it has only origin as common zero if n is odd.

2.2 A family of cubature formulae on B^d. The compact formula of $\mathbf{K}_n(W_\mu;\cdot,\cdot)$ at (2.4) suggests possible choices of $\mathbf{a}_i$ in the method of reproducing kernel. For example, the formula (2.6) suggests that we choose $\mathbf{a}_k$ to have norm one. Since the Gegenbauer polynomial $C_n^{(\mu+(d+1)/2)}(t)$ has all its zeros inside $(-1,1)$, we denote its zeros by

$$-1 < t_{1,n} < t_{2,n} < \ldots < t_{n,n} < 1.$$

It is known that these zeros are symmetric with respect to the origin, that is, they satisfy the relation $t_{i,n} = -t_{n-i+1,n}$.

First we give a choice of $\mathbf{a}_1, \ldots, \mathbf{a}_d$ so that the method of reproducing kernel works for W_μ and all n in every dimension. Let n be fixed and let $t_{*,n}$ be a fixed zero of $C_n^{(\mu+(d+1)/2)}(t)$. We define $\mathbf{a}_1, \ldots, \mathbf{a}_d$ as follows:

$$\mathbf{a}_1 = (1, 0, \ldots, 0), \qquad \mathbf{a}_k = (b_1, \ldots, b_{k-1}, \sqrt{1 - b_1^2 - \ldots - b_{k-1}^2}, 0, \ldots, 0), \quad 2 \le k \le d,$$

where the numbers $b_1, \ldots, b_{d-1}$ are determined inductively by $\mathbf{a}_k \cdot \mathbf{a}_{k+1} = t_{*,n}$, or equivalently,

$$b_1^2 + \ldots + b_{k-1}^2 + \sqrt{1 - b_1^2 - \ldots - b_{k-1}^2}\ \ b_k = t_{*,n}$$

for $k = 1, 2, \ldots, d-1$. Thus, $b_1 = t_{*,n}$, $b_2 = (t_{*,n} - b_1^2)/\sqrt{1 - b_1^2}$, ... Since $t_{*,n} < 1$, the above formula implies that $b_k \le \sqrt{1 - b_1^2 - \ldots - b_{k-1}^2}$; hence, $\mathbf{a}_{k+1}$ is well-defined. It follows that

$$H_1 \cap \cdots \cap H_k = \{\mathbf{x} \in \mathbb{R}^d : \mathbf{x} \cdot \mathbf{a}_1 = t_{i_1,n}, \ \ldots, \ \mathbf{x} \cdot \mathbf{a}_k = t_{i_k,n}, \quad 1 \le i_1, \ldots, i_k \le n\}$$

for $k = 2, \ldots, d$. We claim that $\mathbf{a}_{k+1} \in H_1 \cap \cdots \cap H_k$. Clearly, $\mathbf{a}_2 = (t_{*,n}, \sqrt{1 - t_{*,n}^2}, 0, \ldots, 0)$ belongs to H_1. Assume that we have established $\mathbf{a}_k \in H_1 \cap \cdots \cap H_{k-1}$, that is, $\mathbf{a}_1 \cdot \mathbf{a}_k = t_{*,n}, \ldots, \mathbf{a}_{k-1} \cdot \mathbf{a}_k = t_{*,n}$. Since the first $k-1$ components of $\mathbf{a}_k$ and $\mathbf{a}_{k+1}$ are the same, it follows that $\mathbf{a}_i \cdot \mathbf{a}_{k+1} = t_{*,n}$, $1 \le i \le k-1$. Since $\mathbf{a}_{k+1}$ satisfies $\mathbf{a}_k \cdot \mathbf{a}_{k+1} = t_{*,n}$, it follows that $\mathbf{a}_i \cdot \mathbf{a}_{k+1} = t_{*,n}$, $1 \le i \le k$, that is, $\mathbf{a}_{k+1} \in H_1 \cap \cdots \cap H_k$.

It is easy to see that $H_1 \cap \cdots \cap H_d$ contains n^d distinct points, which are given by

$$\mathbf{x} \cdot \mathbf{a}_1 = t_{i_1,n}, \quad \ldots, \quad \mathbf{x} \cdot \mathbf{a}_d = t_{i_d,n}, \quad 1 \le i_1, \ldots, i_d \le n. \tag{2.7}$$

Hence , we have verified the conditions of the method of reproducing kernel. We summarize the result in the following theorem.

Theorem 2.1. *Let $\mathbf{a}_1, \ldots, \mathbf{a}_d$ be defined as above and let H_k be the hypersurface defined by $\tilde{K}_n(W_\mu; \mathbf{x}, \mathbf{a}_k) = 0$. Then the modified method of the reproducing kernel yields a cubature formula of degree $2n+1$, based on $\mathbf{a}_1, \ldots, \mathbf{a}_d$ and n^d distinct points determined by (2.7).*

The formula is necessarily of the form (1.5), and we have by (2.6) that

$$\lambda_i = 1/\tilde{K}_n(W_\mu; \mathbf{a}_i, \mathbf{a}_i) = 1/C_n^{(\mu+(d+1)/2)}(1) = 1/\binom{n + 2\mu + d}{n}.$$

For fixed n and d, the other weights μ_j in (1.5) can be determined by solving a linear system of equations.

The above construction shows that the method of the reproducing kernel works for all n in every dimension. Moreover, the construction shows that we can get many different formulae from this method. For example, in defining $\mathbf{a}_k$ by $\mathbf{a}_{k-1} \cdot \mathbf{a}_k = t_{*,n}$, we can choose $t_{*,n}$ to be any zero of $C_n^{(\mu+(d+1)/2)}(t)$.

One particular interesting choice appears when n is an odd integer. Since $C_n^{(\mu+(d+1)/2)}$ is an odd polynomial for odd n, it follows that $t = 0$ is an zero of this polynomial. If we take $t_{*,n} = 0$ in the definition of $\mathbf{a}_k$ in the above construction, then we end up with $\mathbf{a}_1 = \mathbf{e}_1, \ldots, \mathbf{a}_d = \mathbf{e}_d$, where $\mathbf{e}_i$ is the standard basis of $\mathbb{R}^d$, that is,

$$\mathbf{e}_1 = (1, 0, \ldots, 0), \ \mathbf{e}_2 = (0, 1, 0, \ldots, 0), \ \ldots, \ \mathbf{e}_d = (0, \ldots, 0, 1).$$

Since it follows from (2.6) that

$$\tilde{K}_n(W_\mu; \mathbf{x}, \mathbf{e}_k) = C_n^{(\mu+(d+1)/2)}(x_k), \qquad k = 1, \ldots, d,$$

we conclude that the n^d intersection points of $H_1 \cap \ldots \cap H_d$ are just tenser product of the zeros in (2.6). Hence we end up with the following formula.

Let n be an odd integer. Let $t_{1,n}, \ldots, t_{n,n}$ be the zeros of $C_n^{(\mu+(d+1)/2)}(t)$. Then there is a cubature formulae of degree $2n+1$ in the form of

$$\begin{aligned}(2.8) \qquad \int_{B^d} f(\mathbf{x}) W_\mu(\mathbf{x}) d\mathbf{x} = & \frac{1}{\binom{n+2\mu+d}{n}} \sum_{k=1}^{n} [f(\mathbf{e}_k) + f(-\mathbf{e}_k)]/2 \\ & + \sum_{k_1=1}^{n} \cdots \sum_{k_d=1}^{n} \mu_{k_1,\ldots,k_d} f(t_{k_1,n}, \ldots, t_{k_d,n}), \qquad f \in \Pi_{2n+1}^d.\end{aligned}$$

The weights μ_i in the formula (2.8) can be computed by solving a linear system equations for a given n and d. For $d = 2$, however, they can be given by a compact formula as follows. Let $\ell_{k,n}$ be defined by

$$\ell_{k,n}(t) = \prod_{i=1, i\neq k}^{n} \frac{x - t_{i,n}}{t_{k,n} - t_{i,n}} = \frac{C_n^{(\mu+(d+1)/2)}(t)}{(2\mu + d + 1) C_{n-1}^{(\mu+(d+3)/2)}(t_{k,n})(x - t_{k,n})},$$

the fundamental interpolation polynomials based on the zeros of $C_n^{(\mu+(d+1)/2)}(t)$; it satisfies $\ell_{k,n}(t_{j,n}) = \delta_{j,k}$. Since the polynomial

$$\ell_{k_1,n}(x_1)\ell_{k_2,n}(x_2)(1 - x_1^2 - x_2^2)$$

is of degree $2(n-1)+2 = 2n$, it will be integrated exactly by the cubature formula (2.8). Therefore, it follows from the interpolation property of $\ell_{k,n}$ that

$$\mu_{k_1,k_2} = \int_{B^2} \ell_{k_1,n}(x_1)\ell_{k_2,n}(x_2)(1 - x_1^2 - x_2^2) W_\mu(x_1, x_2) dx_2 dx_2.$$

In the first sight, the formula (2.8) is a bit peculiar, since it uses tensor product of one variable nodes on a rotation invariant region. Let us examine the related product formula. The points $\{t_{1,n},\ldots,t_{n,n}\}$ are nodes of a Gaussian quadrature formula of degree $2n-1$ on $[-1,1]$ for the measure $(1-x^2)^{\mu+d/2}dx$ on $[-1,1]$, while $\{-1,t_{1,n},\ldots,t_{n,n},1\}$ form the nodes of a Gaussian-Radau quadrature formula of degree $2n+1$,

$$\int_{-1}^{1} f(x)(1-x^2)^{\mu+d/2}dx = Af(-1) + \sum_{k=1}^{n} \lambda_k f(t_{i,n}) + Af(1), \qquad f \in \Pi^1_{2n+1}.$$

As a consequence, the tenser product of $\{-1,t_{1,n},\ldots,t_{n,n},1\}$ serves as the nodes of a product formula of degree $2n+1$ for the product weight function $\prod_{k=1}^{d}(1-x_k)^{\mu+d/2}$ on $[-1,1]^d$; that is,

$$\int_{[-1,1]^d} f(\mathbf{x}) \prod_{k=1}^{d}(1-x_k)^{\mu+d/2}d\mathbf{x} = \sum_{k_1=0}^{n+1}\cdots\sum_{k_d=0}^{n+1} \lambda_{k_1}\cdots\lambda_{k_d} f(t_{k_1,n},\ldots,t_{k_d,n})$$

for $f \in \Pi^d_{2n+1}$, where $t_{0,n}=-1$, $t_{n+1,n}=1$ and $\lambda_0=\lambda_{n+1}=A$. Comparing with (2.8), we see that the nodes of (2.8) is a subset of the nodes of the above formula.

We should point out that some nodes of the cubature formulae constructed above lie outside of the unit ball B^d. By choosing different $\mathbf{a}_k$, one may be able to construct formulae with all nodes inside, but it is not clear how to select $\mathbf{a}_k$ in general.

2.3 Cubature formulae of lower degree with nodes inside B^d. We use the modified method of the reproducing kernel to construct cubature formula of lower degree with nodes inside B^d.

Formulae of degree 5: First we consider formulae of degree 5 and we choose $\mathbf{a}_1$ to be the origin, the resulted formula has one node less than the formula constructed in the previous subsection.

Let $\mathbf{a}_1=0$ and define $\mathbf{a}_{k+1}$, $1\le k\le d-1$, by

$$\mathbf{a}_{k+1} = \Big(\sqrt{\frac{1}{2\mu+d+3}},\ldots,\sqrt{\frac{1}{2\mu+d+3}},\sqrt{\frac{d+3-k}{2\mu+d+3}},0,\ldots,0\Big),$$

which has $d-k$ zero components. Then there exist weights μ_ε such that the following cubature formula is of degree 5 for W_μ on B^d:

(2.9)

$$\int_{B^d} f(\mathbf{x})W_\alpha(\mathbf{x})d\mathbf{x} = \frac{2(2\mu+1)}{(2\mu+d+1)(d+2)}f(0) + \frac{2\mu+d+3}{2\mu+d+1}\sum_{k=1}^{d-1}\frac{f(\mathbf{a}_{k+1})+f(-\mathbf{a}_{k+1})}{(d+2-k)(d+3-k)}$$
$$+ \sum_{\varepsilon\in\{-1,1\}^d} \mu_\varepsilon f\Big(\varepsilon_1\sqrt{\frac{1}{2\mu+d+3}},\ldots,\varepsilon_{d-1}\sqrt{\frac{1}{2\mu+d+3}},\varepsilon_d\sqrt{\frac{3}{2\mu+d+3}}\Big).$$

The weights μ_ε can be determined by the fact that the formula is exact for polynomials of degree 5. We give the explicit formula for the case of $d=2$ and $d=3$ as follows:

$$\int_{B^2} f(\mathbf{x})W_\alpha(\mathbf{x})d\mathbf{x} = \frac{2(2\mu+1)}{4(2\mu+3)}f(0) + \frac{2\mu+5}{12(2\mu+3)}\Big[f(2/\sqrt{2\mu+5},0)+f(-2/\sqrt{2\mu+5},0)\Big] + \frac{2\mu+5}{12(2\mu+3)}\sum f(\pm 1/\sqrt{2\mu+5}, \pm\sqrt{3}/\sqrt{2\mu+5}).$$

$$\int_{B^3} f(\mathbf{x})W_\alpha(\mathbf{x})d\mathbf{x} = \frac{2(2\mu+1)}{5(2\mu+4)}f(0) + \frac{2\mu+6}{2\mu+3}\sum_{k=1}^{2}\frac{f(\mathbf{a}_{k+1})+f(-\mathbf{a}_{k+1})}{(d+2-k)(d+3-k)}$$
$$+ \frac{(\mu+3)(4\mu+9)}{144(\mu+2)}\sum\Big[f(\sqrt{1/(2\mu+6)}, \sqrt{1/(2\mu+6)}, \pm\sqrt{3/(2\mu+6)}) + f(-\sqrt{1/(2\mu+6)}, -\sqrt{1/(2\mu+6)}, \pm\sqrt{3/(2\mu+6)})\Big]$$
$$+ \frac{(\mu+3)(4\mu+15)}{144(\mu+2)}\sum\Big[f(-\sqrt{1/(2\mu+6)}, \sqrt{1/(2\mu+6)}, \pm\sqrt{3/(2\mu+6)}) + f(\sqrt{1/(2\mu+6)}, -\sqrt{1/(2\mu+6)}, \pm\sqrt{3/(2\mu+6)})\Big].$$

The formula on B^d uses 2^d+2d-1 nodes. It is known that the cubature formula of degree 5 must have at least $N \ge d(d+1)+1$ nodes (Möller's lower bound [9]). It follows that for $d=2$ and 3, the above formula for W_μ is minimal. For $d=4$, the formula uses 23 points, just 2 more than the lower bound of 21. For $d=5$, the formula uses 41 points, 10 more than the lower bound of 31. In the case of $\mu=0$, that is, the unit weight function on B^d, several families of cubature formulae are given in the [15]; one family uses $2d^2+1$ nodes, which is less than our formula when $d>5$, and another family uses d^2+d+2 nodes but holds only for $d=4,5,6,7$, which uses less nodes than our formulae for these dimensions. Our formula uses less nodes than the rest of the families given in [15].

Let us show how to construct the above formula by the method of reproducing kernel. From [16, p. 80] we have

$$C_2^{(\lambda)}(t) = \lambda[2(\lambda+1)x^2-1].$$

Therefore, with $\lambda = \mu+(d+1)/2$, it follows from the formula (2.4) that

$$\tilde K_2(W_\mu;\mathbf{x},\mathbf{y}) = \lambda[2\mu+d+3)(\mathbf{x}\cdot\mathbf{y})^2 + 2(\mu+d+3)(1-|\mathbf{x}|^2)(1-|\mathbf{y}|^2)/(2\mu+1)-1].$$

Since we choose $\mathbf{a}_1=0$, the origin, it follows from the formula of $\tilde K_2(W_\mu;\mathbf{x},\mathbf{y})$ that

$$H_1 = \{\mathbf{x} : |\mathbf{x}|^2 = (d+2)/(2\mu+d+3)\}.$$

All $\mathbf{a}_k$ in the method of reproducing kernel need to belong to H_1. Using our choices of $\mathbf{a}_{k+1}$ and the formula of $\tilde K_2(W_\mu)$ it is easy to verify that

$$\tilde K_2(W_\mu;\mathbf{x},\mathbf{a}_{k+1}) = \Big(\mu+\frac{d+1}{2}\Big)[x_1^2+\ldots+x_{k-1}^2+(d+3-k)x_k^2-|\mathbf{x}|^2],$$

from which we can conclude that $\mathbf{a}_{k+1} \in H_1 \cap \cdots \cap H_k$ and we end up with

$$H_1 \cap \cdots \cap H_d = \Big\{ \Big(\pm \sqrt{\frac{1}{2\mu+d+3}}, \ldots, \pm\sqrt{\frac{1}{2\mu+d+3}}, \pm\sqrt{\frac{3}{2\mu+d+3}} \Big) \Big\}.$$

That is, $H_1 \cap \cdots \cap H_d$ has 2^d intersection points. By the modified method of reproducing kernel, we have that these points, together with $\mathbf{a}_1, \ldots, \mathbf{a}_d$, generate a cubature formula of degree 5 on B^d in the form of (2.9). Using the formula of $\tilde{K}_2(W_0; \cdot, \cdot)$, we can verify that the weights of the formulae associated with $\mathbf{a}_{k+1}$ are given by

$$\lambda_1 = 1/\tilde{K}_2(W_\mu; 0, 0) = \frac{2(2\mu+1)}{(2\mu+d+1)(d+2)},$$

$$\lambda_{k+1} = 1/\tilde{K}_2(W_\mu; \mathbf{a}_{k+1}, \mathbf{a}_{k+1}) = \frac{2(2\mu+d+3)}{(2\mu+d+1)(d+2-\mu)(d+3-\mu)}, \quad 2 \le k \le d.$$

To determine the other weights, we use the fact that the formula is exact for polynomials of degree 5. For example, to determine the weights for $d = 3$, we first recognize that some of the weights are equal and then determine the weights by solving the linear system equations that are obtained from setting $f(\mathbf{x})$ to be x_3, x_1x_2, x_3^2, and $x_1x_2x_3$. The general case of $d > 3$ can be done similarly but looks to be rather complicated. Let us record the following formula for the nonzero moments of the weight function,

$$\int_{B^d} x_1^{2k_1} \cdots x_d^{2k_d} W_\mu(\mathbf{x}) d\mathbf{x} = \frac{\Gamma(\mu + (d+1)/2)\Gamma(k_1+1/2)\cdots\Gamma(k_d+1/2)}{\pi^{d/2}\Gamma(\mu+(d+1)/2+k_1+\ldots k_d)},$$

which is useful for setting up the equations that determine the weights.

Formulae of degree 9: We can choose $\mathbf{a}_1 = 0$ in the modified method of the reproducing kernel only when n is even, since the origin is a common zeros of orthogonal polynomials of degree n when n is odd. In the case of $n = 4$, we derive from the compact formula (2.4) that

$$\begin{aligned}\tilde{K}_4(W_\mu; \mathbf{x}, \mathbf{y}) = {} & \frac{1}{2} - \frac{2\mu+7}{2\mu+1}\Big[(2\mu+1)(\mathbf{x}\cdot\mathbf{y})^2 + (1-|\mathbf{x}|^2)(1-|\mathbf{y}|^2)\Big] \\ & + \frac{(2\mu+7)(2\mu+9)}{6(2\mu+1)(2\mu+3)}[(2\mu+1)(2\mu+3)(\mathbf{x}\cdot\mathbf{y})^4 + (2\mu+3)(\mathbf{x}\cdot\mathbf{y})^2(1-|\mathbf{x}|^2)(1-|\mathbf{y}|^2) \\ & + 3(1-|\mathbf{x}|^2)^2(1-|\mathbf{y}|^2)^2].\end{aligned}$$

In particular, it follows that

$$\tilde{K}_4(W_\mu; \mathbf{x}, 0) = \frac{1}{2} - \frac{2\mu+7}{2\mu+1}(1-|\mathbf{x}|^2) + \frac{(2\mu+7)(2\mu+9)}{2(2\mu+1)(2\mu+3)}(1-|\mathbf{x}|^2)^2.$$

Hence, the hypersurface H_1, determined by $\tilde{K}_4(W_\mu; \mathbf{x}, 0) = 0$, consists of two spheres, $H_1 = \{\mathbf{x} : |\mathbf{x}| = z_1 \text{ or } |\mathbf{x}| = z_2\}$, where

$$z_1 = \frac{2}{9+2\mu}\Big(3 + \sqrt{3}\sqrt{\frac{3+2\mu}{7+2\mu}}\Big), \quad z_2 = \frac{2}{9+2\mu}\Big(3 - \sqrt{3}\sqrt{\frac{3+2\mu}{7+2\mu}}\Big).$$

For $d=2$, we can take $\mathbf{a}_2 = (0, \sqrt{z_1})$ so that $\mathbf{a}_2 \in H_1$. Then

$$\tilde{K}_4(W_\mu; \mathbf{x}, \mathbf{a}_2) = \frac{1}{2} - \frac{(7+2\mu)(1-z_1)(1-|\mathbf{x}|^2)}{1+2\mu} + \frac{(7+2\mu)(9+2\mu)(1-z_1)^2(1-|\mathbf{x}|^2)^2}{2(1+2\mu)(3+2\mu)}$$
$$+\Big(-z_1(7+2\mu) + \frac{(7+2\mu)(9+2\mu)z_1(1-z_1)(1-|\mathbf{x}|^2)}{1+2\mu}\Big)x_1^2 + \frac{(7+2\mu)(9+2\mu)}{6}z_1^2x_1^4.$$

It follows that $\tilde{K}_4(W_\mu; \mathbf{x}, \mathbf{a}_2)$ becomes a polynomial of degree 2 in x_1^2 upon substituting either $|\mathbf{x}| = z_1$ or $|\mathbf{x}| = z_2$, which leads to, in either case, 8 distinct nodes for the formula (4 distinct x_1 implies 8 distinct points by $|\mathbf{x}| = z_i$). Hence, all together, $H_1 \cap H_2$ contains 16 distinct points in the form of $(\pm a_i, \pm b_i)$, $i = 1, 2, 3, 4$, which, together with $(0,0)$, $\mathbf{a}_2$ and $-\mathbf{a}_2$, generate a cubature formula of degree 9 with 19 nodes. The nodes of the formula for W_μ can be determined by solving the equation $\tilde{K}_4(W_\mu; \mathbf{x}, \mathbf{a}_2) = 0$ under the condition $|\mathbf{x}| = z_1$ or $|\mathbf{x}| = z_2$. Once the nodes are determined, we can find the weights of the formula by solving a linear system of equations. The task can be easily carried out by using one of the computer algebra system; however, the explicit formulae of the nodes and the weights are rather complicated, we choose not to give them here. For the unit weight function, such a formula has been given by Möller [8].

Formulae of higher degree: Using the compact formula of $\tilde{K}_n(W_\mu; \mathbf{x}, \mathbf{y})$, we have also tried the method of reproducing kernel for $n = 5, 6, 7, 8$, which leads to, if works, cubature formulae of degree $11, 13, 15, 17$. For $n = 6, 8$, we choose $\mathbf{a}_1 = (0,0)$ and $\mathbf{a}_2 = (z, u)$, where z is a zero of $\tilde{K}_n(W_\mu; \mathbf{x}, \mathbf{a}_1)$. For $n = 5, 7$, we can no longer choose $\mathbf{a}_1 = (0,0)$. Instead we choose $\mathbf{a}_1 = (1, 0)$, which implies that H_1 is determined by $x_1 = t_{k,n}$, where $t_{k,n}$ is one of the zeros of $C_n^{(\mu+3/2)}$. Since we need to choose $\mathbf{a}_2 \in H_1$, this implies that we take $\mathbf{a}_2 = (z, u)$ where $z = t_{k,n}$ for a fixed k.

In each case, we tried $\mu = 0$ and $\mu = 1/2$, and tested various values of u. In the case of $n = 5$ and $W_{1/2}(\mathbf{x}) = 1/\pi$, we found a cubature formula of degree 11 with all nodes inside B^2. Since the formula uses 29 nodes, more than several known formulae, and has two negative weights, we do not give it here explicitly. For $n = 5$ and W_0, however, it turns out that whenever a solution is found for $u < 1$, some of the intersection points of $H_1 \cap H_2$ lie outside of the unit disk. On the other hand, if u is large, say, $u > 1.5$, then we always found a solution with all intersection points of $H_1 \cap H_2$ lie inside the disk, which leads to a cubature formula for W_0 with two nodes $\pm\mathbf{a}_2$ outside the disk. For $n = 6, 7, 8$, we did not find any formula with all nodes inside the unit disk for either $W_{1/2}$ or W_0. What is of interest is the fact that in all cases there is a formula for u large ($u > 2$, say), and all nodes in $H_1 \cap H_2$ lie inside the disk. In fact, it can be proved that a cubature formula exists when u is sufficiently large, and the formula has all nodes inside B^2 except $\mathbf{a}_2 = (z, u)$ and $-\mathbf{a}_2$.

3. Cubature formulae on the triangle

We consider cubature formulae on the triangle using the compact formula for the reproducing kernel found in [19]. The compact formula holds for a family of weight functions and it holds on a d-dimensional simplex. For $\mathbf{x} \in \mathbb{R}^d$, we use the notation $|\mathbf{x}|_1 = |x_1| + \ldots + |x_d|$, the ℓ^1 norm of $\mathbf{x}$. We consider the standard simplex

$$\Sigma^d = \{\mathbf{x} \in \mathbb{R}^d : x_1 \geq 0, \ldots, x_d \geq 0, 1 - |\mathbf{x}|_1 \geq 0\}.$$

For $d = 2$, it is the triangle with vertices at $(0,0)$, $(1,0)$ and $(0,1)$.

3.1 Reproducing kernel. In [19], we found the compact formula for the reproducing kernel with respect to the weight function

$$W_\alpha(\mathbf{x}) = w_\alpha x_1^{\alpha_1 - 1/2} \cdots x_d^{\alpha_d - 1/2} (1 - |\mathbf{x}|_1)^{\alpha_{d+1} - 1/2}, \qquad \alpha_i \geq 0, \tag{3.1}$$

where w_α is the normalization constant such that $\int_{\Sigma^d} W_\alpha d\mathbf{x} = 1$,

$$w_\alpha = \frac{\Gamma(|\alpha|_1 + (d+1)/2)}{\Gamma(\alpha_1 + 1/2) \ldots \Gamma(\alpha_{d+1} + 1/2)}.$$

The reproducing kernel $K_n(W_\alpha)$ is again given in terms of Gegenbauer polynomials,

$$\begin{aligned} K_n(W_\alpha; \mathbf{x}, \mathbf{y}) = \int_{[-1,1]^{d+1}} & C_{2n}^{(|\alpha|_1 + (d+1)/2)}(\sqrt{x_1 y_1}\, t_1 + \ldots + \sqrt{x_{d+1} y_{d+1}}\, t_{d+1}) \\ & \times \prod_{i=1}^{d+1} c_{\alpha_i} (1 - t_i^2)^{\alpha_i - 1} d\mathbf{t}, \end{aligned} \tag{3.2}$$

where $\mathbf{x}, \mathbf{y} \in \Sigma^d$ and $x_{d+1} = 1 - |\mathbf{x}|_1$ and $y_{d+1} = 1 - |\mathbf{y}|_1$. where we use limit (2.5) whenever one $\alpha_i = 0$. Since the Gegenbauer polynomial involved is of degree $2n$, it will still take some work to get $K_n(W_\alpha)$ explicitly. If, however, $\mathbf{y} = \mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, the ith element of the standard basis of $\mathbb{R}^d$, $1 \leq i \leq d$, then we have the following explicit formula

$$K_n(W_\alpha; \mathbf{x}, \mathbf{e}_i) = A_{\alpha,i} P_n^{(|\alpha|_1 + d/2 - \alpha_i, \alpha_i - 1/2)}(2x_i - 1) \tag{3.3}$$

where $A_{\alpha,i} = C_{2n}^{(|\alpha|_1 + (d+1)/2)}(0) / P_n^{(|\alpha|_1 + d/2 - \alpha_i, \alpha_i - 1/2)}(-1)$. This formula is derived from (3.2) using a product formula for Jacobi polynomials proved in [4]. We remark that for $\mathbf{e}_i$ is not a common zero of $\mathbb{P}_n$. This follows from a formula for $\mathbb{P}_n^T(\mathbf{x})\mathbb{P}_n(\mathbf{y}) = \sum_k P_k^n(\mathbf{x}) P_k^n(\mathbf{y})$ which takes the similar form as (2.4) ([19]).

We will use the method of reproducing kernel only for $d = 2$. When $\alpha_1 = \alpha_2 = \alpha_3 = 1/2$, the weight function W_α becomes a multiple of unit weight function. We denote this weight function by $W_{1/2}$, that is, $W_{1/2}(\mathbf{x}) = 2$. In this case, the reproducing kernel takes the form

$$K_n(W_{1/2}; \mathbf{x}, \mathbf{y}) = \frac{1}{\pi^3} \int_{[-1,1]^3} C_{2n}^{(3)}(\sqrt{x_1 y_1}\, t_1 + \sqrt{x_2 y_2}\, t_2 + \sqrt{x_3 y_3}\, t_3) \prod_{i=1}^{3} (1 - t_i^2)^{-1/2} d\mathbf{t}.$$

Another interesting case is when $\alpha = 0$, that is, $W_0(\mathbf{x}) = (x_1x_2x_3)^{-1/2}/2\pi$. In [22] it is shown that any cubature formula for W_0 with all nodes inside Σ^2 corresponds to a cubature formula on the sphere S^2. In this case, the reproducing kernel takes a simple form

$$K_n(W_0;\mathbf{x},\mathbf{y}) = \frac{1}{4}\sum C_{2n}^{(3/2)}(\sqrt{x_1y_1} \pm \sqrt{x_2y_2} \pm \sqrt{x_3y_3}),$$

where the sum is over all possible sign changes; the formula follows from (3.2) by taking limits (2.5) repeatly.

3.2 Lower degree formula on triangle. We consider lower degree formulae for the unit weight function $W_{1/2}$.

Formulae of degree 4**:** For $n = 2$, we get the explicit formula for $K_n(W_{1/2};\mathbf{x},\mathbf{y})$ as

$$\begin{aligned} K_2(W_{1/2};\mathbf{x},\mathbf{y}) = 6(1 &- 10(x_1y_1 + x_2y_2 + x_3y_3) \\ &+60(x_1x_2y_1y_2 + x_1x_3y_1y_3 + x_2x_3y_2y_3) + 15(x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2)). \end{aligned}$$

In this case, $K_2(W_{1/2},\mathbf{x},(1,0))$ has two zeros,

$$z_1 = (5-\sqrt{10})/15, \qquad z_2 = (5+\sqrt{10})/15.$$

It follows easily that $K(W_{1/2};\mathbf{x},(1,0))$ and $K(W_{1/2};\mathbf{x},z_1,0)$ have 4 distinct common zeros,

$$\Big((5-\sqrt{10})/15, (70 - 7\sqrt{10} \pm \sqrt{10(233-62\sqrt{10})}/90\Big),$$
$$\Big((5+\sqrt{10})/15, (30 - 3\sqrt{10} \pm \sqrt{3(110-20\sqrt{10})}/90\Big).$$

Together with the nodes $(1,0)$ and $(z_1,0)$, these points serve as nodes of a cubature formula of degree 4. The nodes and the weights of this formula is given in the table below.

Table 3.1. Formula of degree 4

x_i	y_i	λ_i
0	1	0.027777777777778
0	0.1225148226554413	
0.122514822655441	0.318273232972619	0.235005848023034
0.122514822655441	0.745372464334522	0.150642079844394
0.544151844011225	0.096332050209531	0.296338375148580
0.544151844011225	0.359516105779244	0.229407791094891

We do not claim that this formula is new. In fact, there are many formulae of degree 4 with 6 nodes that have been found, see the Table 3.5 in [3] and the references therein.

Formulae of degree 6: For $n = 3$, we get the explicit formula for $K_n(W_{1/2}; \mathbf{x}, \mathbf{y})$ as

$$\begin{aligned} K_3(W_{1/2};\mathbf{x},\mathbf{y}) &= 10(-1 + 2016x_1x_2x_3y_1y_2y_3 + 18(x_1y_1 + x_2y_2 + x_3y_3) \\ &-252(x_1x_2y_1y_2 + x_1x_3y_1y_3 + x_2x_3y_2y_3) - 63(x_1^2y_1^2 + x_2^2y_2^2 + x_3^2y_3^2) \\ &+504(x_1^2x_2y_1^2y_2 + x_1x_2^2y_1y_2^2 + x_1^2x_3y_1^2y_3 + x_2^2x_3y_2^2y_3 + x_1x_3^2y_1y_3^2 + x_2x_3^2y_2y_3^2) \\ &+56(x_1^3y_1^3 + x_2^3y_2^3 + x_3^3y_3^3)). \end{aligned}$$

In this case, $K_3(W_{1/2}, \mathbf{x}, (1, 0))$ has three zeros,

$$z_1 = 0.0729940240731497; \quad z_2 = 0.347003766038351; \quad z3 = 0.705002209888495.$$

It turns out that $K(W_{1/2}; \mathbf{x}, (1, 0))$ and $K(W_{1/2}; \mathbf{x}, z_1, z_3)$ have 9 distinct common zeros. Together with $(1, 0)$ and (z_1, z_3), these points serve as nodes of a cubature formula of degree 6 with 11 nodes, which is just one more than the lower bound of 10. In [13], a formula with 10 nodes is found, some of its nodes lie outside the triangle. As far as we know, this formula is the first formula with 11 nodes, and all its nodes are inside the triangle and its weights are positive. All formulae list in the Table 3.5 in [3] use at least 12 nodes. In the table below, we give the numerical values of the nodes and the weights of this formula.

Table 3.2. Formula of degree 6

x_i	y_i	λ_i
1	0	0.01
0.0729940240731497	0.7050022098884985	0.114526050150397
0.7050022098884985	0.0427807725086187	0.071499927386723
0.3470037660383518	0.0730201995672642	0.123578656202553
0.0729940240731497	0.0742941361357864	0.068263844196723
0.7050022098884985	0.1781695498960980	0.097790048412146
0.7050022098884985	0.2804106790535491	0.033767158949566
0.3470037660383518	0.3250739144186064	0.199084809386271
0.0729940240731497	0.3546443426156327	0.127669126117898
0.3470037660383517	0.5788366073031974	0.125260309936212
0.0729940240731497	0.9119902946404894	0.028560069261510

In the case of $W_0(\mathbf{x}) = w_0(x_1x_2x_3)^{-1/2}$, we also tried to find a cubature formula of degree 6 by following the same approach. However, all formulae we found have at least one node outside the triangle. It is worth to note that the common zeros of $K(W_0; \mathbf{x}, (1, 0))$ and $K(W_0; \mathbf{x}, z_1, u)$, where z_1 is one of the zeros of $K(W_0; \mathbf{x}, (1, 0))$, all lie inside the triangle provided u is relative large, say $u > 1$.

3.3 A limit case. Let $\mathbf{a}_1 = (1, 0)$. By (3.2) $K_n(W_\alpha; \mathbf{x}, \mathbf{a}_1) = 0$ if $x_1 = (1 + t_{i,n})/2$, where $t_{1,n}, \ldots, t_{n,n}$ are zeros of the Jacobi polynomial $P_n^{(\alpha_2+\alpha_3+1,\alpha_1-1/2)}(t)$. Next we take $\mathbf{a}_2 = (t_{1,n}, u)$ so that $\mathbf{a}_2 \in H_1$ as required by the method of reproducing kernel.

Let $\{P_k^n\}$ be the orthonormal polynomials of degree n with respect to W_α in [6],

$$P_k^n(x_1,x_2) = h_k^n P_{n-k}^{(2k+\alpha_2+\alpha_3,\alpha-1/2)}(2x_1-1)(1-x_1)^k P_k^{(\alpha_3-1/2,\alpha_2-1/2)}(2\frac{x_2}{1-x_1}-1),$$

where h_k^n is a normalization constant so that the integral of $(P_k^n)^2$ with respect to $W_\alpha dxdy$ on Σ^2 is 1. It follows from the definition of $K_n(W_\alpha;\mathbf{x},\mathbf{a}_2)$ in (1.1) that

$$K_n(W_\alpha;\mathbf{x},\mathbf{a}_2) = h_n^n \gamma_n^{(\alpha_3-1/2,\alpha_2-1/2)} 2^n\, u^n\, P_n^n(x_1,x_2) + \mathcal{O}(u^{n-1}). \tag{3.4}$$

where we use the usual notation $\gamma_n^{(\lambda,\mu)}$ for the leading coefficient of $P_n^{(\lambda,\mu)}$. From this formula and the method of reproducing kernel, we can conclude that there is a cubature formula of degree $2n$ for u sufficiently large. In particular, if we take $u\to\infty$, we end up with the following formula,

$$\begin{aligned}\int_{\Sigma^2} f(\mathbf{x})W_\alpha(\mathbf{x})d\mathbf{x} &= \lambda_1 f(1,0) + \sum_{i=1}^n\sum_{j=1}^n \mu_{i,j} f(\mathbf{x}_{i,j}) \\ &\quad + (h_n^n)^{-1}[\gamma_n^{(\alpha_3-1/2,\alpha_2-1/2)}]^{-2} 2^{-2n} \lim_{u\to\infty} f((1+t_{1,n})/2,u)/u^{2n},\end{aligned} \tag{3.5}$$

where $\mathbf{x}_{i,j}$ are the common zeros of $\mathbf{K}_n(W_\alpha;\mathbf{x},(1,0))$ and the function that is the limit of $\mathbf{K}_n(W_\alpha;\mathbf{x},\mathbf{a}_2)/u^n$ as $u\to\infty$. The latter function is equal to a constant multiple of $P_n^n(\mathbf{x})$ by (3.4). Since $P_n^n((1+t_{i,n})/2,x_2)=0$ implies that $x_2=(1-t_{i,n})(1+s_{j,n})/4$, where $s_{1,n},\ldots,s_{n,n}$ are zeros of the Jacobi polynomial $P_n^{(\alpha_3-1/2,\alpha_2-1/2)}(t)$. Hence, the nodes $\mathbf{x}_{i,j}$ are of the form $((1+t_{i,n})/2,(1-t_{i,n})(1+s_{j,n})/4)$.

Since the term involving limit in (3.5) will be zero whenever $f\in\Pi_{2n}^2$ does not contain the term x_2^{2n}, we end up with the following formula

$$\int_{\Sigma^2} f(\mathbf{x})W_\alpha(\mathbf{x})d\mathbf{x} = \lambda_1 f(1,0) + \sum_{i=1}^n\sum_{j=1}^n \mu_{i,j} f((1+t_{i,n})/2,(1-t_{i,n})(1+s_{j,n})/4), \tag{3.6}$$

which is exact for $f\in\Pi_{2n}^d\setminus\{x_2^{2n}\}$, the subspace of polynomials of degree $2n$ that does not contain the term x_2^{2n}. It turns out, however, that this formula is a product type formula for Σ^2. In fact, if we write

$$\begin{aligned}\int_{\Sigma^2} f(x_1,x_2)W_\alpha(x_1,x_2)dx_1dx_2 &= \int_0^1\int_0^{1-x_1} f(x_1,x_2)W_\alpha(x_1,x_2)dx_1dx_2 \\ &= w_\alpha\int_0^1\int_0^1 f(x_1,(1-x_1)y)x_1^{\alpha_1-1/2}(1-x_1)^{\alpha_2+\alpha_3}y^{\alpha_2-1/2}(1-y)^{\alpha_3-1/2}dx_1dy,\end{aligned}$$

and use Gaussian quadrature formula for $t^{\alpha_2-1/2}(1-t)^{\alpha_3-1/2}$ on $[0,1]$ and Gauss-Radau type formula of degree $2n$ for $t^{\alpha_1-1/2}(1-t)^{\alpha_2+\alpha_3}$ on $[0,1]$, then we obtain a product formula which can be seen to be the formula (3.6).

In general, the method of reproducing kernel yields formulae of degree $2n$ that are not product type. We have just showed, however, that in a limit setting the method yields a cubature formula that agrees with a special product type formula, and the formula is of degree $2n-1$.

Acknowledgement. The author thanks a referee for his careful review and for his valuable comments.

References

1. Kh. A. Abbas, *Application of the reproducing kernel method to the construction of cubature formulas for a cube and a simplex*, Vestnik S.-Peterburg. Univ. Mat. Mekh. Astronom. (1992), 3–7, 117 (Russian); English transl. in, Vestnik St. Petersburg Univ. Math. **25** (1992), 1–4.
2. Kh. A. Abbas and I. P. Mysovskikh, *The method of reproducing kernels for the construction of cubature formulas*, Vestnik S.-Peterburg. Univ. Mat. Mekh. Astronom. (1991), 3–11, 91 (Russian); English transl. in, Vestnik St. Petersburg Univ. Math. **24** (1991), 1–8.
3. R. Cools and P. Rabinowitz, *Monomial cubature rules since "Stroud": a compilation*, J. Comp. Appl. Math. **48** (1992), 309-326.
4. A. Dijksma and T. Koornwinder, *Spherical harmonics and the product of two Jacobi polynomials*, Indag. Math. **33** (1971), 191-196.
5. H. Engels, *Numerical quadrature and cubature*, Academic Press, New York, 1980.
6. T. Koornwinder, *Two-variable analogues of the classical orthogonal polynomials*, Theory and Application of Special Functions (R. A. Askey, ed.), Academic Press, New York, 1975, pp. 435–495.
7. J.N. Lyness and R. Cools, *A survey of numerical cubature over triangles*, Mathematics of Computation 1943–1993: a half-century of computational mathematics (Vancouver, BC, 1993), Proc. Sympos. Appl. Math., 48, Amer. Math. Soc., Providence, RI, 1994, pp. 127–150.
8. H. M. Möller, *Polynomideale und Kubaturformeln*, Thesis, Univ. Dortmund, 1973.
9. H. M. Möller, *Kubaturformeln mit minimaler Knotenzahl*, Numer. Math. **35** (1976), 185–200.
10. I. P. Mysovskikh, *The approximation of multiple integrals by using interpolatory cubature formulae*, Quantitative Approximation (R. A. DeVore and K. Scherer, ed.), Academic Press, New York, 1980.
11. I. P. Mysovskikh, *Interpolatory cubature formulas*, "Nauka", Moscow, 1981. (Russian)
12. I. P. Mysovskikh, *A representation of the reproducing kernel of a sphere*, Comp. Math. Math. Phys. **36** (1996), 303-308.
13. G. G. Rasputin, *On the construction of cubature formulas containing preassigned knots*, Metody Vychisl. **13** (1983), 122-128. (Russian)
14. G. G. Rasputin, *The construction of cubature formulas containing previously specified nodes*, Soviet Math. **. 30** (1986), 58-67.
15. A. Stroud, *Approximate calculation of multiple integrals*, Prentice Hall, Englewood Cliffs, NJ, 1971.
16. Szegő, H., *Orthogonal polynomials*, 4th ed., Amer. Math. Soc. Colloq. Publ. vol.23, Providence, RI, 1975.
17. Y. Xu, *On orthogonal polynomials in several variables*, Special functions, q-series, and related topics, Fields Institute Communications, vol. 14, 1997, pp. 247 - 270.
18. Y. Xu, *Summability of Fourier orthogonal series for Jacobi weight on a ball in* $\mathbb{R}^d$, Trans. Amer. Math. Soc. (to appear).
19. Y. Xu, *Summability of Fourier orthogonal series for Jacobi weights on the simplex in* $\mathbb{R}^d$, Proc. Amer. Math. Soc. **126** (1998), 3027-3036.
20. Y. Xu, *Asymptotics for orthogonal polynomials and Christoffel functions on a ball*, Methods Appl. Anal. **3** (1996), 257-272.
21. Y. Xu, *Orthogonal polynomials and cubature formulae on spheres and on balls*, SIAM J. Math. Anal. **29** (1998), 779-793.
22. Y. Xu, *Orthogonal polynomials and cubature formulae on spheres and on simplices*, Methods Appl. Analysis **5** (1998), 169-184.

Yuan Xu, Department of Mathematics, University of Oregon, Eugene, Oregon 97403-1222.

E-mail address: `yuan@math.uoregon.edu`