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Orthogonal Polynomials and Cubature Formulae on Balls, Simplices and Spheres

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Abstract

We report on recent development on orthogonal polynomials and cubature formulae on the unit ball B^d, the standard simplex T^d and the unit sphere S^d. The main result shows that the orthogonal structures and cubature formulae on these three regions are closely related. This provides a way to study the structure of orthogonal polynomials; for example, it allows us to use the theory of h-harmonics to study orthogonal polynomials on B^d and on T^d. It also provides a way to construct new cubature formulae on these regions.

1 Introduction

The structure of orthogonal polynomials in several variables is significantly more complicated than that of orthogonal polynomials in one variable. Among the reasons for the complication, some are generic; for example, there are many distinct orders among polynomials in several variables, and consequently, the orthogonal bases are not unique. Others depend on the specific problems under consideration, for example, on weight functions and regions that define the orthogonality. The regions that have attracted most attention are regular ones, such as cubes, balls, simplices, and the surface of spheres. In the literature, orthogonal polynomials on these regions are mostly studied separately.

In the first part of the paper we report some recent results that reveal a close relation between orthogonal polynomials on the unit ball B^d, the standard simplex T^d, and the surface of the unit sphere S^d of the Euclidean space. The main results state, roughly speaking, that a basis of orthogonal polynomials on the simplex T^d is equivalent, under a simple transformation, to a basis

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of orthogonal polynomials on B^d that are invariant under sign changes; and a basis on B^d is equivalent to a basis of orthogonal polynomials on S^d that are even in one of their variables. The forerunner of the results is a relation between the spherical harmonics on S^d and a family of orthogonal polynomials on B^d that was observed and used in the work of Hermite, Didon, Appel and de Fériet, and Koschmieder (see, for example, [1] and [17, Chapt. 12, Vol. II]). The relation between spherical harmonics and a family of orthogonal polynomials on a triangle was used in [10], and also in [7] in an application related to the method of finite elements. The general results are proved in [52,53] for large classes of weight functions that subject only to mild assumptions. As an important application, this allows us to use Dunkl's theory of h-harmonics associated to the reflection groups ([11–15]) to derive compact formulae for the reproducing kernel and to study summability of several families of orthogonal polynomials on B^d and on T^d.

The study of the structures of polynomials on the sphere, the ball and the simplex leads us to a close relation between cubature formulae on these regions, which is discussed in the second part of the paper. The main results state that, roughly speaking, cubature formulae on T^d are equivalent to cubature formulae that are invariant under sign changes on B^d, and formulae on B^d are equivalent to formulae that are invariant under the sign change of one fixed variable on S^d. In the literature, cubature formulae on different regions are mostly studied separately; all three regions have attracted their share of attentions over decades of investigation (see, for example, [16,38,43] and the references therein). The fact that these relations are revealed only recently ([52,53]) seems rather remarkable. This allows us to construct formulae on one region and uses the relations to obtain formulae for the other two regions. In this way, a number of new formulae on these regions have been derived ([20–22]). Most notably a family of cubature formulae for the surface measure on S^d that exists for all d and all degrees (see Section 5). Moreover, these relations may shed lights on an outstanding conjecture of Lebedev about cubature formulae on S^2, which we will discuss in some detail in the paper.

The paper is organized as follows. The next section is devoted to preliminaries. Section 3 deals with relations between orthogonal polynomials on the three regions. Applications on reproducing kernel and orthogonal series are discussed in Section 4. The connections between cubature formulae on the three regions are addressed in Section 5. Lebedev's conjecture is discussed in Section 6. Several open problems are discussed in Section 4 and in Section 6.

2 Preliminary

Basic notation. For $\mathbf{x} \in \mathbb{R}^d$ we denote by $|\mathbf{x}| = \sqrt{x_1^2 + \ldots + x_d^2}$ the usual

Euclidean norm and by $|\mathbf{x}|_1 = x_1 + \ldots + x_d$ the ℓ^1 norm. Let $\mathbb{N}_0$ be the set of nonnegative integers. For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$, we write $|\alpha|_1 = \alpha_1 + \cdots + \alpha_d$, consistent with the notation $|\mathbf{x}|_1$. Throughout the paper we denote by B^d the unit ball of $\mathbb{R}^d$ and S^d the unit sphere on $\mathbb{R}^{d+1}$; that is,

$$B^d = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| \le 1\} \qquad \text{and} \qquad S^d = \{\mathbf{y} \in \mathbb{R}^{d+1} : |\mathbf{y}| = 1\}.$$

We also denote by T^d the standard simplex in $\mathbb{R}^d$; that is,

$$T^d = \{\mathbf{x} \in \mathbb{R}^d : x_1 \ge 0, \ldots, x_d \ge 0, 1 - |\mathbf{x}|_1 \ge 0\}.$$

For $d = 2$, the ball B^2 is the unit disc and the simplex T^2 is the triangle with vertices at $(0,0)$, $(1,0)$ and $(0,1)$.

Polynomial spaces. For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ and $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we write $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. The number $|\alpha|_1$ is called the total degree of $\mathbf{x}^\alpha$. We denote by Π^d the set of polynomials in d variables on $\mathbb{R}^d$ and by Π_n^d the subset of polynomials of total degree at most n. We also denote by $\mathcal{P}_n^d$ the space of homogeneous polynomials of degree n on $\mathbb{R}^d$ and we let $r_n^d = \dim \mathcal{P}_n^d$. It is well-known that

$$\dim \Pi_n^d = \binom{n+d}{n} \quad \text{and} \quad r_n^d = \dim \mathcal{P}_n^d = \binom{n+d-1}{d}.$$

Invariance under a finite group. The region B^d and S^d are evidently invariant under the rotation group. The simplex T^d is invariant under the symmetric group of its vertices. If G is a subgroup of the rotation group of $\mathbb{R}^d$, we define the group action on a function f on $\mathbb{R}^d$ by $R(a)f(\mathbf{x}) = f(\mathbf{x}a)$, $\mathbf{x} \in \mathbb{R}^d$, $a \in G$. If $R(a)f = f$ for all $a \in G$, we say that f is invariant under G. For example, for the simple abelian group $\mathbb{Z}_2^d$ consisting of elements $a = (\varepsilon_1, \ldots, \varepsilon_m)$, where $\varepsilon_i = \pm 1$, $R(a)f(\mathbf{x}) = f(\varepsilon_1 x_1, \ldots, \varepsilon_d x_d)$. For a function f defined on T^d, we say that f is invariant on T^d if it is invariant under the symmetric group of T^d; that is, if it is invariant under permutations among the variables $\{x_1, \ldots, x_d, 1 - |\mathbf{x}|_1\}$. We will deal with polynomials invariant under G. We denote by $\Pi_n^d(G)$ the space of polynomials in Π_n^d that are invariant under G, and by $\mathcal{P}_n^d(G)$ the space of homogeneous polynomials in $\mathcal{P}_n^d$ invariant under G.

Orthogonal polynomials on B^d and T^d. Let Ω denote either B^d or T^d. Let W be a weight function on Ω, which is assumed to be nonnegative and have finite moments. We often normalize W so that it has unit integral over Ω. Given an order among the monomials $\{\mathbf{x}^\alpha\}$, we can use the Gram-Schmidt process to generate a sequence of orthogonal polynomials with respect to the inner product of $L^2(W d\mathbf{x})$. It is known that for each $n \in \mathbb{N}_0$ the set of polynomials of degree n that are orthogonal to all polynomials of lower degree forms a vector space, denoted by $\mathcal{V}_n^d(W)$, whose dimension is r_n^d. We denote by $\{P_\alpha^n\}$, $|\alpha|_1 = n$ and $n \in \mathbb{N}_0$, one family of orthonormal polynomials with respect to

W on Ω that forms a basis of $\mathcal{V}_n^d(W)$, where the superscript n means that $P_\alpha^n \in \Pi_n^d$. The orthonormality means that

$$\int_\Omega P_\alpha^n(\mathbf{x}) P_\beta^m(\mathbf{x}) W(\mathbf{x}) d\mathbf{x} = \delta_{\alpha,\beta}\delta_{m,n}.$$

We note that there are many bases of $\mathcal{V}_n^d(W)$; if Q is an invertible matrix of size r_n^d, then the components of $Q\mathbb{P}_n$ form another basis of $\mathcal{V}_n^d(W)$, where $\mathbb{P}_n$ denotes the vector $(P_\alpha^n)_{|\alpha|_1=n}$, which is orthonormal if Q is an orthogonal matrix. For results on the general structure of orthogonal polynomials in several variables, we refer to the survey [49] and the references there.

Classical Orthogonal polynomials on T^d and B^d. On T^d they are orthogonal with respect to the weight functions

$$W_{\kappa,\mu}^T(\mathbf{x}) = w_{\kappa,\mu}^T x_1^{\kappa_1 - 1/2} \dots x_d^{\kappa_d - 1/2}(1 - |\mathbf{x}|_1)^{\mu - 1/2}, \quad \mathbf{x} \in T^d, \tag{2.1}$$

where $\kappa_i > -1/2$, $\mu > -1/2$ and the normalized constant $w_{\kappa,\mu}^T$ is defined by

$$w_{\kappa,\mu}^T = \frac{\Gamma(|\kappa|_1 + \mu + (d+1)/2)}{\Gamma(\kappa_1 + 1/2)\cdots\Gamma(\kappa_d + 1/2)\Gamma(\mu + 1/2)}.$$

Related to $W_{\kappa,\mu}^T$ are weight functions $W_{\kappa,\mu}^B$ on B^d, defined by

$$W_{\kappa,\mu}^B(\mathbf{x}) = w_{\kappa,\mu}^B |x_1|^{2\kappa_1} \dots |x_d|^{2\kappa_d}(1 - |\mathbf{x}|_2)^{\mu - 1/2}, \quad \mathbf{x} \in B^d, \tag{2.2}$$

where $\kappa_i > -1/2$, $\mu > -1/2$ and $w_{\kappa,\mu}^B = w_{\kappa,\mu}^T$. The classical orthogonal polynomials on B^d are orthogonal with respect to $W_\mu^B := W_{0,\mu}^B$; that is, the case $\kappa = 0$. For $\mu = 1/2$, $W_{1/2}^B = 1/\operatorname{vol} B^d$ is the normalized Lebesgue measure.

We refer to [1] and [17, Chapt. 12] for an account of earlier results on the classical orthogonal polynomials; they are characterized as eigenfunctions of certain second order partial differential equation (see also [26]). Bases of the classical orthogonal polynomials can be constructed explicitly in terms of Jacobi polynomials, and we will see that they can be derived from orthogonal polynomials on the sphere S^d.

Ordinary spherical harmonics. The most important orthogonal polynomials on the sphere are the spherical harmonics, which are orthogonal with respect to the surface measure $d\omega$ on S^d. The harmonic polynomials on $\mathbb{R}^{d+1}$ are homogeneous polynomials satisfying the Laplace equation $\Delta P = 0$, where $\Delta = \partial_1^2 + \cdots + \partial_{d+1}^2$ on $\mathbb{R}^{d+1}$ and ∂_i is the partial derivative with respect to the i-th coordinate. The spherical harmonics are the restriction of harmonic polynomials on S^d. We refer to [36,42,45] for accounts of the theory of spherical harmonics.

h-harmonics associated to reflection groups. The theory of h-harmonics is established recently by Dunkl (see [11–15]). For a nonzero vector $\mathbf{v} \in \mathbb{R}^{d+1}$ define the reflection $\sigma_{\mathbf{v}}$ by $\mathbf{x}\sigma_{\mathbf{v}} := \mathbf{x} - 2\langle \mathbf{x}, \mathbf{v}\rangle \mathbf{v}/|\mathbf{v}|^2$, $\mathbf{x} \in \mathbb{R}^{d+1}$, where $\langle \mathbf{x}, \mathbf{y}\rangle$ denote the usual Euclidean inner product. Let G be a reflection group on $\mathbb{R}^{d+1}$ with positive roots $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$. Assume that $|\mathbf{v}_i| = |\mathbf{v}_j|$ if $\sigma_{\mathbf{v}_i}$ is conjugate to $\sigma_{\mathbf{v}_j}$. The h-harmonics are homogeneous orthogonal polynomials on S^d with respect to $h_\kappa^2 d\omega$, where the weight function h_κ is defined by

$$h_\kappa(\mathbf{x}) := \prod_{i=1}^{m} |\langle \mathbf{x}, \mathbf{v}_i\rangle|^{\kappa_i}, \qquad \kappa_i \geq 0, \tag{2.3}$$

with $\kappa_i = \kappa_j$ whenever σ_{vb_i} is conjugate to $\sigma_{\mathbf{v}_j}$ in the reflection group G generated by the reflections $\{\sigma_{\mathbf{v}_i} : 1 \leq i \leq m\}$. The function h_κ is a positively homogeneous G-invariant function. The key ingredient of the theory is a family of first-order differential-difference operators, $\mathcal{D}_i$ (Dunkl's operators), which generates a commutative algebra ([13]), defined by

$$\mathcal{D}_i f(\mathbf{x}) := \partial_i f(\mathbf{x}) + \sum_{j=1}^{m} \kappa_j \frac{f(\mathbf{x}) - f(\mathbf{x}\sigma_j)}{\langle \mathbf{x}, \mathbf{v}_j\rangle} \langle \mathbf{v}_j, \mathbf{e}_i\rangle, \qquad 1 \leq i \leq d.$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_d$ are the standard unit vectors of $\mathbb{R}^d$. The h-Laplacian is defined by $\Delta_h = \mathcal{D}_1^2 + \ldots + \mathcal{D}_{d+1}^2$, which plays the role of Laplacian in the theory of the ordinary harmonics. In particular, the h-harmonics are the homogeneous polynomials satisfying the equation $\Delta_h P = 0$. The h-spherical harmonics are the restriction of h-harmonics on the sphere. The structure of the space of h-harmonics, such as dimensionality and decomposition, is parallel to that of ordinary harmonics. In particular, there is an intertwining operator V between the algebra of differential operators and the commuting algebra of Dunkl's operators, which helps us to transform certain properties of ordinary harmonics to the h-harmonics. The operator V is the unique linear operator defined by

$$V\mathcal{P}_n \subset \mathcal{P}_n, \qquad V1 = 1, \qquad \mathcal{D}_i V = V\partial_i, \qquad 1 \leq i \leq d.$$

It is also proved in [39] that V is a positive operator. The closed form of V is known, however, only in the case of abelian group $\mathbb{Z}_2^d$ ([14,50]) and the symmetric group S_3 ([15]). For further properties and results of h-harmonics, we refer to [11–15] and the references there.

Examples of reflection invariant weight functions. The group $\mathbb{Z}_2^{d+1}$ is one of the simplest reflection groups. The weight function invariant under $\mathbb{Z}_2^{d+1}$ is

$$h_{\kappa,\mu}(\mathbf{x}) = a_{\kappa,\mu}|x_1|^{\kappa_1} \cdots |x_d|^{\kappa_d}|x_{d+1}|^{\mu}, \qquad \mathbf{x} \in \mathbb{R}^{d+1}, \tag{2.4}$$

where $a_{\kappa,\mu}^2 = 2w_{\kappa,\mu}^T$. We use μ for the power of the last component of $\mathbf{x}$ to emphasis the connection between $h_{\kappa,\mu}$ and $W_{\kappa,\mu}^B$ and $W_{\kappa,\mu}^T$ that will be-come clear in the next section. For $\kappa = 0$ and $\mu = 0$, we have that $a_0^2 = 2\Gamma((d+1)/2)/\pi^{(d+1)/2} = 1/\omega_d$, where ω_d denote the surface area of S^d. Another interesting case is the the hyper-octahedral group G generated by the reflections in $x_i = 0$, $1 \le i \le d+1$ and $x_i \pm x_j = 0$, $1 \le i, j \le d+1$; it is the Weyl group of type B_{d+1}. There are two conjugacy classes of reflections, hence two parameters for h_κ. We have

$$h_\kappa(\mathbf{x}) = \prod_{i=1}^{d} |\mathbf{x}|^{\kappa_1} \prod_{i<j} |x_i^2 - x_j^2|^{\kappa_0}. \tag{2.5}$$

The integral, hence the normalization constant, of h_κ can be computed by use of Selberg's integral. In fact, the integral of h_κ in (2.3) for every reflection group has been computed in the work of Askey, Heckman, McDonald, Opdam and several others. See the references in [14].

3 Relation between orthogonal polynomials on the three regions

Throughout this section we fix the following notation: For $\mathbf{y} \in \mathbb{R}^{d+1}$, we write

$$\mathbf{y} = (y_1, \ldots, y_d, y_{d+1}) = (\mathbf{y}', y_{d+1}) = r(\mathbf{x}, x_{d+1}), \tag{3.1}$$

where $\mathbf{y} \in S^d$, $r = |\mathbf{y}| = \sqrt{y_1^2 + \cdots + y_{d+1}^2}$ and $\mathbf{x} = (x_1, \ldots, x_d) \in B^d$. We call a weight function H, defined on $\mathbb{R}^{d+1}$, S-symmetric if it is symmetric with respect to y_{d+1} and centrally symmetric with respect to the variables $\mathbf{y}' = (y_1, \ldots, y_d)$; *i.e.*,

$$H(\mathbf{y}', y_{d+1}) = H(\mathbf{y}', -y_{d+1}) \quad \text{and} \quad H(\mathbf{y}', y_{d+1}) = H(-\mathbf{y}', y_{d+1}),$$

and we assume that H is not a zero function when restricted to S^d. For example, the weight functions of the form $H(\mathbf{y}) = W(y_1^2, \ldots, y_d^2)$ are S-symmetric, which include $H = h_\kappa^2$ for both h_κ in (2.4) and in (2.5).

In [52] we proved that there are orthonormal bases of homogeneous polynomials with respect to the inner product of $L^2(Hd\omega, S^d)$ for S-symmetric H. Let us denote by $\mathcal{H}_n^{d+1}(H)$ the space of homogeneous orthogonal polynomials of degree n. When $H(\mathbf{y}) = 1$, we write $\mathcal{H}_n^{d+1}$, which is the space of ordinary harmonics of degree n. For $H = h_\kappa^2$ with h_κ as in (2.3), $\mathcal{H}_n^{d+1}(h_\kappa^2)$ is the space of h-harmonics. It is shown in [52] that

$$\dim \mathcal{H}_n^{d+1}(H) = \binom{n+d}{d} - \binom{n+d-2}{d} = \dim \mathcal{P}_n^{d+1} - \dim \mathcal{P}_{n-2}^{d+1}, \tag{3.2}$$

same as the ordinary harmonics, and there is a unique decomposition of $\mathcal{P}_n^{d+1}$,

$$\mathcal{P}_n^{d+1} = \bigoplus_{k=0}^{[n/2]} |\mathbf{y}|^{2k} \mathcal{H}_{n-2k}^{d+1}(H). \tag{3.3}$$

These results are proved using the relation between $\mathcal{H}_n^{d+1}(H)$ and orthogonal polynomials on B^d, which we describe below.

3.1 Orthogonal polynomials on balls and on spheres

In association to a weight function H defined on $\mathbb{R}^{d+1}$, we define a weight function W_H^B on B^d by

$$W_H^B(\mathbf{x}) = H(\mathbf{x}, \sqrt{1-|\mathbf{x}|^2}), \qquad \mathbf{x} \in B^d. \tag{3.4}$$

If H is S-symmetric, then the assumption that H is centrally symmetric with respect to the first d variables implies that W_H^B is centrally symmetric on B^d. Recall the notation $\mathcal{V}_n^d(W)$ for the space of orthonormal polynomials of degree n. We denote by $\{P_\alpha^n\}$ and $\{Q_\alpha^n\}$ systems of orthonormal polynomials that form a basis for $\mathcal{V}_n^d(W_1^B)$ and $\mathcal{V}_n^d(W_2^B)$ with respect to the weight functions

$$W_1^B(\mathbf{x}) = 2W_H^B(\mathbf{x})/\sqrt{1-|\mathbf{x}|^2} \qquad \text{and} \qquad W_2^B(\mathbf{x}) = 2W_H^B(\mathbf{x})\sqrt{1-|\mathbf{x}|^2},$$

respectively. Keeping in mind the notation (3.1) we define

$$Y_{\alpha,n}^{(1)}(\mathbf{y}) = r^n P_\alpha^n(\mathbf{x}) \qquad \text{and} \qquad Y_{\beta,n}^{(2)}(\mathbf{y}) = r^n x_{d+1} Q_\beta^{n-1}(\mathbf{x}), \tag{3.5}$$

where $|\alpha|_1 = n$, $|\beta|_1 = n-1$ and we define $Y_{\beta,0}^{(2)}(\mathbf{y}) = 0$. It is proved in [52] that, as functions of $\mathbf{y}$, $Y_{\alpha,n}^{(1)}$ and $Y_{\beta,n}^{(2)}$ are in fact elements of $\mathcal{H}_n^{d+1}(H)$. The proof of (3.2) and (3.3) follows from this fact. On the other hand, if H is S-symmetric, then $\mathcal{H}_n^{d+1}(H)$ must have a basis that consists of homogeneous polynomials that are either even in y_{d+1} or odd in y_{d+1}. Using $x_{d+1}^2 = 1-|\mathbf{x}|^2$, we can write those that are even in y_{d+1} as $r^n P_\alpha^n(\mathbf{x})$ and those that are odd as $r^n x_{d+1} Q_\beta^{n-1}(\mathbf{x})$, where P_α^n and Q_β^{n-1} are polynomials in $\mathbf{x}$ of degree n and $n-1$, respectively. Then the polynomials P_α^n and Q_β^n are orthogonal polynomials with respect to W_1^B and W_2^B, respectively. We summarize the result as,

Theorem 3.1 *Let H be a S-symmetric weight function defined on $\mathbb{R}^{d+1}$ and let W_1^B and W_2^B be defined as above. Then the relation (3.5) defines a one-to-one correspondence between an orthonormal basis of $\mathcal{H}_n^{d+1}(H)$ and an orthonormal basis of $\mathcal{V}_n^d(W_1^B) \oplus x_{d+1}\mathcal{V}_{n-1}^d(W_2^B)$.*

In particular, if $H(\mathbf{y}) = 1$, then the theorem states that the ordinary spherical harmonics correspond to orthogonal polynomials with respect to the weight functions $1/\sqrt{1-|\mathbf{x}|^2}$ and $\sqrt{1-|\mathbf{x}|^2}$, respectively. In the case of $d = 1$, we have $\mathcal{H}_n^2 = \mathrm{span}\{r^n \cos n\theta, r^n \sin n\theta\}$, using the polar coordinates $y_1 = r\cos\theta$ and $y_2 = r\sin\theta$. The correspondence in the theorem is then the well-known fact that $T_n(x) = \cos n\theta$ and $U_n(x) = \sin n\theta / \sin\theta$, where $x = \cos\theta$, are orthogonal with respect to $1/\sqrt{1-x^2}$ and $\sqrt{1-x^2}$ on $[-1, 1]$, respectively.

Under the correspondence, the orthogonal polynomials for the weight function $W_{\kappa,\mu}^B$ in (2.2) are related to the h-harmonics associated to $h_{\kappa,\mu}$ in (2.4); in particular, the classical orthogonal polynomials on B^d are related to h-harmonics associated to $h_\mu(\mathbf{y}) = |y_{d+1}|^\mu$. Compact formulae of an orthonormal basis of these polynomials on B^d can be obtained accordingly from formulae in [50]. Moreover, the second order partial differential equation satisfied by the classical orthogonal polynomials can be derived from the h-Laplacian by a simple change of variables ([59]).

3.2 Orthogonal polynomials on balls and on simplices

Let $W^B(\mathbf{x}) := W(x_1^2, \ldots, x_d^2)$ be a weight function defined on B^d. Associated to W^B we define a weight function W^T on the simplex T^d by

$$W^T(\mathbf{u}) = W(u_1, \ldots, u_d)/\sqrt{u_1 \cdots u_d}, \quad \mathbf{u} \in T^d, \tag{3.6}$$

and we normalize the weight function W so that W^B has unit integral on B^d. It follows from a simple change of variables that W^T has unit integral on T^d.

A polynomial P is invariant under the group $\mathbb{Z}_2^d$, if P is even in each of its variables; such a polynomial must be of even degree. We denote by $\mathcal{V}_{2n}^d(W^B, \mathbb{Z}_2^d)$ the space of polynomials in $\mathcal{V}_{2n}^d(W^B)$ that are invariant under the group $\mathbb{Z}_2^d$. That is, $\mathcal{V}_{2n}^d(W^B, \mathbb{Z}_2^d)$ contains orthogonal polynomials of degree n that are even in their variables.

Let P_α^{2n} be a polynomial in $\mathcal{V}_{2n}^d(W^B, \mathbb{Z}_2^d)$. Since it is even in each of its variables, we can write it in the form of

$$P_\alpha^{2n}(\mathbf{x}) = R_\alpha^n(x_1^2, \ldots, x_d^2), \qquad |\alpha|_1 = n, \tag{3.7}$$

where R_α^n is a polynomial of degree n. It turns out that R_α^n is a polynomial in $\mathcal{V}_n^d(W^T)$. In fact, the relation defines a one-to-one correspondence.

Theorem 3.2 *Let W^B and W^T be weight functions defined as above. Then the relation (3.7) defines a one-to-one correspondence between an orthonormal basis of $\mathcal{V}_{2n}^d(W^B, \mathbb{Z}_2^d)$ and an orthonormal basis of $\mathcal{V}_n^d(W^T)$.*

Under the correspondence, the classical orthogonal polynomials on T^d associated to the weight function $W^T_{\kappa,\mu}$ in (2.1) correspond to orthogonal polynomials with respect to $W^B_{\kappa,\mu}$ in (2.2); compact formulae of an orthonormal basis can be obtained from those in [50]. We note that the unit Lebesgue measure (unit weight function) on T^d corresponds to $|x_1 \dots x_d|$ on B^d and the Lebesgue measure on B^d corresponds to $1/\sqrt{x_1 \dots x_d}$ on T^d, because the Jacobian of the map $(x_1, \dots, x_d) \mapsto (x_1^2, \dots, x_d^2)$ is $|x_1 \dots x_d|$. From the results in the previous subsection, polynomials in $\mathcal{V}^d_{2n}(W^B_{\kappa,\mu})$ satisfy a differential-difference equation that follows from the h-Laplacian and a change of variables. When we restrict to the elements of $\mathcal{V}^d_{2n}(W^B_{\kappa,\mu}, \mathbb{Z}_2^d)$, the difference part in the equation disappears due to the invariance under $\mathbb{Z}_2^d$ and we end up with a second order partial differential equation. Upon changing variables as in the correspondence (3.7), we then recover the second order partial differential equation satisfied by the classical orthogonal polynomials on T^d ([59]).

3.3 Orthogonal polynomials on spheres and on simplices

Put the results in the Section 3.1 and 3.2 together, we also have a relation between orthogonal polynomials on S^d and those on T^d. The relation can be derived from the previous two subsections, we formulate them below for better reference.

On S^d we need to restrict to the weight function $H(\mathbf{y}) = W(y_1^2, \dots, y_{d+1}^2)$, which is evidently S-symmetric. We denote by $\mathcal{H}^{d+1}_{2n}(H, \mathbb{Z}_2^{d+1})$ the space of orthogonal polynomials in $\mathcal{H}^{d+1}_{2n}(H)$ that are invariant under the group $\mathbb{Z}_2^{d+1}$. Associated to H we define a weight function on T^d by

$$W^T_H(\mathbf{x}) = 2W(x_1, \dots, x_d, 1 - |\mathbf{x}|_1)/\sqrt{x_1 \dots x_d(1 - |\mathbf{x}|_1)}, \qquad \mathbf{x} \in T^d.$$

The constant 2 is there so that if H has unit integral on S^d then W^T_H has unit integral on T^d. Let $\{S^{2n}_\alpha\}$ denote a basis for $\mathcal{H}^{d+1}_{2n}(H, \mathbb{Z}_2^{d+1})$. Since S^{2n}_α is homogeneous and even in each of its variables, we can use the notation (3.1) and the fact that $x_{d+1}^2 = 1 - |\mathbf{x}|^2$ to write S^{2n}_α as

$$S^{2n}_\alpha(\mathbf{y}) = r^{2n} R^n_\alpha(x_1^2, \dots, x_d^2), \tag{3.8}$$

where R^n_α is a polynomial of degree n. On the other hand, given polynomials R^n_α defined on T^d, we can use (3.8) to define homogeneous polynomials on S^d. This relation connects the orthogonal polynomials on S^d and those on T^d.

Theorem 3.3 *Let H and W^T_H be weight functions defined as above. Then the relation (3.8) defines a one-to-one correspondence between an orthonormal basis of $\mathcal{H}^{d+1}_{2n}(H, \mathbb{Z}_2^{d+1})$ and an orthonormal basis of $\mathcal{V}^d_n(W^T_H)$.*

As a consequence of this correspondence, we see that there is a unique decomposition of $\mathcal{P}_n^{d+1}(\mathbb{Z}_2^{d+1})$ in terms of $\mathcal{H}_n^d(H, \mathbb{Z}_2^{d+1})$,

$$\mathcal{P}_{2n}^{d+1}(\mathbb{Z}_2^{d+1}) = \bigoplus_{k=0}^{n} |\mathbf{y}|^{2k} \mathcal{H}_{2n-2k}^d(H, \mathbb{Z}_2^{d+1}).$$

Under the correspondence, the classical orthogonal polynomials on T^d associated to $W_{\kappa,\mu}^T$ in (2.1) correspond to h-spherical harmonics associated to h_κ in (2.4) with $\kappa_{d+1} = \mu$. In particular, the orthogonal polynomials with respect to the weight functions $1/\sqrt{x_1 \cdots x_d(1 - |\mathbf{x}|_1)}$ are related to the ordinary spherical harmonics, and those with respect to the unit weight function on T^d correspond to h-harmonics for $|x_1 \cdots x_{d+1}|$ on S^d.

4 Reproducing Kernel and Fourier Orthogonal Expansion

The relations stated in the previous section allow us to derive properties for orthogonal polynomials on one region from those on the other two regions. In this section, we examine the case of h-harmonics on S^d and their counterpart on B^d and on T^d. As we shall see, this approach will reveal several hidden symmetric properties of orthogonal polynomials on B^d and on T^d by relating them to the rich structure of h-harmonics. Some of the properties are new even for classical orthogonal polynomials.

We start with definition on Fourier orthogonal expansion. Let $\{S_{\alpha,n}^h\}$ denote an orthonormal basis of h-harmonics. The reproducing kernel of $\mathcal{H}_n^{d+1}(h_\kappa^2)$ is defined by the formula

$$P_n(h_\kappa^2; \mathbf{x}, \mathbf{y}) = \sum_\alpha S_{\alpha,n}^h(\mathbf{x}) S_{\alpha,n}^h(\mathbf{y}),$$

where the summation is over all h-harmonics of degree n. For $f \in L^2(h_\kappa^2, S^d)$, we consider the Fourier expansion of f in terms of the orthonormal basis $S_{\alpha,n}^h$. The partial sum of such an expansion with respect to $\mathcal{H}_n^{d+1}(h_\kappa^2)$ is given by

$$P_n(f, h_\kappa^2; \mathbf{x}) = \int_{S^d} f(\mathbf{y}) P_n(h_\kappa^2; \mathbf{x}, \mathbf{y}) h_\kappa^2(\mathbf{y}) d\omega(\mathbf{y}) := (f * P_n(h_\kappa^2))(\mathbf{x}). \quad (4.1)$$

It is not hard to see that the reproducing kernel is independent of the choice of the particular bases. In fact, for h-harmonics, the kernel enjoys a compact formula in terms of the intertwining operator [51],

$$P_n(h_\kappa^2; \mathbf{x}, \mathbf{y}) = \frac{n + |\kappa|_1 + (d-1)/2}{|\kappa|_1 + (d-1)/2} V[C_n^{(|\kappa|_1 + (d-1)/2)}(\langle \mathbf{x}, \cdot \rangle)](\mathbf{y}), \quad (4.2)$$

where $\mathbf{x}, \mathbf{y} \in S^d$ and $C_n^{(\lambda)}$ is the Gegenbauer polynomial of degree n. Here and in the following the reader may want to keep in mind that if $h_\kappa(\mathbf{x}) = 1$, then the h-harmonics are just the classical spherical harmonics and $V = id$ is the identity operator. In that case, (4.2) is just the compact formula for the ordinary zonal polynomials (cf. [36, p. 19] or [42, p. 149]). In the case of $\mathbb{Z}_2^{d+1}$ and the weight function (2.4), the closed form of the intertwining operator V is given by ([14,50])

$$Vf(\mathbf{x}) = \int\limits_{[-1,1]^{d+1}} f(t_1 x_1, \ldots, t_{d+1} x_{d+1}) \prod_{i=1}^{d+1} c_{\kappa_i}(1 + t_i)(1 - t_i^2)^{\kappa_i - 1} d\mathbf{t}, \quad (4.3)$$

where $c_\lambda = 1/\int_{-1}^{1}(1 - t^2)^{\lambda-1}dt$ for $\lambda > 0$, and we have taken $\mu = k_{d+1}$. Moreover, if some $\kappa_i = 0$, then the above formula holds under the limit relation

$$\lim_{\lambda \to 0} c_\lambda \int\limits_{-1}^{1} f(t)(1 - t^2)^{\lambda-1}dt = [f(1) + f(-1)]/2.$$

In this case, we can write down the reproducing kernel $P_n(h_\kappa^2; \mathbf{x}, \mathbf{y})$ explicitly.

Although a closed form of the intertwining operator is not known in general, its average over the sphere can be computed as shown in [51].

Theorem 4.1 *Let h_κ be defined as in (2.3) associated to a reflection group. Let V be the intertwining operator. Then*

$$\int\limits_{S^d} Vf(\mathbf{x}) h_\kappa^2(\mathbf{x}) d\omega = A_\kappa \int\limits_{B^{d+1}} f(\mathbf{x})(1 - |\mathbf{x}|^2)^{|\kappa|_1 - 1} d\mathbf{x} \quad (4.4)$$

for $f \in \Pi^d$, where A_κ is a constant that can be determined by setting $f(\mathbf{x}) = 1$.

The equation (4.4) is not trivial even in the case of $\mathbb{Z}_2^{d+1}$; the reader may try to verify it with V given by (4.3). This result allows us to prove a general convergence theorem for the Fourier orthogonal expansion in h-harmonics. For $\delta > 0$, the Cesàro (C, δ) means, s_n^δ, of a sequence $\{s_n\}$ are defined by

$$s_n^\delta = \frac{1}{\binom{n+\delta}{n}} \sum_{k=0}^{n} \binom{n-k+\delta-1}{n-k} s_k = \frac{1}{\binom{n+\delta}{n}} \sum_{k=0}^{n} \binom{n-k+\delta}{n-k} c_k,$$

where the second equality holds if s_n is the n-th partial sum of the series $\sum_{k=0}^{\infty} c_k$. We say that $\{s_n\}$ is Cesàro (C, δ) summable to s if s_n^δ converges to s as $n \to \infty$. Let $P_n^\delta(h_\kappa^2, f)$ denote the Cesàro (C, δ) means of the Fourier series in h-harmonics. By (4.1), we can write $P_n^\delta(f, h_\kappa^2) = f * P_n^\delta(h_\kappa^2)$, where $P_n^\delta(h_\kappa^2)$ denote the (C, δ) means of $P_n(h_\kappa^2)$, which can be written as $V[k_n^\delta(\langle \mathbf{x}, \cdot \rangle)](\mathbf{y})$ by (4.2), where k_n^δ is the (C,δ) means of the reproducing kernel for the Gegenbauer series of order $|\kappa|_1+(d-1)/2$. Since V is a positive operator, we have $|Vf(\mathbf{x})| \le V(|f|)(x)$. Hence, if we apply Theorem 4.1, then we conclude that

$$\int_{S^d} |P_n^\delta(h_\kappa^2;\mathbf{x},\mathbf{y})| h_\kappa^2(\mathbf{y}) d\omega \le A_\kappa \int_{B^{d+1}} |k_n^\delta(\langle \mathbf{x},\mathbf{y}\rangle)|(1-|\mathbf{y}|^2)^{|\kappa|_1+(d-2)/2} d\mathbf{y}.$$

A standard change of variables shows that the the last integral can be reduced to the integral over $[-1,1]$. As a consequence, the (C,δ) summability of the h-harmonics follows from that of Gegenbauer series. We have ([51])

Theorem 4.2 *Let h_κ be defined as in (2.3). Let $f \in L^p(h_\kappa^2, S^d)$. Then the expansion of f as the Fourier series with respect to h_κ^2 is (C,δ) summable in $L^p(h_\kappa^2, S^d)$, $1 \le p \le \infty$, provided $\delta > |\kappa|_1 + (d-1)/2$.*

Together with the relation between orthogonal polynomials on S^d, B^d and T^d, the above results on h-harmonics allow us to derive results for the reproducing kernel and for the summability of orthogonal expansion on B^d and on T^d. Instead of stating the results for the most general weight functions on these regions, we shall restrict us to the weight functions $W_{\kappa,\mu}^T$ in (2.1) and $W_{\kappa,\mu}^B$ in (2.2), both are related to $h_{\kappa,\mu}$ in (2.3), for which all formulae can be written down in closed form. The restricted cases include those of the classical orthogonal polynomials. First we define the reproducing kernel and the Fourier orthogonal series. Let $\{P_\alpha^n\}$ denote a sequence of orthonormal polynomials with respect to a weight function W defined on Ω, where Ω is either B^d or T^d. The reproduction kernel of $\mathcal{V}_n^d(W)$, denoted by $P_n(W;\cdot,\cdot)$, is defined by

$$P_n(W;\mathbf{x},\mathbf{y}) = \sum_{|\alpha|_1=n} P_\alpha^n(\mathbf{x}) P_\alpha^n(\mathbf{y}).$$

This kernel is in fact independent of the choice of the bases (see, for example, [49]). For f in $L^2(W,\Omega)$, we consider the Fourier orthogonal series whose n-th partial sum $S_n(f,W)$ is defined by

$$S_n(f,W;\mathbf{x}) = \sum_{k=1}^n P_k(f,W;\mathbf{x}), \quad P_k(f,W;\mathbf{x}) = \int_\Omega f(\mathbf{y}) P_k(W;\mathbf{x},\mathbf{y}) W(\mathbf{y}) d\mathbf{y}.$$

The relation on orthogonal polynomials on the three regions lead to relations between the reproducing kernels. In the case of $W = W_{\kappa,\mu}$ on B^d and $W = W_{\kappa,\mu}^T$ on T^d, this allows us to derive a compact formula for the reproducing kernel. First we state the formula for the classical orthogonal polynomials on B^d, that is, for $W_\mu^B = W_{0,\mu}^B$.

Theorem 4.3 *For the classical weight function $W_\mu^B = W_{0,\mu}$ in (2.2), where $\mu \ge 0$, defined on B^d,*

$$P_n(W_\mu^B;\mathbf{x},\mathbf{y}) = \frac{n++\mu+(d-1)/2}{\mu+(d-1)/2}$$
$$\times \int_{-1}^{1} C_n^{(\mu+\frac{d-1}{2})}(\langle \mathbf{x},\mathbf{y}\rangle + s\sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2})(1-s^2)^{\mu-1}ds. \tag{4.5}$$

We can also write down the compact formula for $W_{\kappa,\mu}^B$ as a multiple integrals using (4.2), (4.3) and Theorem 3.1. The formula looks similar to (4.6) below. For $d=1$, the formula (4.5) reduces to the product formula of the Gegenbauer polynomials (cf. [17, Sec. 3.15.1, (20)]); in fact, in the first proof of (4.5) in [55], we wrote $P_n(W_\mu^B)$ as a multiple sum of an explicit orthonormal basis in terms of Gegenbauer polynomials and used the product formula repeatedly to add up the sums. Essentially the same elementary but tedious proof is given in [50] for the compact formula of $P_n(h_\kappa^2)$, from which the formula for $W_{\kappa,\mu}^B$ can be derived. However, without using the relation between orthogonal polynomials on T^d and S^d, it is unlikely that the compact formulae for $W_{\kappa,\mu}^T$ can be discovered. The formula is given as follows.

Theorem 4.4 *For $W_{\kappa,\mu}^T$ in (2.1), where $\kappa_i \ge 0$ and $\mu \ge 0$, defined on T^d,*

$$P_n(W_{\kappa,\mu};\mathbf{x},\mathbf{y}) = \frac{2n+|\kappa|_1+\mu+(d-1)/2}{|\kappa|_1+\mu+(d-1)/2}\times$$
$$\int_{-1}^{1}\int_{[-1,1]^d} C_{2n}^{(|\kappa|_1+\mu+\frac{d-1}{2})}(\sqrt{x_1y_1}\,t_1+\ldots+\sqrt{x_dy_d}t_d+s\sqrt{1-|\mathbf{x}|_1^2}\sqrt{1-|\mathbf{y}|_1^2})$$
$$\times \prod_{i=1}^d c_{\kappa_i}(1-t_i^2)^{\kappa_i-1}d\mathbf{t}(1-s^2)^{\mu-1}ds. \tag{4.6}$$

These remarkable formulae have already been used in several occasions. In [48] and [56], they are used to derive asymptotics of the Christoffel functions for the ball and for the simplex. In [57], they are used to construct cubature formulae via the method of reproducing kernel. They also allow us to prove various results on summability of Fourier orthogonal expansions. For example, we have ([54] and [55]),

Theorem 4.5 *The Cesàro $(C,|\kappa|_1+\mu+(d+1)/2)$ means of the Fourier orthogonal expansion of a function with respect to $W_{\kappa,\mu}^\Omega$, where $\Omega = B$ or T, define a positive linear polynomial approximation identity on $C(\Omega^d)$; the order of summability is best possible in the sense that the (C,δ) means are not positive for $0<\delta<|\kappa|_1+\mu+(d+1)/2$ in the case of $\Omega=T$ and also in the case of $\Omega = B$ when at least one $\kappa_i = 0$.*

The sufficient part of the positivity follows from the positivity of the sums of

Gegenbauer polynomials, (see [18] or [2]). In fact, since V is positive, the sufficient part holds for all h-harmonics and their orthogonal polynomial counterparts on B^d and T^d. The positivity of the (C,δ) means implies that the means converge in norm. However, the positivity is not necessary for convergence. From Theorem 4.2, the compact formulae of the reproducing kernel and the the relation to h-harmonics, we have the following theorem ([54,55,58]).

Theorem 4.6 *Let $f \in L^p(W^{\Omega}_{\kappa,\mu}, \Omega^d)$, where $\Omega = B$ or $\Omega = T$. Then the expansion of f as the Fourier orthogonal series with respect to $W^{\Omega}_{\kappa,\mu}$ is (C,δ) summable in $L^p(W_{\kappa,\mu}, \Omega^d)$, $1 \le p \le \infty$, provided $\delta > |\kappa|_1 + \mu + (d-1)/2$. Moreover, if at least one $\kappa_i = 0$, then the expansion is not (C,δ) summable in $L^p(W_{\kappa,\mu}, \Omega^d)$ for $p = 1$ or $p = \infty$ provided $\delta \le |\kappa|_1 + \mu + (d-1)/2$.*

Some remarks are in order. If at least one $\kappa_i = 0$, the index $\delta_0 := |\kappa|_1 + \mu + (d-1)/2$ is the analogy of the critical index in the Fourier analysis (cf. [42]). Indeed, as the results in the previous section show, the case $\kappa = 0$ and $\mu = 0$ corresponds to expansion in the classical spherical harmonics, and the index $\delta_0 = (d-1)/2$ in this case is the critical index there. The result in the theorem states that the critical index for the classical orthogonal polynomial expansions on the ball B^d and on the simplex T^d are the same. On the other hand, the critical index for orthogonal expansions on the cube $[-1,1]^d$ is very different from these cases. Indeed, the orthogonal expansions for $1/\prod_{i=1^d}\sqrt{1-x_i^2}$ on $[-1,1]^d$ is the same as the ℓ-1 summability of multiple Fourier series, which has no critical index; that is, $\delta_0 = 0$ ([3,4] and the reference therein). The summability of the general product Jacobi expansions is studied in [31].

There are many open questions in this direction. For example, finding a closed form of the intertwining operator V, and finding explicit orthonormal basis for h-harmonics associated to reflection groups other than $\mathbb{Z}_2^{d+1}$. We discuss some of them below.

Question 4.1. Critical index. If none of $\kappa_i = 0$ in the set up of Theorem 4.6, then we believe that the critical index should be $\delta_0 := |\kappa|_1 - \min_i\{\kappa_i\}$, where we take $\kappa_{d+1} = \mu$. Indeed, it can be shown that (C,δ) means fails to converge in $L^1(W_{\kappa,\mu}, T^d)$ if $\delta \le \delta_0$. To prove that (C,δ) means converge above this critical index, however, will require a method different from what we used to prove Theorem 4.6. For convergence below the critical index, we expect results like those for the classical spherical harmonics. We have, for example, for the Lebesgue measure on B^d the following result.

Theorem 4.7 *Let p satisfies $|1/2 - 1/p| \ge 1/(d+2)$, $d \ge 2$. Then the Cesàro (C,δ) means of the Fourier orthogonal series with respect to the Lebesgue measure converges in $L^p(B^d)$ provided $\delta > \max\{(d+1)|1/p - 1/2| - 1/2, 0\}$.*

There is a further relation between orthogonal polynomials on spheres and on balls, which relates orthogonal polynomials on S^{d+m} to those on B^d with respect to $W^B_{(m-1)/2}$. This relation allows us to derive convergence result for $W^B_{(m-1)/2}$ from that of the spherical harmonics, leading to the above theorem. Some of the difficulties in these questions may come from the fact that we are dealing with reflection group symmetry instead of rotational symmetry, so that many techniques developed in the context of classical harmonic analysis have to be modified.

Question 4.2. Convergence inside the region. The proof of Theorem 4.6 can also be modified to prove that the partial sum $S_n(f, W_{\kappa,\mu})$ converges in norm if f is in $C^{[r]}$, where $r = |\kappa|_1 + \mu + (d-1)/2$, and the modulus of continuity of $f^{[r]}$ satisfies $\omega(f^{[r]}, \varepsilon) = o(\varepsilon^{r-[r]})$, and these conditions are sharp. However, if we are interested in the pointwise convergence, then we can weaken the conditions substantially. Indeed, using [47, Corollary 5.3] and an estimate of the Christoffel function, we can show that $S_n(f, W_{\kappa,\mu})$ converge in the interior of B^d or T^d when the similar condition holds with $r = d/2$. Moreover, we conjecture that the sharp condition is $r = (d-1)/2$, which is the same as the convergence of the ordinary harmonics. We also conjecture that for continuous functions the (C, δ) means $S_n^\delta(f, W_{\kappa,\mu})$ will converge inside B^d and T^d if $\delta > (d-1)/2$. Naturally, we expect that the maximum of $S_n(f, W_{\kappa,\mu})$ or $S_n^\delta(f, W_{\kappa,\mu})$ is attained on the boundary of B^d or T^d, but this has yet to be proved.

Question 4.3. Estimate and asymptotics of the kernel. The reproducing kernels (4.5) and (4.6) deserve a careful study. In the one variable case, a useful form of the kernel is given in terms of the Christoffel-Darboux formula, which is no longer available in several variables. A detail estimate of the kernel will be crucial in the study of various convergence questions. More difficult is to find the asymptotics of the kernel. For the case $\mathbf{x} = \mathbf{y}$, the asymptotics of $P_n(W; \mathbf{x}, \mathbf{x})$ have been studied for $W = W^B_\mu$ on B^d in [48], and for $W = W^T_{\kappa,\mu}$ in [56]. However, the study is incomplete in the case of $W^T_{\kappa,\mu}$.

5 Cubature formulae on the three regions

In this section we discuss the connection between cubature formulae on B^d, T^d and S^d. For a given integral $\mathcal{L}(f) := \int f W d\mathbf{x}$, where W is a weight function on T^d or B^d, a cubature formula of degree M is a linear functional

$$\mathcal{I}_M(f) = \sum_{k=1}^{N} \lambda_k f(\mathbf{x}_k) \ , \quad \lambda_k \in \mathbb{R}, \quad \mathbf{x}_k \in \mathbb{R}^d,$$

defined on Π^d, such that $\mathcal{L}(f) = \mathcal{I}_M(f)$ whenever $f \in \Pi^d_M$, and $\mathcal{L}(f^*) \neq \mathcal{I}_M(f^*)$ for at least one $f^* \in \Pi^d_{M+1}$. When the weight function is supported on S^d, we need to replace Π^d_M by Π^{d+1}_M in the above formulation and require $\mathbf{x}_k \in$

S^d. The points $\mathbf{x}_1, \ldots, \mathbf{x}_N$ in the formula are called *nodes* and the numbers $\lambda_1, \ldots, \lambda_N$ are called *weights*. If all weights of a cubature formula are positive, we call the formula positive.

Each of the three regions B^d, T^d and S^d have drawn their share of attentions over the years; a great number of cubature formulae, most lower degrees, have been constructed on them; see, for example, [6,16,38,43] and reference there). However, the fact that cubature formulae on these three regions are related, in fact often equivalent, is revealed only recently ([52,53]). We state first the correspondence between cubature formulae on balls and on spheres. For a weight function H defined on $\mathbb{R}^{d+1}$ we define W_H^B on B^d as in (3.4).

Theorem 5.1 *If there is a cubature formula of degree M on S^d giving by*

$$\int_{S^d} f(\mathbf{y})H(\mathbf{y})d\omega = \sum_{i=1}^{N} \lambda_i f(\mathbf{y}_i), \tag{5.1}$$

whose nodes are all located on S^d, then there is a cubature formula of degree M on B^d for W_H^B,

$$\int_{B^d} g(\mathbf{x})W_H^B(\mathbf{x})d\mathbf{x} = \sum_{i=1}^{N} \lambda_i g(\mathbf{x}_i), \tag{5.2}$$

where $\mathbf{x}_i \in B^d$ are the first d components of $\mathbf{y}_i$. On the other hand, if there is a cubature formula of degree M in the form (5.2) whose N nodes lie on B^d, then there is a cubature formula of degree M on S^d giving by

$$\int_{S^d} f(\mathbf{y})H(\mathbf{y})d\omega = \sum_{i=1}^{N} \lambda_i \Big[f(\mathbf{x}_i, \sqrt{1-|\mathbf{x}_i|^2}) + f(\mathbf{x}_i, -\sqrt{1-|\mathbf{x}_i|^2}) \Big]/2. \tag{5.3}$$

To state the correspondence between cubature formulae on B^d and on T^d, we need the notion of invariant cubature formulae. A linear functional $\mathcal{I}(f)$ is called invariant under a group G if $\mathcal{I}(R(a)f) = \mathcal{I}(f)$ for all $a \in G$. For $\mathbf{u} \in \mathbb{R}^d$, we denote its G-*orbit* by $G(\mathbf{u})$, which is defined by $G(\mathbf{u}) = \{\mathbf{u}a | a \in G\}$; we also denote by $|G(\mathbf{u})|$ the number of distinct elements in $G(\mathbf{u})$. A cubature formula is invariant under G if the set of its nodes is a union of G-orbits and the nodes belonging to the same G-orbit have the same weight. In the case of $G = \mathbb{Z}_2^d$, the invariant cubature formula, denoted by $I_M(f, \mathbb{Z}_2^d)$, takes the form

$$\mathcal{I}_M(f; \mathbb{Z}_2^d) = \sum_{k=1}^{N} \lambda_k \sum_{\varepsilon \in \{-1,1\}^d} f(\varepsilon_1 u_{i,1}, \ldots \varepsilon_d u_{i,d})/2^{k(\mathbf{u}_i)}$$

where $k(\mathbf{u}) = |\mathbb{Z}_2^d(\mathbf{u})|$, which is equal to the number of non-zero elements of $\mathbf{u}$. We use the notation of W^B and W^T in the Section 3.2.

Theorem 5.2 *If there is a cubature formula of degree M on T^d giving by*

$$\int_{\Sigma^d} f(\mathbf{u}) W^T(\mathbf{u}) d\mathbf{u} = \sum_{i=1}^{N} \lambda_i f(\mathbf{u}_i), \tag{5.4}$$

with all $\mathbf{u}_i \in \mathbb{R}_+^d$, then there is a cubature formula of degree $2M+1$ on the unit ball B^d giving by,

$$\int_{B^d} g(\mathbf{x}) W^B(\mathbf{x}) d\mathbf{x} = \sum_{i=1}^{N} \lambda_i \sum_{\varepsilon \in \{-1,1\}^d} f(\varepsilon_1 \sqrt{u_{i,1}}, \ldots \varepsilon_d \sqrt{u_{i,d}}) / 2^{k(\mathbf{u}_i)} \tag{5.5}$$

Moreover, a cubature formula of degree $2M+1$ in the form of (5.5) implies a cubature formula of degree M in the form of (5.4).

We note that the degree of the formula (5.5) is $2M+1$, more than twice that of formula (5.4). From these two theorems, we can also write down a correspondence between cubature formulae on T^d and on S^d, which we shall not formulate here; see [53].

The importance of these correspondences is evident. They allow us to construct cubature formulae on one region from those on another region. Even from the existing list of cubature formulae, they lead to many new formulae. However, it should be pointed out that most of the cubature formulae in the literature are constructed for the Lebesgue measure (unit weight function). The correspondences on these regions show that the Lebesgue measure $d\mathbf{x}$ on T^d corresponds to $|x_1 \cdots x_{d+1}| d\omega$ on S^d and $|x_1 \ldots x_d| d\mathbf{x}$ on B^d, and the Lebesgue measure on B^d corresponds to $|x_{d+1}| d\omega$ on S^d and $d\mathbf{x}/\sqrt{x_1 \cdots x_d}$ on B^d. In the discussion below, we will concentrate on the cubature formulae for the surface measure $d\omega$, which corresponds to formulae for the weight function $1/\sqrt{1-|\mathbf{x}|^2}$ on B^d and the weight function $1/\sqrt{x_1 \cdots x_d(1-|\mathbf{x}|_1)}$ on T^d. These two weight functions are special cases of W_μ^B and $W_{\kappa,\mu}^T$, we shall call them Chebyshev weight function on B^d and on T^d, respectively.

One way to use the correspondences is to revisit the existing methods of constructing cubature formulae on B^d or T^d for the unit weight function, use them to construct formulae for the Chebyshev weight function, and then obtain new cubature formulae for the surface measure on S^d. This approach has been used in [20–22]. We present one notable family of cubature formulae on S^d obtained in [22] below, and discuss symmetric formulae on S^2 in the following section. We need to introduce the following notation.

Let f be defined on $\mathbb{R}^{d+1}$. Since the hyper-octahedral group $\mathcal{B}_{d+1}$ is the semi-product of the symmetric group $\mathcal{S}_{d+1}$ and $\mathbb{Z}_2^{d+1}$, we can write

$$\sum_{\sigma\in\mathcal{B}_{d+1}} f(\mathbf{x}\sigma) = \sum f(\pm x_{\tau_0}, \pm x_{\tau_1}, \ldots, \pm x_{\tau_d}),$$

where we write $\mathbf{x} = (x_0, \ldots, x_d)$ and the sum in the right hand side is over all choices of signs and over all $\tau \in \mathcal{S}_{d+1}$; that is, we write $\tau = (\tau_0, \ldots, \tau_d)$ to denote a permutation of $(0, 1, \ldots, d)$. Cubature formulae that consist entirely of sums as these are invariant under $\mathcal{B}_{d+1}$, they are also called fully symmetric, see [43, p. 128]. For $\alpha \in \mathbb{N}_0^{d+1}$, we will write $\alpha = (\alpha_0, \ldots, \alpha_d)$ in the rest of this section. We have the following result.

Theorem 5.3 *Let $s \in \mathbb{N}_0$ and $n = 2s + 1$. Then the following is a cubature formula of degree $2n + 1$ on S^d,*

$$\begin{aligned}
\int_{S^d} g(\mathbf{y})d\omega = {} & \frac{\pi^{(d+1)/2}}{2^{2s+d}}\Big[\sum_{i=0}^{s}(-1)^i \frac{(n+(d-1)/2-2i)^n}{i!\Gamma(n+(d+1)/2-i)} \\
& \times \sum_{|\beta|_1=s-i,\beta_0\geq\cdots\geq\beta_d} \binom{\beta_0-1/2}{\beta_0}\cdots\binom{\beta_d-1/2}{\beta_d} \\
& \times \sum_{\sigma\in\mathcal{B}_{d+1}} g\Big(\Big(\frac{\sqrt{2\beta_0+1/2}}{\sqrt{n+(d-1)/2-2i}}, \cdots, \frac{\sqrt{2\beta_d+1/2}}{\sqrt{n+(d-1)/2-2i}}\Big)\sigma\Big)\Big].
\end{aligned}$$

This formula is apparently the first known family of cubature formulae that exist for higher degrees and for all d. Its number of nodes is also relatively small. The drawback of the formula, however, is that it is not positive and its conditional number grows rapidly as s increases. The formula is discovered using the correspondence in Theorem 5.1 and Theorem 5.2 as follows. In [19], a family of cubature formulae was established for the unit weight function on T^d by proving a combinatorial formula. In [22], we establish the analogous of these formulae for the weight function $W^T_{\kappa,\mu}$ on T^d for general κ and μ by proving a combinatorial formula that contains a number of parameters. The combinatorial formula is as follows.

$$\begin{aligned}
2^{2s}\frac{\prod_{j=0}^d \Gamma(\alpha_j+\mu_j+1)}{\prod_{j=0}^d \Gamma(\mu_j+1)} = {} & \sum_{j=0}^{s}(-1)^j\binom{2s+|\mu|_1+d+1}{j} \\
& \times \sum_{|\beta|_1=s-j}\binom{\beta_0+\mu_0}{\beta_0}\cdots\binom{\beta_d+\mu_d}{\beta_d}\prod_{i=0}^{d}(2\beta_i+\mu_i+1)^{\alpha_i},
\end{aligned} \tag{5.6}$$

which holds for all $\alpha \in \mathbb{N}_0^d$, $|\alpha| = 2s + 1$, and $\mu_i > -1$ for $0 \leq i \leq d$. For $\mu_0 = \ldots = \mu_d = 1/2$, this is the formula proved in [19]. Let $X^\alpha =$

$(1-|\mathbf{x}|_1)^{\alpha_0}x_1^{\alpha_1}\cdots x_d^{\alpha_d}$. Using the fact that $\{X^\alpha\}_{|\alpha|_1=n}$ forms a basis for Π_n^d and the fact that the integral of X^α with respect to $W_{\kappa,\mu}^T$ can be derived from

$$\int_{T^d} X^{\alpha+\mu} d\mathbf{x} = \frac{\prod_{j=0}^d \Gamma(\alpha_j+\mu_j+1)}{\Gamma(|\alpha|_1+|\mu|_1+d+1)},$$

it follows that (5.6) yields a cubature formula of degree $2s+1$ for $W_{\kappa,\mu}^T$ on T^d (taking $\kappa_i = \mu_i + 1/2$ for $0 \le i \le d$ and $\mu = \mu_0 + 1$. Using the correspondence in Theorem 5.1 and Theorem 5.2 we then get cubature formulae for $W_{\kappa,\mu}^B$ on B^d and $h_{\kappa,\mu}^2$ on S^d. The Theorem 5.3 is the special case of $\kappa = 0$ and $\mu = 0$ on S^d.

6 Cubature formulae on S^2 invariant under the octahedral group

Numerical integration on the sphere S^2 has attracted a lot of attention, we refer to [27–30,34,38,41,43] and the references there. Most of the cubature formulae on S^2 are constructed by solving moment equations under the assumption that the formulae are symmetric under a finite group. The symmetry helps to reduce the number of moment equations that have to be solved, due to a fundamental result of Sobolev which states that a cubature formula invariant under a finite group is exact for all polynomials in a subspace $\mathcal{P}$ if, and only if, it is exact for all polynomials in $\mathcal{P}$ that are invariant under the same group. The groups that have been employed previously in the context are mainly the octahedral group and the icosahedral group. In particular, Lebedev constructed in [27–30] cubature formulae of degree up to 59, many of which have the smallest number of nodes among all formulae that are known; he also made an outstanding conjecture that we shall address in this section.

Under the correspondences in the previous section, formulae on S^2 invariant under the octahedral group (which is the symmetric group of the unit cube $\{\pm1,\pm1,\pm1\}$ in $\mathbb{R}^3$) correspond to formulae on T^2 invariant under the symmetric group of T^2. Since the octahedral group is the semi-direct product of symmetric group and $\mathbb{Z}_2^3$ and only the action of symmetric group appears on T^2 under our correspondence, there is certain advantage in dealing with symmetric formulae on T^2 instead of octahedral symmetric formulae on S^2. We shall state Lebedev's conjecture in terms of symmetric cubature formulae on T^2. To do so, we follow the set-up in [32] and use the equilateral triangle

$$\triangle = \{(x,y) : x \le 1/2, \quad \sqrt{3}y - x \le 1, \quad -\sqrt{3}y - x \le 1\},$$

which can be transformed to T^2 by a simple affine transformation. The symmetric group $\mathcal{S}_3(\triangle)$ of $\triangle$ is generated by a rotation through an angle $2\pi/3$ and a reflection about the x-axis. It is sometimes convenient to use the polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$ to denote points on $\triangle$. Let Λ denote the triangle

$$\Lambda = \{(x,y) : 0 \le x \le 1/2, \quad 0 \le x \le \sqrt{3}y\}.$$

Then Λ is one of the fundamental regions of $\triangle$ under $\mathcal{S}_3(\triangle)$. To describe a symmetric cubature formula on $\triangle$, it suffices to determine its nodes inside Λ. We say that a symmetric cubature formula is of type $[m_0; m_1, m_2, m_3; m4, m5]$, if it has m_0 node at the origin, m_1 node at the vertex $(1/2, \sqrt{3}/2)$, m_2 node at $(1/2, 0)$, m_3 nodes on the two sides $\theta = 0$ and $\theta = \pi/3$ (not at the vertices) of Λ, m_4 nodes on the side $x = 1/2$ (not at the vertices) of Λ, and m_5 nodes in the interior of Λ. We also call the corresponding formula on S^2 type $[m_0; m_1, m_2, m_3; m4, m5]$.

Moment equations for the type $[m_0; m_1, m_2, m_3; m4, m5]$ formulae were set up in [32], and used to construct cubature formulae of degree up to 20 in [32,8] for the unit weight function. In [21] the moment equations are solved for the Chebyshev weight function (recall that it relates to the surface measure on S^2), which yields formulae of degree up to 41 for the surface measure on S^2, including those found by Lebedev. To set up the moment equation, one usually requires that the number of parameters matches the number of equations. For a type $[m_0; m_1, m_2, m_3; m4, m5]$ formula of degree M, this leads to

$$m_0 + m_1 + m_2 + 2m_3 + 2m_4 + 3m_5 = [(M^2 + 6M + 12)/12],$$

where $[x]$ denote the greatest integer less than or equal to x. For each M, there can be a number of integer solutions to the equation, leading to different types of cubature formulae. However, since the moment equations are nonlinear, many types of formulae do not exist. Based on his computation, Lebedev made the following conjecture.

Conjecture 6.1 *Cubature formulae of type* $[1; 0, 1, 3m; m, m(m-1)]$ *and* $[1; 1, 1, 3m+1; m, m^2]$ *exist.*

Formulae of these types on $\triangle$ are of degree $6m + 2$ and $6m + 5$, and they correspond to formulae of degree $12m + 5$ and $12m + 11$ on S^2, respectively. Lebedev have constructed formulae for $m = 1, 2, 3, 4$ on S^2, whose nodes turn out to be rather uniformly distributed on the sphere.

The work of [27–30], as well as that of [21], is essentially numerical computation. Being so, it gives little hint on how to prove the conjecture. The formulae invariant under the octahedral group are also called fully symmetric formulae ([43]). The structure of the fully symmetric formulae or that of the associated moment equations have been studied in [23,24,32,33] and the references therein, but the study appears to be still in the initial stage. In particular, the intrinsic structure of the symmetric formulae of the types in Lebedev's conjecture has not been studied. In the following we discuss some observations and other problems related to this conjecture.

Because of the fundamental result of Sobolev, it is essential to understand the structure of polynomials invariant under the symmetric group. Let us denote by $\Pi_n^2(\mathcal{S}_3)$ the space of polynomials of degree n invariant under the group $\mathcal{S}_3(\triangle)$. It is easy to see that

$$\Pi_n^2(\mathcal{S}_3) = \mathrm{span}\{(x^2+y^2)^k(x^3-3xy^2)^j : \quad 2k+3j \le n\}.$$

We can change variables $t = x^2 + y^2$ and $s = x^3 - 3xy^2$ so that the space $\Pi_n^2(\mathcal{S}_3)$ becomes a subspace spanned by monomials $\{t^k s^j : 2k+3j \le n\}$. However, the order of this polynomial subspace is messed up; for example, the multiplication by t or s is no longer a mapping from $\Pi_n^2(\mathcal{S}_3)$ to $\Pi_{n+1}^2(\mathcal{S}_3)$. On the other hand, for special values of n, we have the following observation.

Proposition 6.2 *Let* $\Pi_m^* := \Pi_{3m-1}^2(\mathcal{S}_3)$ *for* $m = 1, 2, \ldots$ *Then* $s\Pi_m^* \subset \Pi_{m+1}^*$ *and* $t\Pi_m^* \subset \Pi_{m+1}^*$.

Using the fact that $\Pi_0^* = \mathrm{span}\{1\}$ and $\Pi_1^* = \mathrm{span}\{1, s\}$, we can also write down the decomposition of Π_m^* in terms of space of homogeneous polynomials. The observation suggests that the space Π_m^* has a structure similar to the space Π_m^2. What prompts us to consider this space is explained as follows. A formula in Lebedev's conjecture, if exists, is like a Gaussian quadrature formulae in the sense that its number of parameters matches up the number of moment equations. One possible way to establish the conjecture is to show that the nodes of the formula are common zeros of a sequence of symmetric orthogonal polynomials. We will not discuss the connection between cubature formulae and common zeros of orthogonal polynomials here, but refer to [35,37,38,46,49] and the references there. Like the case of Gaussian quadrature, for cubature formulae of degree $2n-1$, we need to look at common zeros of orthogonal polynomials of degree n or higher. For Lebedev's formulae of degree $6m+5$, this suggests us to look at the polynomial space of $\Pi_{3m+2}^2(\mathcal{S}_3) = \Pi_{m+1}^*$.

The mapping $t \to x^2 + y^2$ and $s \to x^3 - 3xy^2$ is nonsingular on Λ and it maps Λ to a curved triangular region which we denote by Λ^*. Symmetric cubature formulae on $\triangle$ corresponds to formulae on Λ^* for the space Π_n^*. Using a computer algebra system doing symbolic computation, we found that the degree 5 formula on Λ^*, which corresponds to the degree 11 formula ($m = 1$ of the case $6m+5$) in Lebedev's conjecture, is indeed generated by common zeros of orthogonal polynomials.

To prove the conjecture following this line, we need compact formulae for an orthonormal basis of Π_m^*, which can be obtained from the symmetric basis of orthogonal polynomials on the triangle, or from spherical harmonics on S^2 by the correspondence. However, a compact formulae of orthonormal basis for symmetric polynomials on T^2 is the same as a basis for the spherical harmonics invariant under the octahedral group, which is not easy to find. Only [10] seems to contain some results in this direction. Moreover, the cubature formulae in

Lebedev's conjecture have nodes on the boundary of $\triangle$. Using a procedure that resembles the passage from Gaussian-Radau type quadrature to Gaussian quadrature, we can reduce the problem to finding cubature formulae with all nodes inside $\triangle$ for integrals with respect to a modified weight function. The new weight function is obtained by multiplying the Chebyshev weight with the polynomial that is quadratic on each side of the boundary of $\triangle$. The corresponding weight function on the sphere S^2 is $h(\mathbf{x}) = (x_1^2 - x_2^2)^2(x_2^2 - x_3^2)^2(x_2^2 - x_3^2)^2$, which is a special case of h_κ in (2.5). Thus, this calls to study the h-harmonics associated to $h^2 d\omega$ that are invariant under the octahedral group. At the time of this writing, no compact formulae of an orthonormal basis for these h-harmonics are known.

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