

MONOMIAL ORTHOGONAL POLYNOMIALS OF SEVERAL VARIABLES

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ABSTRACT. A monomial orthogonal polynomial of several variables is of the form $x^\alpha - Q_\alpha(x)$ for a multiindex $\alpha \in \mathbb{N}_0^{d+1}$ and it has the least L^2 norm among all polynomials of the form $x^\alpha - P(x)$, where P and Q_α are polynomials of degree less than the total degree of x^α . We study monomial orthogonal polynomials with respect to the weight function $\prod_{i=1}^{d+1} |x_i|^{2\kappa_i}$ on the unit sphere S^d as well as for the related weight functions on the unit ball and on the standard simplex. The results include explicit formula, L^2 norm, and explicit expansion in terms of known orthonormal basis. Furthermore, in the case of $\kappa_1 = \dots = \kappa_{d+1}$, an explicit basis for symmetric orthogonal polynomials is also given.

1. INTRODUCTION

The purpose of this paper is to study monomial orthogonal polynomials of several variables. Let W be a weight function defined on a set Ω in \mathbb{R}^d . Let $\alpha \in \mathbb{N}_0^d$. The monomial orthogonal polynomials are of the form $R_\alpha(x) = x^\alpha - Q_\alpha(x)$ with Q_α being a polynomial of degree less than $n = |\alpha| := \alpha_1 + \dots + \alpha_d$, and it is orthogonal to all polynomials of degree less than n in $L^2(W, \Omega)$; in other words, they are orthogonal projections of x^α onto the subspace of orthogonal polynomials of degree n . In the case of one variable, such a polynomial is just an orthogonal polynomial normalized with a unit leading coefficient and its explicit formula is known for many classical weight functions. For several variables, there are many linearly independent orthogonal polynomials of the same degree and the explicit formula of R_α is not immediately known.

Let Π_n^d denote the space of polynomials of degree at most n in d variables. The polynomial R_α can be considered as the error of the best approximation of x^α by polynomials from Π_{n-1}^d , $n = |\alpha|$, in $L^2(W, \Omega)$. Indeed, a standard Hilbert space argument shows that

$$\|R_\alpha\|_2 = \|x^\alpha - Q_\alpha\|_2 = \inf_P \{\|x^\alpha - P\|_2, \quad P \in \Pi_{n-1}^d, \quad n = |\alpha|\},$$

where $\|\cdot\|_2$ is the $L^2(W, \Omega)$ norm. In other words, R_α has the least L^2 norm among all polynomials of the form $x^\alpha - P$, where $P \in \Pi_{n-1}^d$.

Let $d\omega$ be the surface measure on the unit sphere $S^d = \{x : \|x\| = 1\}$, where $\|x\|$ is the Euclidean norm of $x \in \mathbb{R}^{d+1}$. In the present paper we consider the monomial

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orthogonal polynomials in $L^2(h_\kappa^2 d\omega, S^d)$, where

$$(1.1) \quad h_\kappa(x) = \prod_{i=1}^{d+1} |x_i|^{\kappa_i}, \quad \kappa_i \geq 0, \quad x \in \mathbb{R}^{d+1}.$$

The homogeneous orthogonal polynomials with respect to this weight function are called h -harmonics; they are the simplest examples of the h -harmonics associated with the reflection groups (see, for example, [4, 5, 7] and the references therein). The weight function in (1.1) is invariant under the group \mathbb{Z}_2^{d+1} . Let \mathcal{P}_n^{d+1} denote the space of homogeneous polynomials of degree n in $d+1$ variables. The monomial homogeneous polynomials are of the form $R_\alpha(x) = x^\alpha - \|x\|^2 Q_\alpha(x)$, where $Q_\alpha \in \mathcal{P}_{n-2}^{d+1}$ and $n = |\alpha|$. In this case, we define R_α through a generating function and derive their various properties. Using a correspondence between the h -harmonics and orthogonal polynomials on the unit ball $B^d = \{x : \|x\| \leq 1\}$ of \mathbb{R}^d , this also gives the monomial orthogonal polynomials with respect to the weight function

$$(1.2) \quad W_\kappa^B(x) = \prod_{i=1}^d |x_i|^{2\kappa_i} (1 - \|x\|^2)^{\kappa_{d+1}-1/2}, \quad x \in B^d, \quad \kappa_i \geq 0.$$

In the case $\kappa_i = 0$ for $1 \leq i \leq d$ and $\kappa_{d+1} = \mu$, the weight function W_κ^B is the classical weight function $(1 - \|x\|^2)^{\mu-1/2}$ for which the monomial polynomials are known already to Hermite (in special cases); see [8, Vol. 2, Chapt 12]. There is also a correspondence between the h -harmonics and orthogonal polynomials on the simplex $T^d = \{x : x_i \geq 0, 1 - |x| \geq 0\}$ of \mathbb{R}^d , where $|x| = x_1 + \dots + x_d$, which allows us to derive properties of the monomial orthogonal polynomials with respect to the weight function

$$(1.3) \quad W_\kappa^T(x) = \prod_{i=1}^d |x_i|^{\kappa_i-1/2} (1 - |x|)^{\kappa_{d+1}-1/2}, \quad x \in T^d, \quad \kappa_i \geq 0.$$

For these families of the weight functions, we will define the monomial orthogonal polynomials using generating functions, and give explicit formulae for these polynomials in the next section.

If $\kappa_1 = \dots = \kappa_{d+1}$, then the weight function is invariant under the action of the symmetric group. We can consider the subspace of h -harmonics invariant under the symmetric group. Recently, in [6], Dunkl gave an explicit basis in terms of monomial symmetric polynomials. Another explicit basis can be derived from the explicit formulae of R_α , which we give in Section 3.

Various explicit bases of orthogonal polynomials for the above weight functions have appeared in [7, 11, 16], some can be traced back to [2, 8] in special cases. Our emphasis is on the monomial bases and explicit computation of the L^2 norm. The L^2 norms or the monomial orthogonal polynomials give the error of the best approximation to monomials by polynomials of lower degrees. We compute the norms in Section 4. They are expressed as integrals of the product of the Jacobi or Gegenbauer polynomials. We mention two special cases of our general results, in which $P_n(t)$ denotes the Legendre polynomial of degree n :

Theorem 1.1. *For $\alpha \in \mathbb{N}_0^d$, let $n = |\alpha|$. Then*

$$\min_{Q \in \Pi_{n-1}^d} \frac{1}{\text{vol } B^d} \int_{B^d} |x^\alpha - Q(x)|^2 dx = \frac{d! \alpha!}{2^n (d/2)_n} \int_0^1 \prod_{i=1}^d P_{\alpha_i}(t) t^{n+d-1} dt,$$

where $\text{vol } B^d = \pi^{d/2}/\Gamma(d/2 + 1)$ is the volume of B^d , and

$$\min_{Q \in \Pi_{n-1}^d} \frac{1}{d!} \int_{T^d} |x^\alpha - Q(x)|^2 dx = \frac{d \alpha!^2}{(d)_{2n}} \int_0^1 \prod_{i=1}^d P_{\alpha_i}(2r-1) r^{n+d-1} dr.$$

As the best approximation to x^α , the monomial orthogonal polynomials with respect to the unit weight function (Lebesgue measure) on B^2 have been studied recently in [3]. Let us also mention [1], in which certain invariant polynomials with the least L^p norm on S^d are studied.

For h -harmonics, the set $\{R_\alpha : |\alpha| = n\}$ contains an orthogonal basis of h -harmonics of degree n but the set itself is not a basis. In general, two monomial orthogonal polynomials of the same degree are not orthogonal to each other. On the other hand, for each of the three families of the weight functions, an orthonormal basis can be given in terms of the Jacobi polynomials or the Gegenbauer polynomials. We will derive an explicit expansion of R_α in terms of this orthonormal basis in Section 5, the coefficients of the expansion are given in terms of Hahn polynomials of several variables.

Finally in Section 6, we discuss another property of the polynomials defined by the generating function. It leads to an expansion of monomials in terms of monomial orthogonal polynomials.

2. MONOMIAL ORTHOGONAL POLYNOMIALS

Throughout this paper we use the standard multiindex notation. For $\alpha \in \mathbb{N}_0^m$ we write $|\alpha| = \alpha_1 + \dots + \alpha_m$. For $\alpha, \beta \in \mathbb{N}_0^m$ we also write $\alpha! = \alpha_1! \dots \alpha_m!$ and $(\alpha)_\beta = (\alpha_1)_{\beta_1} \dots (\alpha_m)_{\beta_m}$, where $(a)_n = a(a+1) \dots (a+n-1)$ is the Pochhammer symbol. Furthermore, for $\alpha \in \mathbb{N}^m$ and $a, b \in \mathbb{R}$, we write $a\alpha + b\mathbf{1} = (a\alpha_1 + b, \dots, a\alpha_m + b)$ and denote $\mathbf{1} := (1, 1, \dots, 1)$. For $\alpha, \beta \in \mathbb{N}_0^m$, the inequality $\alpha \leq \beta$ means that $\alpha_i \leq \beta_i$ for $1 \leq i \leq m$.

2.1. Monomial h -harmonics. First we recall relevant part of the theory of h -harmonics; see [4, 5, 7] and the reference therein. We shall restrict ourself to the case of h_κ defined in (1.1); see also [16].

Let $\mathcal{H}_n^{d+1}(h_\kappa^2)$ denote the space of homogeneous orthogonal polynomials of degree n with respect to $h_\kappa^2 d\omega$ on S^d . If all $\kappa_i = 0$, then $\mathcal{H}_n^{d+1}(h_\kappa^2)$ is just the space of the ordinary harmonics. It is known that

$$\dim \mathcal{H}_n^{d+1}(h_\kappa^2) = \dim \mathcal{P}_n^{d+1} - \dim \mathcal{P}_{n-2}^{d+1} = \binom{n+d}{d} - \binom{n+d-2}{d}.$$

The elements of $\mathcal{H}_n^{d+1}(h_\kappa^2)$ are called h -harmonics since they can be defined through an analog of Laplacian operator. The essential ingredient is Dunkl's operators, which are a family of first order differential-difference operators defined by

$$(2.1) \quad \mathcal{D}_i f(x) = \partial_i f(x) + \kappa_i \frac{f(x) - f(x_1, \dots, -x_i, \dots, x_{d+1})}{x_i}, \quad 1 \leq i \leq d+1.$$

These operators commute; that is, $\mathcal{D}_i \mathcal{D}_j = \mathcal{D}_j \mathcal{D}_i$. The h -Laplacian is defined by $\Delta_h = \mathcal{D}_1^2 + \dots + \mathcal{D}_{d+1}^2$. Then $\Delta_h P = 0$, $P \in \mathcal{P}_n^{d+1}$ if and only if $P \in \mathcal{H}_n^{d+1}(h_\kappa^2)$. The structure of the h -harmonics and that of ordinary harmonic polynomials are parallel. Some of the properties of h -harmonics can be expressed using the *intertwining*

operator, V_κ , which is a linear operator that acts between ordinary harmonics and h -harmonics. It is uniquely determined by the properties

$$\mathcal{D}_i V_\kappa = V_\kappa \partial_i, \quad V_\kappa 1 = 1, \quad V_\kappa \mathcal{P}_n^{d+1} \subset \mathcal{P}_n^{d+1}.$$

For the weight function h_κ in (1.1), V_κ is an integral operator defined by

$$(2.2) \quad V_\kappa f(x) = \int_{[-1,1]^{d+1}} f(x_1 t_1, \dots, x_{d+1} t_{d+1}) \prod_{i=1}^{d+1} c_{\kappa_i} (1+t_i)(1-t_i^2)^{\kappa_i-1} dt,$$

where $c_\mu = \Gamma(\mu + 1/2)/(\sqrt{\pi}\Gamma(\mu))$. If any one of $\kappa_i = 0$, the formula holds under the limit

$$(2.3) \quad \lim_{\mu \rightarrow 0} c_\mu \int_{-1}^1 f(t)(1-t^2)^{\mu-1} d\mu = [f(1) + f(-1)]/2.$$

The Poisson kernel, or reproducing kernel, $P(h_\kappa^2; x, y)$ of the h -harmonics is defined by the property

$$(2.4) \quad f(x) = c'_h \int_{S^d} f(y) P(h_\kappa^2; x, y) f(y) h_\kappa^2(y) d\omega(y), \quad c'_h = \frac{\Gamma(|\kappa| + \frac{d+1}{2})}{2 \prod_{i=1}^{d+1} \Gamma(\kappa_i + \frac{1}{2})},$$

for $f \in \mathcal{H}_n^d(h_\kappa^2)$ and $\|y\| \leq 1$, where c'_h is the normalization constant of the weight function h_κ^2 on the unit sphere S^d , $c'_h \int_{S^d} h_\kappa^2 d\omega = 1$ and $d\omega$ is the surface measure. Using the intertwining operator, the Poisson kernel of the h -harmonics can be written as

$$P(h_\kappa^2; x, y) = V_\kappa \left[\frac{1 - \|y\|^2}{(1 - 2\langle y, \cdot \rangle + \|y\|^2)^{\rho+1}} \right] (x), \quad \rho = |\kappa| + \frac{d-1}{2},$$

for $\|y\| < 1 = \|x\|$. If all $\kappa_i = 0$, then $V_\kappa = id$ is the identity operator and $P(h_0^2; x, y)$ is the classical Poisson kernel, which is related to the Poisson kernel of the Gegenbauer polynomials

$$\frac{1 - r^2}{(1 - 2rt + r^2)^{\lambda+1}} = \sum_{n=0}^{\infty} \frac{n + \lambda}{\lambda} C_n^\lambda(t) r^n.$$

The above function can be viewed as a generating function for the Gegenbauer polynomials $C_n^\lambda(t)$. The usual generating function of C_n^λ , however, takes the following form: $(1 - 2rt + r^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(t) r^n$. Our definition of the monomial orthogonal polynomials is the analog of the generating function of C_n^λ in several variables.

Definition 2.1. Let $\rho = |\kappa| + \frac{d-1}{2} > 0$. Define polynomials $\tilde{R}_\alpha(x)$ by

$$V_\kappa \left[\frac{1}{(1 - 2\langle b, \cdot \rangle + \|b\|^2 \|x\|^2)^\rho} \right] (x) = \sum_{\alpha \in \mathbb{N}_0^{d+1}} b^\alpha \tilde{R}_\alpha(x), \quad x \in \mathbb{R}^{d+1}.$$

Let F_B be the Lauricella hypergeometric series of type B , which generalizes the hypergeometric function ${}_2F_1$ to several variables (cf. [10]),

$$F_B(\alpha, \beta; c; x) = \sum_{\gamma} \frac{(\alpha)_\gamma (\beta)_\gamma}{(c)_{|\gamma|} \gamma!} x^\gamma, \quad \alpha, \beta \in \mathbb{N}_0^{d+1}, \quad c \in \mathbb{R} \quad \max_{1 \leq i \leq d+1} |x_i| < 1,$$

where the summation is taken over $\gamma \in \mathbb{N}_0^{d+1}$. We derive properties of \tilde{R}_α in the following.

Proposition 2.2. *The polynomials \tilde{R}_α satisfy the following properties:*

(1) $\tilde{R}_\alpha \in \mathcal{P}_n^{d+1}$ and

$$\tilde{R}_\alpha(x) = \frac{2^{|\alpha|}(\rho)_{|\alpha|}}{\alpha!} \sum_{\gamma} \frac{(-\alpha/2)_{\gamma}(-(\alpha+1)/2)_{\gamma}}{(-|\alpha|-\rho+1)_{|\gamma|}\gamma!} \|x\|^{2|\gamma|} V_{\kappa}(x^{\alpha-2\gamma}),$$

where the series terminates as the summation is over all γ such that $2\gamma \leq \alpha$;

(2) $\tilde{R}_\alpha \in \mathcal{H}_n^{d+1}(h_{\kappa}^2)$ and $\tilde{R}_\alpha(x) = \frac{2^{|\alpha|}(\rho)_{|\alpha|}}{\alpha!} V_{\kappa}[S_{\alpha}(\cdot)](x)$ for $\|x\| = 1$, where

$$S_{\alpha}(y) = y^{\alpha} F_B \left(-\frac{\alpha}{2}, \frac{-\alpha+1}{2}; -|\alpha|-\rho+1; \frac{1}{y_1^2}, \dots, \frac{1}{y_{d+1}^2} \right).$$

Furthermore,

$$\sum_{|\alpha|=n} b^{\alpha} \tilde{R}_{\alpha}(x) = \frac{\rho}{n+\rho} P_n(h_{\kappa}^2; b, x), \quad \|x\| = 1,$$

where $P_n(h_{\kappa}^2; y, x)$ is the reproducing kernel of $\mathcal{H}_n^{d+1}(h_{\kappa}^2)$ in $L^2(h_{\kappa}^2, S^{d-1})$.

Proof. Using the multinomial and binomial formula, we write

$$\begin{aligned} (1 - 2\langle a, y \rangle + \|a\|^2)^{-\rho} &= (1 - a_1(2y_1 - a_1) - \dots - a_d(2y_{d+1} - a_{d+1}))^{-\rho} \\ &= \sum_{\beta} \frac{(\rho)_{|\beta|}}{\beta!} a^{\beta} (2y_1 - a_1)^{\beta_1} \dots (2y_{d+1} - a_{d+1})^{\beta_{d+1}} \\ &= \sum_{\beta} \frac{(\rho)_{|\beta|}}{\beta!} \sum_{\gamma} \frac{(-\beta_1)_{\gamma_1} \dots (-\beta_{d+1})_{\gamma_{d+1}}}{\gamma!} 2^{|\beta|-|\gamma|} y^{\beta-\gamma} a^{\gamma+\beta}. \end{aligned}$$

Changing summation indices $\beta_i + \gamma_i = \alpha_i$ and using the formulae

$$(\rho)_{m-k} = \frac{(-1)^k(\rho)_m}{(1-\rho-m)_k} \quad \text{and} \quad \frac{(-m+k)_k}{(m-k)!} = \frac{(-1)^k(-m)_{2k}}{m!}$$

as well as $2^{-2k}(-m)_{2k} = (-m/2)_k((1-m)/2)_k$, we can rewrite the formula as

$$\begin{aligned} (1 - 2\langle a, y \rangle + \|a\|^2)^{-\rho} &= \sum_{\alpha} a^{\alpha} \frac{2^{|\alpha|}(\rho)_{|\alpha|}}{\alpha!} \sum_{\gamma} \frac{(-\alpha/2)_{\gamma}((-\alpha+1)/2)_{\gamma}}{(-|\alpha|-\rho+1)_{|\gamma|}\gamma!} y^{\alpha-2\gamma} \\ &= \sum_{\alpha} a^{\alpha} \frac{2^{|\alpha|}(\rho)_{|\alpha|}}{\alpha!} y^{\alpha} F_B \left(-\frac{\alpha}{2}, \frac{1-\alpha}{2}; -|\alpha|-\rho+1; \frac{1}{y_1^2}, \dots, \frac{1}{y_d^2} \right), \end{aligned}$$

Using the first equal sign of the expansion with the function

$$(1 - 2\langle b, y \rangle + \|x\|^2\|b\|^2)^{-\rho} = (1 - 2\langle \|x\|b, y/\|x\| \rangle + \|\|x\|b\|^2)^{-\rho}$$

and applying V with respect to y gives the expression of \tilde{R}_{α} in the part (1). If $\|x\| = 1$, then the second equal sign gives the expression of \tilde{R}_{α} in the part (2). We still need to show that $\tilde{R}_{\alpha} \in \mathcal{H}_n^{d+1}(h_{\kappa}^2)$. Let $\|x\| = 1$. For $\|y\| \leq 1$ the generating function of the Gegenbauer polynomials gives

$$(1 - 2\langle b, y \rangle + \|b\|^2)^{-\rho} = (1 - 2\|b\|\langle b/\|b\|, y \rangle + \|b\|^2)^{-\rho} = \sum_{n=0}^{\infty} \|b\|^n C_n^{\rho}(\langle b/\|b\|, y \rangle).$$

Consequently, applying V_{κ} on y in the above equation gives

$$\sum_{|\alpha|=n} b^{\alpha} \tilde{R}_{\alpha}(x) = \|b\|^n V_{\kappa}[C_n^{\rho}(\langle b/\|b\|, \cdot \rangle)](x), \quad \|x\| = 1.$$

On the other hand, it is known that the reproducing kernel $P_n(h_\kappa^2; x, y)$ of $\mathcal{H}_n^{d+1}(h_\kappa^2)$ is given by [7, p. 190]

$$P_n(h_\kappa^2; x, y) = \frac{n + \rho}{\rho} \|y\|^n V_\kappa[C_n^\rho(\langle y/\|y\|, \cdot \rangle)](x), \quad \|y\| \leq \|x\| = 1,$$

so that $\sum_{|\alpha|=n} b^\alpha \tilde{R}_\alpha(x)$ is a constant multiple of $P_n(h_\kappa^2; x, b)$. Consequently, for any b , $\sum b^\alpha \tilde{R}_\alpha(x)$ is an element in $\mathcal{H}_n^{d+1}(h_\kappa^2)$; therefore, so is \tilde{R}_α . \square

In the following let $[x]$ denote the integer part of x . We also use $[\alpha/2]$ to denote $([\alpha_1/2], \dots, [\alpha_{d+1}/2])$ for $\alpha \in \mathbb{N}_0^{d+1}$.

Proposition 2.3. *Let $\rho = |\kappa| + (d - 1)/2$. Then*

$$\tilde{R}_\alpha(x) = \frac{2^{|\alpha|}(\rho)_{|\alpha|}}{\alpha!} \frac{(1/2)_{\alpha-\beta}}{(\kappa + 1/2)_{\alpha-\beta}} R_\alpha(x), \quad \text{where } \beta = \alpha - \left\lfloor \frac{\alpha + 1}{2} \right\rfloor$$

and

$$R_\alpha(x) = x^\alpha F_B \left(-\beta, -\alpha + \beta - \kappa + \frac{1}{2}; -|\alpha| - \rho + 1; \frac{\|x\|^2}{x_1^2}, \dots, \frac{\|x\|^2}{x_{d+1}^2} \right).$$

Proof. By considering m being even or odd, it is easy to verify that

$$c_\kappa \int_{-1}^1 t^{m-2k} (1+t)(1-t^2)^{\kappa-1} dt = \frac{(\frac{1}{2})_{[\frac{m+1}{2}]}}{(\kappa + \frac{1}{2})_{[\frac{m+1}{2}]}} \frac{(-[\frac{m+1}{2}] - \kappa + \frac{1}{2})_k}{(-[\frac{m+1}{2}] + \frac{1}{2})_k}$$

for $\kappa \geq 0$. Hence, using the explicit formula of V_κ , the formula of \tilde{R}_α in (1) of Proposition 2.2 becomes,

$$\begin{aligned} \tilde{R}_\alpha(x) &= \frac{2^{|\alpha|}(\rho)_{|\alpha|}}{\alpha!} \frac{(1/2)_{[\frac{\alpha+1}{2}]}}{(\kappa + 1/2)_{[\frac{\alpha+1}{2}]}} \\ &\times \sum_\gamma \frac{(-\alpha/2)_\gamma ((-\alpha + 1)/2)_\gamma}{(-|\alpha| - \rho + 1)_{|\gamma|} \gamma!} \frac{(-[(\alpha + 1)/2] - \kappa + 1/2)_\gamma}{(-[(\alpha + 1)/2] + 1/2)_\gamma} \|x\|^{2|\gamma|} x^{\alpha-2\gamma}. \end{aligned}$$

Using the fact that

$$\left(-\frac{\alpha}{2} \right)_\gamma \left(\frac{-\alpha + 1}{2} \right)_\gamma = \left(-\alpha + \left\lfloor \frac{\alpha + 1}{2} \right\rfloor \right)_\gamma \left(-\left\lfloor \frac{\alpha + 1}{2} \right\rfloor + \frac{1}{2} \right)_\gamma,$$

the above expression of \tilde{R}_α can be written in terms of F_B as stated. \square

Note that the F_B function in the proposition is a finite series, since $(-n)_m = 0$ if $m > n$. In particular, this shows that $R_\alpha(x)$ is the monomial orthogonal polynomial of the form $R_\alpha(x) = x^\alpha - \|x\|^2 Q_\alpha(x)$, where $Q_\alpha \in \mathcal{P}_{n-2}^d$.

Another generalization of the hypergeometric series ${}_2F_1$ to several variables is the Lauricella function of type A, defined by (cf. [10])

$$F_A(c, \alpha; \beta; x) = \sum_\gamma \frac{(c)_{|\gamma|} (\alpha)_\gamma}{(\beta)_{|\gamma|} \gamma!} x^\gamma, \quad \alpha, \beta \in \mathbb{N}_0^{d+1}, \quad c \in \mathbb{R},$$

where the summation is taken over $\gamma \in \mathbb{N}_0^{d+1}$. If all components of α are even, then we can write R_α using F_A .

Proposition 2.4. *Let $\beta \in \mathbb{N}_0^{d+1}$. Then*

$$R_{2\beta}(x) = (-1)^{|\beta|} \frac{(\kappa + \mathbf{1}/2)_\beta}{(n + \rho)_{|\beta|}} \|x\|^{2|\beta|} F_A \left(-\beta, |\beta| + \rho; \kappa + \frac{1}{2}; \frac{x_1^2}{\|x\|^2}, \dots, \frac{x_{d+1}^2}{\|x\|^2} \right).$$

Proof. For $\alpha = 2\beta$ the formula in terms of F_B becomes

$$R_{2\beta}(x) = \sum_{\gamma \leq \beta} \frac{(-\beta)_\gamma (-\beta - \kappa + \mathbf{1}/2)_\gamma}{(-2|\beta| - \rho + 1)_{|\gamma|} \gamma!} \|x\|^{2|\gamma|} x^{2\beta - 2\gamma},$$

where $\gamma \leq \beta$ means $\gamma_1 < \beta_1, \dots, \gamma_{d+1} < \beta_{d+1}$; note that $(-\beta)_\gamma = 0$ if $\gamma > \beta$. Changing the summation index by $\gamma_i \mapsto \beta_i - \gamma_i$ and using the formula $(a)_{n-m} = (-1)^m (a)_n / (1 - n - a)_m$ to rewrite the Pochhammer symbols, for example,

$$(\kappa + \mathbf{1}/2)_{\beta - \gamma} = \frac{(-1)^\gamma (\kappa + \mathbf{1}/2)_\beta}{(-\beta - \kappa + \mathbf{1}/2)_\gamma}, \quad (\beta - \gamma)! = (\mathbf{1})_{\beta - \gamma} = \frac{(-1)^{|\gamma|} \beta!}{(-\beta)_\gamma},$$

we can rewrite the summation into the stated formula in F_A . \square

Let $\text{proj}_n : \mathcal{P}_n^{d+1} \mapsto \mathcal{H}_n^{d+1}(h_\kappa^2)$ denote the projection operator of polynomials in \mathcal{P}_n^{d+1} onto $\mathcal{H}_n^{d+1}(h_\kappa^2)$. It follows that R_α is the orthogonal projection of the monomial x^α . Recall that \mathcal{D}_i is the Dunkl operator defined in (2.1). We define $\mathcal{D}^\alpha = \mathcal{D}_1^{\alpha_1} \dots \mathcal{D}_{d+1}^{\alpha_{d+1}}$ for $\alpha \in \mathbb{N}_0^{d+1}$. Let $e_i = (0, \dots, 0, 1, 0, \dots, 0)$ denote the standard basis of \mathbb{R}^{d+1} .

Proposition 2.5. *The polynomials R_α satisfy the following properties:*

(1) $R_\alpha(x) = \text{proj}_n x^\alpha$, $n = |\alpha|$, and

$$R_\alpha(x) = \frac{(-1)^n}{2^n (\rho)_n} \|x\|^{2\rho + 2n} \mathcal{D}^\alpha (\|x\|^{-2\rho}), \quad \rho = |\kappa| + \frac{d-1}{2}.$$

(2) R_α satisfies the relation

$$\|x\|^2 \mathcal{D}_i R_\alpha(x) = -2(n + \rho) [R_{\alpha + e_i}(x) - x_i R_\alpha(x)].$$

(3) The set $\{R_\alpha : |\alpha| = n, \alpha_{d+1} = 0, 1\}$ is a basis of $\mathcal{H}_n^{d+1}(h_\kappa^2)$.

Proof. Since $R_\alpha \in \mathcal{H}_n^{d+1}(h_\kappa^2)$ and $R_\alpha(x) = x^\alpha - \|x\|^2 Q(x)$, where $Q \in \mathcal{P}_{n-2}^{d+1}$, it follows that $R_\alpha(x) = \text{proj}_n x^\alpha$. On the other hand, it is shown in [19] that the polynomials H_α , defined by

$$H_\alpha(x) = \|x\|^{2\rho + 2n} \mathcal{D}^\alpha \|x\|^{-2\rho},$$

satisfy the relation $H_\alpha(x) = (-1)^n 2^n (\rho)_n \text{proj}_{|\alpha|} x^\alpha$, from which the explicit formula in the part (1) follows. The polynomials H_α satisfy the recursive relation

$$H_{\alpha + e_i}(x) = -(2|\alpha| + 2\rho) x_i H_\alpha(x) + \|x\|^2 \mathcal{D}_i H_\alpha(x),$$

which gives the relation in part (2). Finally, it is proved in [19] that $\{H_\alpha : |\alpha| = n, \alpha_{d+1} = 0, 1\}$ is a basis of $\mathcal{H}_n^{d+1}(h_\kappa^2)$. \square

In the case of $\alpha = ne_i$, R_α takes a simple form. Indeed, let $C_n^{(\lambda, \mu)}(t)$ denote the generalized Gegenbauer polynomials defined by

$$C_n^{(\lambda, \mu)}(x) = c_\mu \int_{-1}^1 C_n^\lambda(xt) (1+t)(1-t^2)^{\mu-1} dt.$$

These polynomials are orthogonal with respect to the weight function $w_{\lambda, \mu}(t) = |t|^{2\mu} (1-t^2)^{\lambda-1/2}$ on $[-1, 1]$ and they become Gegenbauer polynomials when $\mu = 0$

(use (2.3)); that is, $C_n^{(\lambda,0)}(t) = C_n^\lambda(t)$. In terms of the Jacobi polynomials $P_n^{(a,b)}(t)$, the generalized Gegenbauer polynomials can be written as

$$(2.5) \quad \begin{aligned} C_{2n}^{(\lambda,\mu)}(x) &= \frac{(\lambda + \mu)_n}{(\mu + \frac{1}{2})_n} P_n^{(\lambda-1/2, \mu-1/2)}(2x^2 - 1), \\ C_{2n+1}^{(\lambda,\mu)}(x) &= \frac{(\lambda + \mu)_{n+1}}{(\mu + \frac{1}{2})_{n+1}} x P_n^{(\lambda-1/2, \mu+1/2)}(2x^2 - 1). \end{aligned}$$

Recall that the Jacobi polynomial $P_n^{(a,b)}(t)$ can be written as a ${}_2F_1$ function

$$(2.6) \quad P_n^{(a,b)}(t) = \frac{(a+1)_n}{n!} {}_2F_1\left(-n, n+a+b+1; a+1; \frac{1-t}{2}\right).$$

For $\alpha = ne_i$, the F_B formula of R_α becomes a single sum since $(-m)_j = 0$ if $m < j$, which can be written in terms of ${}_2F_1$. For example, if $n = 2m+1$, then

$$\begin{aligned} R_{(2m+1)e_1}(x) &= \sum_{j=0}^m \frac{(-m)_j (-m - \kappa_1 - 1/2)_j}{(-2m-1-\rho+1)_j j!} \|x\|^{2j} x_1^{2m-2j+1} \\ &= x_1^{2m+1} {}_2F_1\left(-m, -m - \kappa_1 - 1/2; -2m-1-\rho+1; \frac{\|x\|^2}{x_1^2}\right). \end{aligned}$$

This can be written in terms of Jacobi polynomials (2.6), upon changing the summation index by $j \mapsto m-j$, and further in the generalized Gegenbauer polynomials using (2.5). The result is

Corollary 2.6. *Let $n \in \mathbb{N}_0$. Then $R_{ne_i}(x) = \text{proj}_n x_i^n$ satisfies*

$$R_{ne_i}(x) = \|x\|^n \left[k_n^{(\rho-\kappa_i, \kappa_i)} \right]^{-1} C_n^{(\rho-\kappa_i, \kappa_i)}(x_i \|x\|),$$

where $k_n^{(\lambda,\mu)}$ denote the leading coefficient of $C_n^{(\lambda,\mu)}(t)$ given by

$$(2.7) \quad k_{2n}^{(\lambda,\mu)} = \frac{(\lambda + \mu)_{2n}}{(\mu + \frac{1}{2})_n n!} \quad \text{and} \quad k_{2n+1}^{(\lambda,\mu)} = \frac{(\lambda + \mu)_{2n+1}}{(\mu + \frac{1}{2})_{n+1} n!}.$$

In the case of ordinary harmonics, that is, $\kappa_i = 0$, the polynomials R_{ne_i} are given in terms of the Gegenbauer polynomials.

2.2. Monomial orthogonal polynomials on the unit ball. The h -spherical harmonics associated to (1.1) are closely related to orthogonal polynomials associated to the weight functions W_κ^B in (1.2). In fact, if $Y \in \mathcal{H}_n^{d+1}(h_\kappa^2)$ is an h -harmonic associated with $h_\kappa(y) = \prod_{i=1}^{d+1} |y_i|^{\kappa_i}$ that is even in its $(d+1)$ -th variable, $Y(y', y_{d+1}) = Y(y', -y_{d+1})$, then the polynomial P_α defined by

$$(2.8) \quad Y(y) = r^n P(x), \quad y = r(x, x_{d+1}), \quad r = \|y\|, \quad (x, x_{d+1}) \in S^d,$$

is an orthogonal polynomials with respect to W_κ^B . Moreover, this defines an one-to-one correspondence between the two sets of polynomials ([17]).

Working with polynomials on B^d , the monomials are x^α with $\alpha \in \mathbb{N}_0^d$, instead of \mathbb{N}_0^{d+1} . Since $x_{d+1}^2 = 1 - \|x\|^2$ for $(x, x_{d+1}) \in S^d$, we only consider R_α in Definition 2.1 with $\alpha = (\alpha_1, \dots, \alpha_d, 0)$. The correspondence (2.8) leads to the following definition:

Definition 2.7. Let $\rho = |\kappa| + \frac{d-1}{2}$. Define polynomials $\tilde{R}_\alpha^B(x)$, $\alpha \in \mathbb{N}_0^d$, by

$$\begin{aligned} c_\kappa \int_{[-1,1]^d} \frac{1}{(1 - 2(b_1 x_1 t_1 + \dots + b_d x_d t_d) + \|b\|^2)^\rho} \prod_{i=1}^d (1 + t_i)(1 - t_i^2)^{\kappa_i - 1} dt \\ = \sum_{\alpha \in \mathbb{N}_0^d} b^\alpha \tilde{R}_\alpha^B(x), \quad x \in B^d. \end{aligned}$$

The polynomials \tilde{R}_α^B form a basis of the subspace of orthogonal polynomials of degree n with respect to W_κ^B . It is given by the explicit formula

Proposition 2.8. Let $\rho = |\kappa| + (d-1)/2$. For $\alpha \in \mathbb{N}_0^d$ and $x \in \mathbb{R}^d$,

$$\tilde{R}_\alpha^B(x) = \frac{2^{|\alpha|}(\rho)_{|\alpha|}}{\alpha!} \frac{(1/2)_{\alpha-\beta}}{(\kappa + 1/2)_{\alpha-\beta}} R_\alpha^B(x), \quad \text{where } \beta = \alpha - \left\lfloor \frac{\alpha + \mathbf{1}}{2} \right\rfloor,$$

and

$$R_\alpha(x) = x^\alpha F_B \left(-\beta, -\alpha + \beta - \kappa + \frac{\mathbf{1}}{2}; -|\alpha| - \rho + 1; \frac{1}{x_1^2}, \dots, \frac{1}{x_d^2} \right).$$

In particular, $R_\alpha^B(x) = x^\alpha - Q_\alpha(x)$, $Q_\alpha \in \Pi_{n-1}^d$, is the monomial orthogonal polynomial with respect to W_κ^B on B^d .

Proof. Setting $b_{d+1} = 0$ and $\|x\| = 1$ in the generating function (2.1) shows that the generating function of \tilde{R}_α^B is the same as the one for $\tilde{R}_{(\beta,0)}(x)$. Consequently, $\tilde{R}_\alpha^B(x) = \tilde{R}_{(\alpha,0)}(x, x_{d+1})$ for $(x, x_{d+1}) \in S^d$. Since $\tilde{R}_{(\alpha,0)}(x, x_{d+1})$ is even in its $d+1$ variable, the correspondence (2.8) shows that \tilde{R}_α^B is orthogonal and its properties can be derived from those of \tilde{R}_α . \square

In particular, if $\kappa_i = 0$ for $i = 1, \dots, d$ and $\kappa_{d+1} = \mu$ so that W_κ^B becomes the classical weight function $(1 - \|x\|^2)^{\mu-1/2}$, then the limit relation (2.3) shows that the generating function becomes simply

$$(1 - 2\langle b, x \rangle + \|b\|^2)^{-\mu-(d-1)/2} = \sum_{\alpha \in \mathbb{N}_0^d} b^\alpha R_\alpha^B(x), \quad x \in \mathbb{R}^d.$$

This is the generating function of one family of Appell's biorthogonal polynomials and $R_\alpha^B(x)$ is usually denoted by $V_\alpha(x)$ in the literature (see, for example, [8, Vol. II, Chapt 12] or [7, Chapt 2]).

The definition of R_α^B comes from that of h -harmonics $R_{(\alpha,0)}$ by the correspondence. If we consider R_β with $\beta = (\alpha, \alpha_{d+1})$ and assume that α_{d+1} is an even integer, then R_β leads to the orthogonal projection of the polynomial $x^\alpha(1 - \|x\|^2)^{\alpha_{d+1}/2}$ with respect to W_κ^B on B^d . Furthermore, the correspondence also gives a generating function of these projections.

2.3. Monomial orthogonal polynomials on the simplex. The h -spherical harmonics associated to (1.1) are also related to orthogonal polynomials associated to the weight functions W_κ^T in (1.3). If $Y \in \mathcal{H}_{2n}^{d+1}(h_\kappa^2)$ is an h -harmonic that is even in each of its variables, then Y can be written as

$$(2.9) \quad Y(y) = r^n P(x_1^2, \dots, x_d^2), \quad y = r(x_1, \dots, x_d, x_{d+1}), \quad r = \|y\|.$$

The polynomial $P(x)$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ is an orthogonal polynomial of degree n in d variables with respect to W_κ^T on T^d . Moreover, this defines an one-to-one correspondence between the two sets of polynomials ([18]).

Since the simplex T has a natural symmetry in terms of $(x_1, \dots, x_d, x_{d+1})$, $x_{d+1} = 1 - |x|$, we use the homogeneous coordinates $X := (x_1, \dots, x_d, x_{d+1})$. For the monomial h -harmonics defined in Definition 2.1, the polynomial $R_{2\alpha}$ is even in each of its variables, which corresponds to, under (2.9), monomial orthogonal polynomials R_α^T in $\mathcal{V}_n^d(W_\kappa^T)$ in the homogeneous coordinates X . This leads to the following definition

Definition 2.9. Let $\rho = |\kappa| + \frac{d-1}{2}$. Define polynomials $\tilde{R}_\alpha^T(x)$, $\alpha \in \mathbb{N}_0^d$, by

$$\begin{aligned} c_\kappa \int_{[-1,1]^{d+1}} \frac{1}{(1 - 2(b_1 x_1 t_1 + \dots + b_{d+1} x_{d+1} t_{d+1}) + \|b\|^2)^\rho} \prod_{i=1}^{d+1} (1 - t_i^2)^{\kappa_i - 1} dt \\ = \sum_{\alpha \in \mathbb{N}_0^{d+1}} b^{2\alpha} \tilde{R}_\alpha^T(x), \quad x \in T^d, \quad x_{d+1} = 1 - |x|. \end{aligned}$$

The main properties of R_α^T are summarized in the following proposition.

Proposition 2.10. For each $\alpha \in \mathbb{N}_0^{d+1}$ with $|\alpha| = n$, the polynomials

$$\tilde{R}_\alpha^T(x) = \frac{2^{2|\alpha|}(\rho)_{2|\alpha|}}{(2\alpha)!} \frac{(1/2)_\alpha}{(\kappa + 1/2)_\alpha} R_\alpha^T(x),$$

where

$$\begin{aligned} R_\alpha^T(x) &= X^\alpha F_B\left(-\alpha, -\alpha - \kappa + \frac{1}{2}; -2|\alpha| - \rho + 1; \frac{1}{x_1}, \dots, \frac{1}{x_{d+1}}\right) \\ &= (-1)^n \frac{(\kappa + 1)_\alpha}{(n + |\kappa| + d)_n} F_A(|\alpha| + |\kappa| + d, -\alpha; \kappa + 1; X) \end{aligned}$$

are orthogonal polynomials with respect to W_κ^T on the simplex T^d . Moreover, $R_\alpha^T(x) = X^\alpha - Q_\alpha(x)$, where Q_α is a polynomial of degree at most $n - 1$, and $\{R_\alpha^T, \alpha = (\alpha', 0), |\alpha| = n\}$ is a basis for the subspace of orthogonal polynomials of degree n .

Proof. We go back to the generating function of h -harmonics in Definition 2.1. The explicit formula of R_α shows that $R_\alpha(x)$ is even in each of its variables only if each α_i is even for $i = 1, \dots, d + 1$. Let $\varepsilon \in \{-1, 1\}^{d+1}$. Then $R_\alpha(x\varepsilon) = R_\alpha(\varepsilon_1 x_1, \dots, \varepsilon_{d+1} x_{d+1}) = \varepsilon^\alpha R_\alpha(x)$. It follows that

$$\sum_{\beta \in \mathbb{N}_0^{d+1}} b^{2\beta} R_{2\beta}(x) = \frac{1}{2^{d+1}} \sum_{\alpha \in \mathbb{N}_0^{d+1}} b^\alpha \sum_{\varepsilon \in \{-1, 1\}^{d+1}} R_\alpha(x\varepsilon).$$

On the other hand, using the explicit formula of V_κ , the generating function gives

$$\begin{aligned} \frac{1}{2^{d+1}} \sum_{\varepsilon \in \{-1, 1\}^{d+1}} \sum_{\alpha \in \mathbb{N}_0^{d+1}} b^\alpha R_\alpha(x\varepsilon) = \\ c_\kappa \int_{[-1,1]^{d+1}} \sum_{\varepsilon \in \{-1, 1\}^{d+1}} \frac{\prod_{i=1}^{d+1} (1 + t_i)(1 - t_i^2)^{\kappa_i - 1}}{(1 - 2(b_1 x_1 t_1 \varepsilon_1 + \dots + b_{d+1} x_{d+1} t_{d+1} \varepsilon_{d+1}) + \|b\|^2)^\rho} dt \end{aligned}$$

for $\|x\| = 1$. Changing variables $t_i \mapsto t_i \varepsilon_i$, the fact that $\sum_{\varepsilon} \prod_{i=1}^{d+1} (1 + \varepsilon_i t_i) = 2^{d+1}$ shows that the generating function of $R_{2\beta}(x)$ agrees with the generating function of $R_\beta^T(x_1^2, \dots, x_{d+1}^2)$ in Definition 2.9. Consequently, the formulae of R_α^T follow from the corresponding ones for $R_{2\alpha}$. The polynomial R_α^T is homogeneous in X . Using

the correspondence (2.9) between orthogonal polynomials on S^d and on T^d , we see that R_α^T are orthogonal with respect to W_κ^T . If $\alpha_{d+1} = 0$, then $R_\alpha^T(x) = x^\alpha - Q_\alpha$, which proves the last statement of the proposition. \square

In the case of $\alpha_{d+1} = 0$, the explicit formula of R_α^T shows that $R_{(\alpha,0)}^T(x) = x^\alpha - Q_\alpha(x)$; setting $b_{d+1} = 0$ in Definition 2.9 gives the generating function of $R_{(\alpha,0)}^T$. The explicit formula of $R_{(\alpha,0)}^T$ can be found in [7], which appeared earlier in the literature in some special cases. The generating function of R_α^T appears to be new in all cases. We note that if all $\kappa_i = 0$, then the integrals in the Definition 2.9 disappear, so that the generating function in the case of the Chebyshev weight function $W^T(x) = (x_1 \dots x_d(1 - |x|))^{-1/2}$ is simply $(1 - 2\langle b, x \rangle + \|b\|^2)^{-1}$.

3. SYMMETRIC MONOMIAL ORTHOGONAL POLYNOMIALS

Let \mathcal{S}_{d+1} denote the symmetric group of $d+1$ objects. For a permutation $w \in \mathcal{S}_{d+1}$ we write $xw = (x_{w(1)}, \dots, x_{w(d+1)})$ and define $T(w)f(x) = f(xw)$. If $T(w)f = f$ for all $w \in \mathcal{S}_{d+1}$, we say that f is invariant under \mathcal{S}_{d+1} .

For $\alpha \in \mathbb{N}_0^d$ and $w \in \mathcal{S}_{d+1}$, we define the action of w on α by $(\alpha w)_i = \alpha_{w^{-1}(i)}$. Using this definition we have $(xw)^\alpha = x^{\alpha w}$.

3.1. Symmetric monomial h -harmonics. In this section, we assume that $\kappa_1 = \dots = \kappa_{d+1} = \kappa$. Then the weight function h_κ in (1.1) is invariant under \mathcal{S}_{d+1} . Let $\mathcal{H}_n^{d+1}(h_\kappa^2, \mathcal{S})$ denote the subspace of h -harmonics in $\mathcal{H}_n^{d+1}(h_\kappa^2)$ invariant under the group \mathcal{S}_{d+1} . Our goal is to give an explicit basis for $\mathcal{H}_n^{d+1}(h_\kappa^2, \mathcal{S})$.

A partition λ of $d+1$ parts is an element in \mathbb{N}_0^{d+1} such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{d+1}$. Let Ω_n^{d+1} denote the set of partitions of $d+1$ parts. Let

$$\Omega_n^{d+1} = \{\lambda \in \mathbb{N}_0^{d+1} : \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{d+1}, |\lambda| = n\},$$

the set of $d+1$ parts partitions of size n , and let $\Lambda_n^{d+1} = \{\lambda \in \Omega_n : \lambda_1 = \lambda_2\}$. For a partition λ the monomial symmetric polynomial m_λ is defined by ([14])

$$m_\lambda(x) = \sum \{x^\alpha : \alpha \text{ being distinct permutations of } \lambda\}.$$

Let \mathcal{B}_{d+1} denote the hyperoctahedral group, which is a semi-direct product of \mathbb{Z}_2^{d+1} and \mathcal{S}_{d+1} . A function f is invariant under \mathcal{B}_{d+1} if $f(x) = g(x_1^2, \dots, x_{d+1}^2)$ and g is invariant under \mathcal{S}_{d+1} . Since the weight function $h_\kappa(x) = \prod_{i=1}^{d+1} |x_i|^\kappa$ is invariant under \mathcal{B}_{d+1} , the monomials x^α and x^β are automatically orthogonal whenever α and β are of different parity. Hence, closely related to $\mathcal{H}_n^{d+1}(h_\kappa^2, \mathcal{S})$ is the space $\mathcal{H}_n^{d+1}(h_\kappa^2, \mathcal{B})$, the subspace of h -harmonics in $\mathcal{H}_n^{d+1}(h_\kappa^2)$ invariant under \mathcal{B}_{d+1} . Recently Dunkl [6] gave an explicit basis for $\mathcal{H}_n^{d+1}(h_\kappa^2, \mathcal{B})$ in the form of

$$(3.1) \quad p_\lambda(x) = m_\lambda(x^2) + \sum \{c_\mu m_\mu(x^2) : \mu \in \Omega_n^{d+1}, \mu_i \leq \lambda_i, 2 \leq i \leq d+1, \mu \neq \lambda\},$$

where $x^2 = (x_1^2, \dots, x_{d+1}^2)$ and the coefficients c_μ were determined explicitly, and proved that the set $\{p_\lambda : \lambda \in \Lambda_n^{d+1}\}$ is a basis of $\mathcal{H}_n^{d+1}(h_\kappa^2, \mathcal{B})$.

Using the explicit formula of R_α we give a basis for $\mathcal{H}_n^{d+1}(h_\kappa^2, \mathcal{S})$ in this section. Let $\mathcal{S}_{d+1}(\lambda)$ denote the stabilizer of λ , $\mathcal{S}_{d+1}(\lambda) = \{w \in \mathcal{S}_{d+1} : \lambda w = \lambda\}$. Then we can write $m_\lambda = \sum x^{\lambda w}$ with the summation over all coset representatives of the subgroup $\mathcal{S}_{d+1}(\lambda)$ of \mathcal{S}_{d+1} , which we denote by $\mathcal{S}_{d+1}/\mathcal{S}_{d+1}(\lambda)$, it contains all w such that $\lambda_i = \lambda_j$ and $i < j$ implies $w(i) < w(j)$.

Definition 3.1. Let $\lambda \in \Omega^{d+1}$. Define

$$S_\lambda(x) = \sum_{w \in \mathcal{S}_{d+1}/\mathcal{S}_{d+1}(\lambda)} R_{\lambda w}(x).$$

Proposition 3.2. For $\lambda \in \Omega_n^{d+1}$, the polynomial $S_\lambda = \text{proj}_n m_\lambda$ is an element of $\mathcal{H}_n^{d+1}(h_\kappa^2; \mathcal{S})$. Moreover, the set $\{S_\lambda : \lambda \in \Lambda_n^{d+1}\}$ is a basis of $\mathcal{H}_n^{d+1}(h_\kappa^2; \mathcal{S})$.

Proof. The definition of S_λ and the fact that $R_\alpha(x) = x^\alpha + \|x\|^2 Q_\alpha(x)$ shows

$$S_\lambda(x) = \sum_{w \in \mathcal{S}_{d+1}/\mathcal{S}_{d+1}(\lambda)} (x^{\lambda w} + \|x\|^2 Q_{\lambda w}(x)) = m_\lambda(x) + \|x\|^2 Q(x),$$

where $Q \in \Pi_{n-1}^{d+1}$. Also $S_\lambda \in \mathcal{H}_n^{d+1}(h_\kappa^2)$ since each R_α does. Hence, $S_\lambda = \text{proj}_n m_\lambda$. It follows from the part (2) of Proposition 2.5 that ,

$$S_\lambda(x) = \frac{(-1)^n}{2^n(\lambda)_n} \|x\|^{2\rho+2n} m_\lambda(\mathcal{D})(\|x\|^{-2\rho}),$$

which shows that S_λ is symmetric. Since $\dim \mathcal{H}_n^{d+1}(h_\kappa^2; \mathcal{S}) = \#\Lambda_n^{d+1}$, we see that $\{S_\lambda : \lambda \in \Lambda_n^{d+1}\}$ is a basis of $\mathcal{H}_n^{d+1}(h_\kappa^2; \mathcal{S})$. \square

The fact that S_λ is a symmetric polynomial also follows from a general statement about the best approximation by polynomials, proved in [1] for $L^p(S^d)$ and the proof carries over to the case $L^p(S^d; h_\kappa^2)$. Since the proof is short, we repeat it here. Let

$$\|f\|_p = \left(c'_h \int_{S^d} |f(y)|^p h_\kappa^2(y) d\omega(y) \right)^{1/p},$$

for $1 \leq p < \infty$ and let $\|f\|_\infty$ be the uniform norm on S^d .

Proposition 3.3. If f is invariant under \mathcal{S}_{d+1} then the best approximation of f in the space $L^p(S^d; h_\kappa^2)$ by polynomials of degree less than n is attained by symmetric polynomials.

Proof. Let $P \in \Pi_{n-1}^{d+1}$. Since $\kappa_1 = \dots = \kappa_{d+1}$, h_κ is invariant under the symmetric group, and so is the norms of the space $L^p(S^d; h_\kappa^2)$. Hence the triangle inequality and the fact that f is symmetric gives

$$\begin{aligned} \|f - P\|_p &= \frac{1}{(d+1)!} \sum_{w \in \mathcal{S}_{d+1}} \|f(xw) - P(xw)\|_p \\ &\geq \frac{1}{(d+1)!} \left\| \sum_{w \in \mathcal{S}_{d+1}} f(xw) - \sum_{w \in \mathcal{S}_{d+1}} P(xw) \right\|_p = \|f - P^*\|_p, \end{aligned}$$

where P^* is the symmetrization of P . Since $P^* \in \Pi_n^{d+1}$, this shows that the best approximation of f can be attained by symmetric polynomials of the same degree. \square

The best approximation in $L^2(S^d; h_\kappa^2)$ by polynomials is unique, so that a best approximation polynomial to a symmetric function must be a symmetric polynomial. Thus, the above proposition applies to S_λ , as $S_\lambda - m_\lambda$ is the best approximation to m_λ in $L^2(S^d; h_\kappa^2)$ by polynomials of lower degrees.

From the definition of S_λ , it is not immediately clear that S_λ is symmetric. Next we give an explicit formula of S_λ in terms of monomial symmetric functions and powers of $\|x\|$.

We start with the following simple observation:

Lemma 3.4. *Let $w \in \mathcal{S}_{d+1}$. Then $R_\alpha(xw) = R_{\alpha w}(x)$.*

Proof. This follows from the generating function of $R_\alpha(x)$. Indeed, let $\Phi_\kappa(t) = \prod_{i=1}^{d+1} c_k(1+t_i)(1-t_i^2)^{k-1}$; then $\Phi_\kappa(t)$ is invariant under \mathcal{S}_{d+1} . Hence, using the explicit formula of V_κ in (2.2), it follows from the Definition 2.1 that

$$\begin{aligned} \sum b^\alpha \tilde{R}_\alpha(xw) &= \int_{[-1,1]^{d+1}} \frac{1}{(1 - 2 \sum b_i(xw)_i t_i + \|b\|^2 \|x\|^2)^\rho} \Phi_\kappa(t) dt \\ &= \int_{[-1,1]^{d+1}} \frac{1}{(1 - 2 \sum (bw^{-1})_i x_i t_i + \|b\|^2 \|x\|^2)^\rho} \Phi_\kappa(t) dt \\ &= \sum (bw^{-1})^\alpha \tilde{R}_\alpha(x) = \sum b^{\alpha w^{-1}} \tilde{R}_\alpha(x) = \sum b^\alpha \tilde{R}_{\alpha w}(x), \end{aligned}$$

since the sum is over all $\alpha \in \mathbb{N}_0^{d+1}$. \square

We need one more definition. For any $\alpha \in \mathbb{N}_0^{d+1}$, let α^+ be the unique partition such that $\alpha^+ = \alpha w$ for some $w \in \mathcal{S}_{d+1}$.

Proposition 3.5. *Let $\lambda \in \Omega_n^{d+1}$ and let $\rho = (d+1)\kappa + (d-1)/2$. Then*

$$S_\lambda(x) = m_\lambda(\mathbf{1}) \sum_{2\gamma \leq \lambda} a_{\lambda, \gamma} \|x\|^{2|\gamma|} \frac{m_{(\lambda-2\gamma)^+}(x)}{m_{(\lambda-2\gamma)^+}(\mathbf{1})}, \quad x \in \mathbb{R}^{d+1},$$

where

$$a_{\lambda, \gamma} = \frac{(-\lambda + [(\lambda + \mathbf{1})/2])_\gamma (-[(\lambda + \mathbf{1})/2] - \kappa + \mathbf{1}/2)_\gamma}{(-|\lambda| - \rho + 1)_{|\gamma|} \gamma!}.$$

Proof. Let $d_\lambda = |\mathcal{S}_{d+1}(\lambda)|$. We can write $m_\lambda(x) = d_\lambda^{-1} \sum_{w \in \mathcal{S}_{d+1}} x^{\lambda w}$ and, using Lemma 3.4,

$$S_\lambda(x) = d_\lambda^{-1} \sum_{w \in \mathcal{S}_{d+1}} R_{\lambda w}(x) = d_\lambda^{-1} \sum_{w \in \mathcal{S}_{d+1}} R_\lambda(xw).$$

The coefficients $a_{\lambda, \gamma}$ appear in the explicit formula of R_λ . Indeed, from the formula in Proposition 2.3, $R_\lambda(x) = \sum a_{\lambda, \gamma} \|x\|^{2|\gamma|} x^{\lambda-2\gamma}$. For $w \in \mathcal{S}_{d+1}$ and $\lambda, \gamma \in \mathbb{N}^{d+1}$, we have $(\lambda w)_\gamma = (\lambda)_{\gamma w^{-1}}$. Therefore, as $|\alpha w| = |\alpha|$ for $\alpha \in \mathbb{N}_0^{d+1}$, it follows from the formula of $a_{\lambda, \gamma}$ that $a_{\lambda w, \gamma} = a_{\lambda, \gamma w^{-1}}$. Consequently,

$$\begin{aligned} \sum_{w \in \mathcal{S}_{d+1}} R_\lambda(xw) &= \sum_{w \in \mathcal{S}_{d+1}} \sum_{\gamma} a_{\lambda w, \gamma} \|x\|^{2|\gamma|} x^{\lambda w - 2\gamma} \\ &= \sum_{w \in \mathcal{S}_{d+1}} \sum_{\gamma} a_{\lambda, \gamma w^{-1}} \|x\|^{2|\gamma|} x^{\lambda w - 2\gamma w^{-1} w} \\ &= \sum_{\gamma} a_{\lambda, \gamma} \|x\|^{2|\gamma|} \sum_{w \in \mathcal{S}_{d+1}} x^{(\lambda-2\gamma)w}, \end{aligned}$$

since the summation is over all $\gamma \in \mathbb{N}_0^d$. Note that the coefficients $a_{\lambda, \gamma} = 0$ if $\gamma_i > \lambda_i - [(\lambda_i + 1)/2]$, so that $\lambda_i - 2\gamma_i \geq 0$. Therefore, we can write

$$\sum_{w \in \mathcal{S}_{d+1}} x^{(\lambda-2\gamma)w} = \sum_{w \in \mathcal{S}_{d+1}} x^{(\lambda-2\gamma)^+ w} = d_{(\lambda-2\gamma)^+} m_{(\lambda-2\gamma)^+}(x).$$

Put these formulae together, we get

$$S_\lambda(x) = d_\lambda^{-1} \sum_{\gamma} a_{\lambda, \gamma} \|x\|^{2|\gamma|} d_{(\lambda-2\gamma)^+} m_{(\lambda-2\gamma)^+}(x),$$

which gives the stated formula upon using the fact that $m_\lambda(\mathbf{1}) = (d+1)!/d_\lambda$. \square

In the simplest case of $\lambda = (n, 0, \dots, 0) = ne_1$, we conclude that

$$(3.2) \quad \begin{aligned} S_{ne_1}(x) &= \sum_j \frac{(-n + [\frac{n+1}{2}])_j (-[\frac{n+1}{2}] - \kappa + \frac{1}{2})_j}{(-n - \rho + 1)_j j!} \|x\|^{2j} m_{(n-2j)e_1}(x) \\ &= \|x\|^n \sum_{i=1}^{d+1} \left[k_n^{(\rho-\kappa, \kappa)} \right]^{-1} C_n^{(\rho-\kappa, \kappa)}(x_i/\|x\|), \end{aligned}$$

where $\rho = (d+1)\kappa + (d-1)/2$, and the second equality follows from the definition of $C_n^{(\lambda, \mu)}$ or from Corollary 2.6.

Since the sum in the formula of S_λ is over all $\gamma \in \mathbb{N}_0^{d+1}$, some m_μ may appear several times in the sum. With a little more effort one may write S_λ in a more compact form. Evidently this depends on how many parts of λ are repeated. We shall consider only a simple case of $\lambda = (q, \dots, q)$, in which all parts are equal.

Corollary 3.6. *For $\lambda = (q, \dots, q)$, $q \in \mathbb{N}_0$,*

$$S_\lambda(x) = \sum_{\mu \in \Omega^{d+1}} a_{\lambda, \mu} \|x\|^{2|\gamma|} m_{\lambda-2\mu}(x).$$

Proof. In this case, it is easy to see that $a_{\lambda, \mu w} = a_{\lambda, \mu}$ for each $w \in \mathcal{S}_{d+1}$. Moreover, $d_\lambda = (d+1)!$. Consequently, using the fact that $\sum_\gamma c_\gamma = \sum_{\mu \in \Omega^{d+1}} d_\mu^{-1} \sum_{w \in \mathcal{S}_{d+1}} c_{\mu w}$, it follows that

$$\begin{aligned} S_\lambda(x) &= \frac{1}{(d+1)!} \sum_\gamma a_{\lambda, \gamma} \|x\|^{2|\gamma|} d_{(\lambda-2\gamma)^+} m_{(\lambda-2\gamma)^+}(x) \\ &= \frac{1}{(d+1)!} \sum_{\mu \in \Omega^{d+1}} d_\mu^{-1} \sum_{w \in \mathcal{S}_{d+1}} a_{\lambda, \mu w} \|x\|^{2|\mu|} d_{(\lambda-2\mu w)^+} m_{(\lambda-2\mu w)^+}(x) \\ &= \sum_{\mu \in \Omega^{d+1}} a_{\lambda, \mu} \|x\|^{2|\mu|} \frac{1}{(d+1)!} \sum_{w \in \mathcal{S}_{d+1}} m_{(\lambda-2\mu w)^+}(x), \end{aligned}$$

since $\lambda = (q, \dots, q)$ implies that $d_{(\lambda-2\mu w)^+} = d_{\mu w} = d_\mu$. Also, the special form of λ implies $m_{(\lambda-2\mu w)^+} = m_{\lambda-2(\mu w)^+} = m_{\lambda-2\mu}$, which completes the proof. \square

Since $m_{2\lambda}(x) = m_\lambda(x_1^2, \dots, x_{d+1}^2)$, the theorem shows that the set $\{S_{2\lambda} : \lambda \in \Lambda_n^{d+1}\}$ is a basis for the space $\mathcal{H}_n^{d+1}(h_\kappa^2; \mathcal{B})$. These results are interesting even in the case of the ordinary harmonics ($\kappa = 0$). The only other symmetric orthogonal basis known is given by Dunkl [6] recently for $\mathcal{H}_n^{d+1}(h_\kappa^2; \mathcal{B})$. It should be pointed out, however, that S_λ are not mutually orthogonal for $\lambda \in \Omega_n^{d+1}$. We do not know how to construct an orthonormal basis for $\mathcal{H}_n^{d+1}(h_\kappa^2; \mathcal{S})$ or if there is a compact formula for the L^2 norm of S_λ .

Since $\|x\|^2$ is symmetric, one can write $\|x\|^2 m_\mu$ in terms of symmetric monomial polynomials m_σ so that $S_{2\lambda}$ can be written in terms of $m_\mu(x^2)$ as in Dunkl's basis (3.1). It turns out, however, that the two bases $\{S_{2\lambda} : \lambda \in \Lambda_n^{d+1}\}$ and $\{p_\lambda : \lambda \in \Lambda_n^{d+1}\}$ are quite different and they are in fact biorthogonal ([6]).

3.2. Symmetric monomial orthogonal polynomials on the unit ball. On the unit ball B^d we consider the weight function $W_\kappa^B(x)$ with $\kappa_1 = \dots = \kappa_d = 0$. Writing $\kappa_{d+1} = \mu$, we write $W_{\kappa, \mu}^B$ instead of W_κ^B . That is,

$$W_{\kappa, \mu}^B(x) = \prod_{i=1}^d |x_i|^{2\kappa} (1 - \|x\|^2)^{\mu-1/2}, \quad x \in B^d.$$

This weight function is evidently invariant under the symmetric group \mathcal{S}_d . Let $\mathcal{V}_n^d(W_{\kappa,\mu}^B; \mathcal{S})$ denote the space of symmetric orthogonal polynomials of degree n with respect to $W_{\kappa,\mu}^B$. The dimension of this space is $\dim \mathcal{V}_n^d(W_{\kappa,\mu}^B; \mathcal{S}) = \#\Omega_n^d$, the cardinality of d -parts partitions of size n , since a basis can be obtained by applying Gram-Schmidt process on a basis of symmetric polynomials of degree at most n in d variables.

For symmetric orthogonal polynomials we cannot use the correspondence (2.8) between h -harmonics and orthogonal polynomials on the unit ball, since $R_\alpha^B(x) = R_{(\alpha,0)}(x, x_{d+1})$. On the other hand, the polynomial R_α^B in Definition 2.7 is similar to R_α^B specified in Definition 2.1. The similarity allows us to carry out the study in the previous subsection with little additional effort. We define S_λ^B as in Definition 3.1:

Definition 3.7. Let $\lambda \in \Omega^d$. Define

$$S_\lambda^B(x) = \sum_{w \in \mathcal{S}_{d+1}/\mathcal{S}_d(\lambda)} R_{\lambda w}^B(x).$$

Proposition 3.8. For $\lambda \in \Omega_n^d$, the polynomial S_λ is the orthogonal projection of the symmetric monomial polynomial m_λ onto $\mathcal{V}_n^d(W_{\kappa,\mu}^B; \mathcal{S})$. Moreover, the set $\{S_\lambda : \lambda \in \Omega_n^d\}$ is a basis of $\mathcal{V}_n^d(W_{\kappa,\mu}^B; \mathcal{S})$.

We again have $R_{\alpha w}^B(x) = R_\alpha^B(xw)$ for any $w \in \mathcal{S}_d$ and we can derive an explicit formula of S_λ^B as in Proposition 3.5:

Proposition 3.9. Let $\lambda \in \Omega_n^d$ and let $\rho = d\kappa + \mu + (d-1)/2$. Then

$$S_\lambda^B(x) = m_\lambda(\mathbf{1}) \sum_{2\gamma \leq \lambda} a_{\lambda,\gamma} \frac{m_{(\lambda-2\gamma)^+}(x)}{m_{(\lambda-2\gamma)^+}(\mathbf{1})}, \quad x \in \mathbb{R}^d,$$

where $\mathbf{1} = (1, \dots, 1) \in \mathbb{N}_0^d$ and

$$a_{\lambda,\gamma} = \frac{(-\lambda + [(\lambda + \mathbf{1})/2])_\gamma (-[(\lambda + \mathbf{1})/2] - \kappa + \mathbf{1}/2)_\gamma}{(-|\lambda| - \rho + 1)_{|\gamma|} \gamma!}.$$

In the simplest case of $\lambda = (n, 0, \dots, 0) = ne_1 \in \mathbb{R}^d$, we conclude that

$$\begin{aligned} S_{ne_1}^B(x) &= \sum_j \frac{(-n + [\frac{n+1}{2}])_j (-[\frac{n+1}{2}] - \kappa_1 + \frac{1}{2})_j}{(-n - \rho + 1)_j j!} m_{(n-2j)e_1}(x) \\ &= \sum_{i=1}^d \left[k_n^{(\rho-\kappa_1, \kappa_1)} \right]^{-1} C_n^{(\rho-\kappa_1, \kappa_1)}(x_1), \end{aligned}$$

where $\rho = d\kappa + \mu + (d-1)/2$, since $R_{ne_1}^B(x) = R_{ne_1'}(x, x_{d+1})$ for $(x, x_{d+1}) \in S^d$, where $e_1' = (1, 0, \dots, 0) = (e_1, 0) \in \mathbb{R}^{d+1}$, and Corollary 2.6 shows that $R_{ne_i}^B(x) = [k_n^{(\rho-\kappa_i, \kappa_i)}]^{-1} C_n^{(\rho-\kappa_i, \kappa_i)}(x_i)$.

3.3. Symmetric monomial orthogonal polynomials on the simplex. We can also give explicit formulae for the symmetric monomial orthogonal polynomials with respect to W_κ^T on the simplex.

On the simplex T^d , it is natural to consider the symmetric group \mathcal{S}_{d+1} of the vertices of T^d . A function $f(x)$ on T^d is symmetric if in the homogeneous coordinates $X = (x, x_{d+1})$, $x_{d+1} = 1 - |x|$, $f(x) = g(X)$ is invariant under \mathcal{S}_{d+1} ; that

is, if $g(Xw) = g(X)$ for every $w \in \mathcal{S}_{d+1}$. Let $\mathcal{V}_n^{d+1}(W_\kappa^T; S)$ denote the space of orthogonal polynomials of degree n that are symmetric.

Proposition 3.10. *For each $\lambda \in \Omega_n^{d+1}$, the polynomial*

$$S_\lambda^T(x) = \sum_{w \in \mathcal{S}_{d+1}/\mathcal{S}_{d+1}(\lambda)} R_{\lambda w}^T(x)$$

is a symmetric orthogonal polynomial and $S_\lambda^T(x) = m_\lambda(X) + Q(x)$, $Q \in \Pi_{n-1}^{d+1}$. Moreover,

$$S_\lambda^T(x) = m_\lambda(\mathbf{1}) \sum_{\gamma} \frac{(-\lambda)_\gamma (-\lambda - \kappa + \mathbf{1}/2)_\gamma}{(-2|\lambda| - \rho + 1)_{|\gamma|} \gamma!} \cdot \frac{m_{(\lambda-\gamma)+}(X)}{m_{(\lambda-\gamma)+}(\mathbf{1})},$$

where $\mathbf{1} = (1, \dots, 1) \in \mathbb{N}_0^{d+1}$. Furthermore, the set $\{S_\lambda^T : \lambda \in \Lambda_n^{d+1}\}$ is a basis of $\mathcal{V}_n^{d+1}(W_\kappa^T; S)$.

This follows from the correspondence (2.9) and the properties of $S_{2\lambda}$. Notice that $m_{2\lambda}(x) = m_\lambda(x_1^2, \dots, x_{d+1}^2)$.

4. NORM OF THE MONOMIAL POLYNOMIALS

4.1. Norm of monomial h -harmonics. Since R_α is orthogonal to polynomials in Π_{n-1}^{d+1} with respect to $h_\kappa^2 d\omega$ on S^d and $R_\alpha(x) - x^\alpha$ is a polynomial of lower degree when restricted to S^d , the standard Hilbert space theory shows that the polynomial R_α is the best approximation of x^α in the L^2 norm defined by

$$\|f\|_2 = \left(c'_h \int_{S^d} |f(y)|^2 h_\kappa^2(y) d\omega(y) \right)^{1/2},$$

where c'_h is the normalization constant of h_κ^2 . In other words, the polynomial $x^\alpha - R_\alpha$ has the smallest L^2 norm among all polynomials of the form $x^\alpha - P(x)$, $P \in \Pi_{n-1}^{d+1}$ on S^d . That is,

$$\|R_\alpha\|_2 = \min_{P \in \Pi_{n-1}^{d+1}} \|x^\alpha - P\|_2, \quad |\alpha| = n.$$

In the following we compute the L^2 norm of R_α .

Theorem 4.1. *Let $\rho = |\kappa| + \frac{d-1}{2}$. Let $\alpha \in \mathbb{N}_0^{d+1}$ and denote $\beta = \alpha - [(\alpha + \mathbf{1})/2]$. Then*

$$\begin{aligned} c'_h \int_{S^d} |R_\alpha(x)|^2 h_\kappa^2(x) d\omega &= \frac{\rho (\kappa + \mathbf{1}/2)_\alpha}{(\rho)_{|\alpha|}} \sum_{\gamma} \frac{(-\beta)_\gamma (-\alpha + \beta - \kappa + \mathbf{1}/2)_\gamma}{(-\alpha - \kappa + \mathbf{1}/2)_\gamma \gamma! (|\alpha| - |\gamma| + \rho)} \\ &= 2\rho \frac{\beta! (\kappa + \mathbf{1}/2)_{\alpha-\beta}}{(\rho)_{|\alpha|}} \int_0^1 \prod_{i=1}^{d+1} C_{\alpha_i}^{(\frac{1}{2}, \kappa_i)}(t) t^{|\alpha|+2\rho-1} dt. \end{aligned}$$

Proof. Using the explicit formula of $R_\alpha(x)$ and the Beta type integral,

$$c'_h \int_{S^d} x^{2\sigma} h_\kappa^2(x) d\omega = \frac{\Gamma(|\kappa| + (d+1)/2)}{\Gamma(|\sigma| + |\kappa| + (d+1)/2)} \prod_{i=1}^{d+1} \frac{\Gamma(\sigma_i + \kappa_i + 1/2)}{\Gamma(\kappa_i + 1/2)} = \frac{(\kappa + \mathbf{1}/2)_\sigma}{(\rho + \mathbf{1})_{|\sigma|}},$$

it follows from the explicit formula of R_α in Proposition 2.3 that

$$\begin{aligned} c'_h \int_{S^d} |R_\alpha(x)|^2 h_\kappa^2(x) d\omega &= c'_h \int_{S^d} R_\alpha(x) x^\alpha h_\kappa^2(x) d\omega \\ &= \sum_{\gamma} \frac{(-\beta)_\gamma (-\alpha + \beta - \kappa + \mathbf{1}/2)_\gamma (\kappa + \mathbf{1}/2)_{\alpha-\gamma}}{(-|\alpha| - \rho + 1)_{|\gamma|} \gamma! (\rho + \mathbf{1})_{|\alpha|-|\gamma|}}. \end{aligned}$$

Rewriting the sum using $(a)_{n-m} = (-1)^m (a)_n / (1-n-a)_m$ and $(-a)_n / (-a+1)_n = a/(a-n)$ gives the first stated equation. To derive the second equation, we show that the sum in the first equation can be written as an integral. We define a function

$$F(r) = \sum_{\gamma \leq \beta} \frac{(-\beta)_\gamma (-\alpha + \beta - \kappa + \mathbf{1}/2)_\gamma}{(-\alpha - \kappa + \mathbf{1}/2)_\gamma \gamma! (|\alpha| - |\gamma| + \rho)} r^{|\alpha|-|\gamma|+\rho}.$$

Evidently, $F(1)$ is the sum in the first equation. Moreover, the sum is a finite sum over $\gamma \leq \beta$ as $(-\beta)_\gamma = 0$ for $\gamma > \beta$, it follows that $F(0) = 0$. Hence, the sum $F(1)$ is given by $F(1) = \int_0^1 F'(r) dr$. The derivative of F can be written as

$$\begin{aligned} F'(r) &= \sum_{\gamma} \frac{(-\beta)_\gamma (-\alpha + \beta - \kappa + \mathbf{1}/2)_\gamma}{(-\alpha - \kappa + \mathbf{1}/2)_\gamma \gamma!} r^{|\alpha|-|\gamma|+\rho-1} \\ &= r^{|\alpha|+\rho-1} \prod_{i=1}^{d+1} \sum_{\gamma_i} \frac{(-\beta_i)_{\gamma_i} (-\alpha_i + \beta_i - \kappa_i + \mathbf{1}/2)_{\gamma_i}}{(-\alpha_i - \kappa_i + \mathbf{1}/2)_{\gamma_i} \gamma_i!} r^{-\gamma_i} \\ &= r^{|\alpha|+\rho-1} \prod_{i=1}^{d+1} {}_2F_1 \left(\begin{matrix} -\beta_i, -\alpha_i + \beta_i - \kappa_i + \mathbf{1}/2 \\ -\alpha_i - \kappa_i + \mathbf{1}/2 \end{matrix}; \frac{1}{r} \right) \end{aligned}$$

The Jacobi polynomial $P_n^{(a,b)}$ can be written as ${}_2F_1$ in a different form [15, (4.22.1)],

$$P_n^{(a,b)}(t) = \binom{2n+a+b}{n} \left(\frac{t-1}{2} \right)^n {}_2F_1 \left(\begin{matrix} -n, -n-a \\ -2n-a-b \end{matrix}; \frac{2}{1-t} \right).$$

Use this formula with $n = \beta_i$, $a = \alpha_i - 2\beta_i + \kappa_i - 1/2$, $b = 0$ and $r = (1-t)/2$, and then use $P_n^{(a,b)}(t) = (-1)^n P_n^{(b,a)}(-t)$, we conclude

$$F'(r) = \frac{(\kappa + \mathbf{1}/2)_{\alpha-\beta} \beta!}{(\kappa + \mathbf{1}/2)_\alpha} r^{|\alpha|-|\beta|+\rho-1} \prod_{i=1}^{d+1} P_{\beta_i}^{(0, \alpha_i-2\beta_i+\kappa_i-1/2)}(2r-1).$$

Consequently, it follows that

$$(4.1) \quad F(1) = \frac{(\kappa + \mathbf{1}/2)_{\alpha-\beta} \beta!}{(\kappa + \mathbf{1}/2)_\alpha} \int_0^1 \prod_{i=1}^{d+1} P_{\beta_i}^{(0, \alpha_i-2\beta_i+\kappa_i-1/2)}(2r-1) r^{|\alpha|-|\beta|+\rho-1} dr.$$

From the relation (2.5) it follows that $P_{\beta_i}^{(0, \alpha_i-2\beta_i+\kappa_i-1/2)}(2t^2-1) = C_\alpha^{(1/2, \kappa_i)}(t)$ if α_i is even, and $t P_{\beta_i}^{(0, \alpha_i-2\beta_i+\kappa_i-1/2)}(2t^2-1) = C_\alpha^{(1/2, \kappa_i)}(t)$ if α_i is odd. Hence, changing variables $r \rightarrow t^2$ in the above integral leads to the second stated equation. \square

The constant in the second equal sign can be written in terms of $k_n^{(1/2, \kappa_i)}$, the leading coefficient of $C_n^{(1/2, \kappa_i)}$, by using (2.7) and considering α_i being even and odd separately. As an equivalent statement, the theorem gives

Corollary 4.2. *Let $\alpha \in \mathbb{N}_0^d$ and $n = |\alpha|$. Then*

$$\inf_{Q \in \Pi_{n-1}^d} \|x^\alpha - Q(x)\|_2^2 = \frac{2\rho(\kappa + \frac{1}{2})_\alpha}{(\rho)_{|\alpha|}} \int_0^1 \prod_{i=1}^{d+1} \frac{C_{\alpha_i}^{(\frac{1}{2}, \kappa_i)}(t)}{k_{\alpha_i}^{(\frac{1}{2}, \kappa_i)}} t^{|\alpha|+2\rho-1} dt.$$

In the case of $d = 1$, the integral contains the product of two Jacobi polynomials. Moreover, the parameters satisfy a condition for which the integral can be written as a terminating ${}_3F_2$ and simplified by the known formula (see [9, Vol. 2, p. 286])

$$\begin{aligned} & \int_0^1 P_{\beta_1}^{(0, \sigma_1)}(2r-1) P_{\beta_2}^{(0, \sigma_2)}(2r-1) r^{|\beta|+|\sigma|} dr \\ &= \frac{(|\beta|)! (|\sigma|+1)_{|\beta|} (\sigma_1+1)_{|\beta|} (\sigma_2+1)_{|\beta|}}{(|\sigma|+2)_{2|\beta|} (\sigma_1+1)_{\beta_1} (\sigma_2+1)_{\beta_2} (|\sigma|+2|\beta|+1)}. \end{aligned}$$

Using this formula with an obvious choice of the parameters, the norm of R_α for $d = 1$ can be written in a compact form. Equivalently, this gives

Corollary 4.3. *Let $\alpha = (\alpha_1, \alpha_2)$ and write $\sigma = [(\alpha+1)/2]$. Then*

$$\inf_{Q \in \Pi_{n-1}^2} \|x^\alpha - Q(x)\|_2^2 = \frac{(|\kappa|)_{|\sigma|} (|\alpha| - |\sigma|)!}{(|\kappa|+1)_{|\alpha|} (|\kappa|)_{|\alpha|}} \left(\kappa_1 + \frac{1}{2} \right)_{\sigma_1 + \alpha_2 - \sigma_2} \left(\kappa_2 + \frac{1}{2} \right)_{\sigma_2 + \alpha_1 - \sigma_1}.$$

For $d > 1$ and $\alpha = ne_i$, the sum in Theorem 4.1 is a balanced ${}_3F_2$, which can be summed using the Saalschütz summation formula. Alternatively, we can evaluate the norm of R_{ne_i} by using the explicit formula of R_{ne_i} in Corollary 2.6 and the formula

$$(4.2) \quad \int_{S^d} f(x) d\omega_d = \int_0^\pi \int_{S^{d-1}} f(\cos \theta, \sin \theta x') d\omega_{d-1}(x') (\sin \theta)^{d-1} d\theta.$$

This way, the norm of R_{ne_i} can be derived from the leading coefficient $k_n^{(\lambda, \mu)}$, given in (2.7), of $C_n^{(\lambda, \mu)}$ and the norm of $C_n^{(\lambda, \mu)}(t)$. We denote by $h_n^{(\lambda, \mu)}$ the L^2 norm of $C_n^{(\lambda, \mu)}$ with respect to the normalized weight function $c_{\lambda, \mu} w_{\lambda, \mu}(t)$, where $w_{\lambda, \mu}(t) = |t|^{2\mu} (1-t^2)^{\lambda-1/2}$ and $c_{\lambda, \mu}^{-1} = \Gamma(\mu+1/2)\Gamma(\lambda+1/2)/\Gamma(\lambda+\mu+1)$. It is given by ([7, p. 27])

$$(4.3) \quad \begin{aligned} h_{2m}^{(\lambda, \mu)} &= \frac{(\lambda + \frac{1}{2})_m (\lambda + \mu)_m (\lambda + \mu)}{m! (\mu + \frac{1}{2})_m (\lambda + \mu + 2m)}, \\ h_{2m+1}^{(\lambda, \mu)} &= \frac{(\lambda + \frac{1}{2})_m (\lambda + \mu)_{m+1} (\lambda + \mu)}{m! (\mu + \frac{1}{2})_{m+1} (\lambda + \mu + 2m + 1)}. \end{aligned}$$

We will follow the second approach to evaluate the norm of R_{ne_i} since an intermediate result will be used later in the section.

Corollary 4.4. *For $n \in \mathbb{N}_0^d$, let $m = [(n+1)/2]$. Then*

$$\inf_{Q \in \Pi_{n-1}^d} \|x_i^n - Q(x)\|_2^2 = \frac{(n-m)! (\kappa_i + \frac{1}{2})_n (|\kappa| - \kappa_i + \frac{d}{2})_{n-m}}{(|\kappa| + \frac{d+1}{2})_n (m + |\kappa| + \frac{d-1}{2})_{n-m} (m + \kappa_i + \frac{1}{2})_{n-m}}.$$

Proof. We only need to prove the case $i = 1$. For $x \in S^d$, write $x = (\cos \theta x', \sin \theta)$, $x' \in S^{d-1}$. Let $\lambda_1 = \rho - \kappa_1 = |\kappa| - \kappa_1 + (d-1)/2$. Using the explicit formula of

R_{ne_i} in Corollary 2.6, the equation (4.2) with a change of variable $t = \cos \theta$ shows that

$$\begin{aligned} c'_h \int_{S^d} |R_{ne_1}(t)|^2 dt &= c'_h \int_{-1}^1 \left| \frac{C_n^{(\lambda_1, \kappa_1)}(t)}{k_n^{(\lambda_1, \kappa_1)}} \right|^2 w_{\lambda_1, \kappa_1}(t) dt \int_{S^{d-1}} \prod_{i=2}^{d+1} |x'_i|^{2\kappa_i} d\omega_{d-1}(x') \\ &= h_n^{(\lambda_1, \kappa_1)} / [k_n^{(\lambda_1, \kappa_1)}]^2. \end{aligned}$$

Hence, the stated formula follows from the explicit formulae of $k_n^{(\lambda, \kappa)}$ in (2.7) and $h_n^{(\lambda, \kappa)}$ in (4.3). \square

We note that Corollary 4.2 and the above proof implies the formula

$$\frac{2\rho(\kappa_1 + \frac{1}{2})_n}{(\rho)_n} \int_0^1 \frac{C_n^{(\frac{1}{2}, \kappa_1)}(t)}{k_n^{(\frac{1}{2}, \kappa_1)}} t^{|\alpha|+2\rho-1} dt = c_{\lambda_1, \mu_1} \int_{-1}^1 \left| \frac{C_n^{(\lambda_1, \kappa_1)}(t)}{k_n^{(\lambda_1, \kappa_1)}} \right|^2 w_{\lambda_1, \kappa_1}(t) dt,$$

which does not seem to follow from a simple transformation. This suggests the possibility that the norm of R_α may be expressed in some other, perhaps more illuminating, ways.

In general, however, the norm of R_α may not have a compact formula in the form of a ratio of products of Pochhammer symbols. For example, if $\alpha = (\alpha_1, \alpha_2, 0, \dots, 0)$, then the integral in Theorem 4.1 becomes (see (4.1))

$$(4.4) \quad I(\sigma, \beta) := \int_0^1 P_{\beta_1}^{(0, \sigma_1)}(2r-1) P_{\beta_2}^{(0, \sigma_1)}(2r-1) r^{\sigma_1 + \sigma_2 + \beta_1 + \beta_2 + a} dr$$

with $\sigma_i = \alpha_i - 2\beta_i + \kappa_i - 1/2$, $\beta_i = \alpha_i - [(\alpha_i + 1)/2]$ and $a = |\kappa| - \kappa_1 - \kappa_2 + (d-1)/2$. Using the ${}_2F_1$ formula of the Jacobi polynomials, this integral can be written as a single sum of a balanced ${}_4F_3$ series evaluated at 1,

$$(4.5) \quad I(\sigma, \beta) = \frac{(-1)^{\beta_1} (\sigma_1 + 1)_{\beta_1} (\sigma_1 + a + 1)_{|\beta|}}{\beta_1! (|\beta| + |\sigma| + a + 2)_{|\beta|} (\beta_2 (\sigma_1 + a + 1)_{\beta_1}} \\ \times {}_4F_3 \left(\begin{matrix} -\beta_1, \beta_1 + \sigma_1 + 1, |\beta| + |\sigma| + a + 1, |\beta| + \sigma_1 + a + 1 \\ |\sigma| + |\beta| + \beta_2 + a + 2, \sigma_1 + 1, \sigma_1 + \beta_1 + a + 1 \end{matrix}; 1 \right).$$

This ${}_4F_3$ is a finite sum, but it does not seem to have a compact form.

As a consequence of Theorem (4.1), the integral of the product generalized Gegenbauer polynomials in the theorem is positive, which does not seem to be obvious. It shows, in particular, that the expression $I(\sigma, \beta)$ is positive if $\sigma_i \geq 0$, $\alpha_i \geq 0$ and $a \geq 0$.

For the symmetric orthogonal polynomials, there is one simple case for which we can compute the norm explicitly, the norm of the symmetric polynomials S_{ne_1} in (3.2). Recall that by Corollary 2.6 and (3.2), $S_{ne_1} = R_{ne_1} + \dots + R_{ne_{d+1}}$ when $\kappa_i = \kappa$ for $1 \leq i \leq d+1$, and R_{ne_i} is given in terms of $C_n^{(\rho - \kappa_i, \kappa_i)}(t)$. The key ingredient is the lemma below.

Lemma 4.5. *Let $\lambda_i = |\kappa| - \kappa_i + \frac{d-1}{2}$. Then for $n = 2m$,*

$$c'_h \int_{S^d} \frac{C_n^{(\lambda_1, \kappa_1)}(x_1) C_n^{(\lambda_2, \kappa_2)}(x_2)}{k_n^{(\lambda_1, \kappa_1)} k_n^{(\lambda_2, \kappa_2)}} h_\kappa^2(x) d\omega = (-1)^m \frac{(\kappa_2 + \frac{1}{2})_m}{(\lambda_1 + \frac{1}{2})_m} \frac{h_n^{(\lambda_1, \kappa_1)}}{[k_n^{(\lambda_1, \kappa_1)}]^2}.$$

Proof. For $x \in S^d$, write $x = (\cos \theta, \sin \theta x')$, $x' \in S^{d-1}$ and $0 \leq \theta \leq \pi$. Using the integration formula (4.2) and changing variable $t = \cos \theta$, we see that the left hand side of the stated integral is equal to

$$c'_h \int_{-1}^1 \frac{C_n^{(\lambda_1, \kappa_1)}(t)}{k_n^{(\lambda_1, \kappa_1)}} \left[\int_{S^{d-1}} \frac{C_n^{(\lambda_2, \kappa_2)}(\sqrt{1-t^2}x')}{k_n^{(\lambda_2, \kappa_2)}} \prod_{i=2}^{d+1} |x'_i|^{2\kappa_i} d\omega_{d-1}(x') \right] \\ \times |t|^{2\kappa_1} (1-t^2)^{\lambda_1 - \frac{1}{2}}.$$

Since $n = 2m$, the integral inside the square bracket is a polynomial of degree n in t whose leading term is $(1-t^2)^m = (-1)^m t^{2m} + \dots$. Consequently, by the orthogonality of $C_n^{(\lambda_1, \kappa_1)}(t)$, it follows that the above integral is equal to

$$(-1)^m c'_h \int_{-1}^1 \frac{C_n^{(\lambda_1, \kappa_1)}(t)}{k_n^{(\lambda_1, \kappa_1)}} t^{2m} |t|^{2\kappa_1} (1-t^2)^{\lambda_1 - \frac{1}{2}} dt \int_{S^{d-1}} x_2'^{2m} \prod_{i=2}^{d+1} |x'_i|^{2\kappa_i} d\omega_{d-1}(x') \\ = (-1)^m \frac{(\kappa_2 + \frac{1}{2})_m}{(\lambda_1 + \frac{1}{2})_m} \frac{1}{[k_n^{(\lambda_1, \kappa_1)}]^2} c_{\lambda_1, \kappa_1} \int_{-1}^1 |C_n^{(\lambda_1, \kappa_1)}(t)|^2 w_{\lambda_1, \kappa_1}(t) dt$$

using (2.4), which gives the stated formula. \square

Together with the proof of Corollary 4.4, this lemma allows us to compute the norm of any linear combination $b_1 R_{ne_1} + \dots + b_{d+1} R_{ne_{d+1}}$, without the need of assuming $\kappa_i = \kappa$ for all i . We shall, however, use it only in the case of $\kappa_1 = \dots = \kappa_{d+1}$ to compute the norm of S_{ne_1} . The proof shows clearly how the norm of the general case can be computed.

Proposition 4.6. *Let $\kappa_1 = \dots = \kappa_{d+1} = \kappa$. Let $\lambda = d\kappa + \frac{d-1}{2}$. Then for $n = 2m$,*

$$\inf_{Q \in \Pi_{n-1}^{d+1}} \|x_1^n + \dots + x_{d+1}^n - Q(x)\|_2^2 = (d+1) \left(1 + d(-1)^m \frac{(\kappa + \frac{1}{2})_m}{(\lambda + \frac{1}{2})_m} \right) \\ \times \frac{(\lambda + \frac{1}{2})_m (\lambda + \kappa)_m (\kappa + \frac{1}{2})_m m!}{(\lambda + \kappa)_{2m} (\lambda + \kappa + 1)_{2m}};$$

and for $n = 2m + 1$,

$$\inf_{Q \in \Pi_{n-1}^{d+1}} \|x_1^n + \dots + x_{d+1}^n - Q(x)\|_2^2 = (d+1) \frac{(\lambda + \frac{1}{2})_m (\lambda + \kappa)_{m+1} (\kappa + \frac{1}{2})_{m+1} m!}{(\lambda + \kappa)_{2m+1} (\lambda + \kappa + 1)_{2m+1}}.$$

In particular, the case $\kappa = 0$ and $\lambda = (d-1)/2 > 0$ gives the best approximation of S_{ne_1} in the L^2 norm with respect to the surface measure $d\omega$.

Proof. Since $|\kappa| = (d+1)\kappa$, λ_i in the lemma becomes $\lambda = d\kappa + (d-1)/2$. By (3.2), for $\|x\| = 1$, we have

$$\int_{S^d} [S_{ne_1}(x)]^2 h_\kappa^2(x) d\omega = \int_{S^d} \left[\sum_{i=1}^{d+1} [k_n^{(\lambda, \kappa)}]^{-1} C_n^{(\lambda, \kappa)}(x_i) \right]^2 h_\kappa^2(x) d\omega \\ = \sum_{i=1}^{d+1} \int_{S^d} \left[\frac{C_n^{(\lambda, \kappa)}(x_i)}{k_n^{(\lambda, \kappa)}} \right]^2 h_\kappa^2(x) d\omega + \sum_{i \neq j} \int_{S^d} \frac{C_n^{(\lambda, \kappa)}(x_i) C_n^{(\lambda, \kappa)}(x_j)}{k_n^{(\lambda, \kappa)} k_n^{(\lambda, \kappa)}} h_\kappa^2(x) d\omega.$$

If $n = 2m + 1$, then the integrals in the second sum is zero since $C_{2m+1}^{(\lambda, \mu)}(t)$ is an odd polynomial. Hence, since h_κ is invariant under the symmetric group, it follows

$$c'_h \int_{S^d} [S_{ne_1}(x)]^2 h_\kappa^2(x) d\omega = (d+1) c'_h \int_{S^d} \left[\frac{C_n^{(\lambda, \kappa)}(x_1)}{k_n^{(\lambda, \kappa)}} \right]^2 h_\kappa^2(x) d\omega = \frac{(d+1) h_n^{(\lambda, \kappa)}}{[k_n^{(\lambda, \kappa)}]^2},$$

as in the proof of Corollary 4.4. If $n = 2m$, then the integrals in the second sum can be evaluate as in Lemma 4.5, so that we get

$$\begin{aligned} c'_h \int_{S^d} [S_{ne_1}(x)]^2 h_\kappa^2(x) d\omega &= (d+1) c'_h \int_{S^d} \left[\frac{C_n^{(\lambda, \kappa)}(x_1)}{k_n^{(\lambda, \kappa)}} \right]^2 h_\kappa^2(x) d\omega \\ &\quad + d(d+1) c'_h \int_{S^d} \left[\frac{C_n^{(\lambda, \kappa)}(x_1) C_n^{(\lambda, \kappa)}(x_2)}{k_n^{(\lambda, \kappa)} k_n^{(\lambda, \kappa)}} \right]^2 h_\kappa^2(x) d\omega \\ &= (d+1) \frac{h_n^{(\lambda, \kappa)}}{[k_n^{(\lambda, \kappa)}]^2} \left(1 + d(-1)^m \frac{(\kappa + \frac{1}{2})_m}{(\lambda + \frac{1}{2})_m} \right). \end{aligned}$$

Using the formulae $k_n^{(\lambda, \kappa)}$ in (2.7) and $h_n^{(\lambda, \kappa)}$ in (4.3) completes the proof. \square

In [1] some invariant polynomials of lower degrees with the least $L^p(S^d; d\omega)$ norm on the sphere are studied. In particular, for the L^2 norm, it is computed there that

$$\inf_{Q \in \Pi_3^m} \|x_1^4 + \dots + x_m^4 - Q(x)\|_2^2 = \frac{24(m-1)}{(m+2)^2(m+4)(m+6)}.$$

This is our general result with $\kappa = 0$, $n = 4$ and $m = d + 1$.

The Proposition 4.6 gives the norm of the symmetric monomial polynomial S_{ne_1} . We do not have a compact formula for the norm of the symmetric monomial orthogonal polynomials in general.

4.2. Norm of monomial polynomials on the ball. For $\alpha \in \mathbb{N}_0^d$, the polynomial $R_\alpha^B(x)$ is related to the best approximation to x^α . Let w_κ^B denote the normalization constant of the weight function W_κ^B in (1.2). Define

$$\|f\|_{2,B} = \left(w_\kappa^B \int_{B^d} |f(x)|^2 W_\kappa^B(x) dx \right)^{1/2}, \quad w_\kappa^B = \frac{\Gamma(|\kappa| + (d+1)/2)}{\prod_{i=1}^{d+1} \Gamma(\kappa_i + 1/2)}.$$

As it is shown in the previous section, for $\alpha \in \mathbb{N}_0^d$, the monomial orthogonal polynomials R_α^B is related to the h -harmonic polynomial $R_{(\alpha,0)}$ by the formula $R_\alpha^B(x) = R_{(\alpha,0)}(x, x_{d+1})$, $(x, x_{d+1}) \in S^d$. Using the formula

$$\int_{S^d} f(y) d\omega = \int_{B^d} \left[f(x, \sqrt{1 - \|x\|^2}) + f(x, -\sqrt{1 - \|x\|^2}) \right] \frac{dx}{\sqrt{1 - \|x\|^2}},$$

the norm of R_α follows from that of $R_{(\alpha,0)}$ right away.

Theorem 4.7. *The polynomial R_α^B has the smallest $\|f\|_{2,B}$ norm among all polynomials of the form $x^\alpha - P(x)$, $P \in \Pi_{n-1}^d$. Furthermore, for $\alpha \in \mathbb{N}_0^d$,*

$$\|R_\alpha^B\|_{2,B}^2 = \|R_{(\alpha,0)}\|_2^2 = \frac{2\rho \prod_{i=1}^d (\kappa_i + \frac{1}{2})_{\alpha_i}}{(\rho)_{|\alpha|}} \int_0^1 \prod_{i=1}^d \frac{C_{\alpha_i}^{(\frac{1}{2}, \kappa_i)}(t)}{k_{\alpha_i}^{(\frac{1}{2}, \kappa_i)}} t^{|\alpha|+2\rho-1} dt.$$

For the classical weight function $W_\mu(x) = (1 - \|x\|^2)^{\mu-1/2}$, the norm of R_α can be expressed as the integral of the product Legendre polynomials $P_n(t) = C_n^{1/2}(t)$. Equivalently, as the best approximation in the L^2 norm, it gives the following:

Corollary 4.8. *Let $\rho = \mu + (d-1)/2 > 0$ and $n = |\alpha|$ for $\alpha \in \mathbb{N}_0^d$. For the classical weight function $W_\mu(x) = (1 - \|x\|^2)^{\mu-1/2}$ on B^d ,*

$$\min_{Q \in \Pi_{n-1}^d} \|x^\alpha - Q(x)\|_{2,B}^2 = \frac{\rho \alpha!}{2^{n-1}(\rho)_n} \int_0^1 \prod_{i=1}^d P_{\alpha_i}(t) t^{n+2\rho-1} dt.$$

Proof. Set $\kappa_i = 0$ for $1 \leq i \leq d$ and $\mu = \kappa_{d+1}$ in the formula of Theorem 4.7. The stated formula follows from $(1)_{2n} = 2^{2n}(1/2)_n(1)_n$, $n! = (1)_n$, and the fact that $C_m^{(1/2,0)}(t) = C_m^{1/2}(t) = P_m(t)$. \square

In particular, for $d = 2$, the product involves only two Legendre polynomials. Since $P_n(t) = P_n^{(0,0)}(t)$, the integral for $d = 2$ can be written as a terminating ${}_4F_3$ series using the formula in (4.4) and (4.5). For the unit weight function on B^2 (that is, $W_{1/2}^B(x) = 1$), another formula of R_α is given in [3], writing it in terms of the basis $\{U_n(\cos(k\pi/(n+1))x_1 + \sin(k\pi/(n+1))x_2) : 0 \leq k \leq n\}$, where U_n denotes the Chebyshev polynomial of the second kind, and the norm of R_α , $|\alpha| = n$, is given as follows in [3],

$$\begin{aligned} \min_{P \in \Pi_{n-1}^d} \int_{B^2} |x^\alpha - P(x)|^2 dx \\ = \frac{n+1}{2^{2n+3}} \int_0^{2\pi} \left(\int_{-1}^1 (\sin \theta - is \cos \theta)^{\alpha_1} (\cos \theta + is \sin \theta)^{\alpha_2} ds \right)^2 d\theta, \end{aligned}$$

in which $\alpha = (\alpha_1, \alpha_2)$ and $i = \sqrt{-1}$. This formula is quite different from the one contained in Corollary 4.8. In fact, it is not all clear how to derive one from the other.

Setting $\alpha = ne_i$ in Theorem 4.7 and using Corollary 4.4, it follows that $\|R_{ne_i}^B\|_{2,B}^2 = h_n^{(\lambda_1, \kappa_1)} / [k_n^{(\lambda_1, \kappa_1)}]^2$. Following the proof of Proposition 4.6 we can also compute the norm of $S_{ne_1}^B$ with respect to $W_{\kappa, \mu}^B$. The result is essentially the same as in Proposition 4.6 with $d+1$ replaced by d , $\kappa_1 = \dots = \kappa_d = \kappa$, $\mu = \kappa_{d+1}$ and $\lambda = \mu + (d-1)\kappa + (d-1)/2$.

4.3. Norm of monomial polynomials on the simplex. In the case of simplex, the polynomials R_α^T is the orthogonal projection of $X^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d} (1 - |x|)^{\alpha_{d+1}}$. Let w_κ^T denote the normalization constant of W_κ^T . Define

$$\|f\|_{2,T} = \left(w_\kappa^T \int_{T^d} |f(x)|^2 W_\kappa^T(x) dx \right)^{1/2}, \quad w_\kappa^T = \frac{\Gamma(|\kappa| + (d+1)/2)}{\prod_{i=1}^{d+1} \Gamma(\kappa_i + 1/2)}.$$

Let $F(x) = f(x_1^2, \dots, x_{d+1}^2)$. Then the norm is related to the norm on S^d via

$$c'_h \int_{S^d} f(x_1^2, \dots, x_{d+1}^2) h_\kappa^2(x) d\omega = w_\kappa^T \int_{T^d} f(x_1, \dots, x_d, 1 - |x|) W_\kappa^T(x) dx.$$

Since $R_\alpha^T(x_1^2, \dots, x_{d+1}^2) = R_{2\alpha}(x_1, \dots, x_{d+1})$, the norm of R_α^T can be derived from the norm of $R_{2\alpha}$. We use (2.5) to write $C_{2\beta_i}^{(1/2, \kappa_i)}(t) = P_{\beta_i}^{(0, \kappa_i-1/2)}(2t^2 - 1)$ and change variable $t^2 \mapsto r$ in the integral in Theorem 4.1 to get the following:

Theorem 4.9. *Let $\beta \in \mathbb{N}_0^{d+1}$ and $\rho = |\kappa| + (d-1)/2$. The polynomial R_β^T has the smallest $\|\cdot\|_{2,T}$ norm among all polynomials of the form $X^\beta - P$, $P \in \Pi_{|\beta|-1}^d$, and the norm is given by*

$$\begin{aligned} w_\kappa^T \int_{T^d} |R_\beta(x)|^2 W_\kappa^T(x) dx &= \frac{\rho(\kappa + \frac{1}{2})_{2\beta}}{(\rho)_{2|\beta|}} \sum_\gamma \frac{(-\beta)_\gamma (-\beta - \kappa + \frac{1}{2})_\gamma}{(-2\beta - \kappa + \frac{1}{2})_\gamma \gamma! (2|\beta| - |\gamma| + \rho)} \\ &= \frac{\rho \beta! (\kappa + \frac{1}{2})_\beta}{(\rho)_{2|\beta|}} \int_0^1 \prod_{i=1}^{d+1} P_{\beta_i}^{(0, \kappa_i - 1/2)}(2r-1) r^{|\beta| + \rho - 1} dr. \end{aligned}$$

In particular, if $\beta_{d+1} = 0$, then the norm of $R_{(\beta,0)}(x)$ is the smallest norm among all polynomials of the form $x^\beta - P$, $P \in \Pi_{n-1}^d$.

Corollary 4.10. *Let $\alpha \in \mathbb{N}_0^d$ and $n = |\alpha|$. Then*

$$\inf_{Q \in \Pi_{n-1}^d} \|x^\alpha - Q(x)\|_{2,T}^2 = \frac{\rho \alpha! \prod_{i=1}^d (\kappa_i + \frac{1}{2})_{\alpha_i}}{(\rho)_{2|\alpha|}} \int_0^1 \prod_{i=1}^d P_{\alpha_i}^{(0, \kappa_i - 1/2)}(2r-1) r^{|\alpha| + \rho - 1} dr.$$

The case $\kappa_i = 1/2$ for $1 \leq i \leq d+1$ corresponds to the unit weight function $W_\kappa^T(x) = 1$, for which the norm is computed by an integral of the product of Legendre polynomials $P_n(t) = P_n^{(0,0)}(t)$. Indeed, setting $\kappa_i = 1/2$ in the above theorem gives $\rho = d$ and the following:

Corollary 4.11. *For $\alpha \in \mathbb{N}_0^d$, $n = |\alpha|$,*

$$\min_{Q \in \Pi_{n-1}^d} \frac{1}{d!} \int_{T^d} |x^\alpha - Q(x)|^2 dx = \frac{d \alpha!^2}{(d)_{2n}} \int_0^1 \prod_{i=1}^d P_{\alpha_i}(2r-1) r^{n+d-1} dr.$$

For $d = 2$, the product involves only two Jacobi polynomials, and its integral can be written using the formula in (4.4) and (4.5) in terms of a terminating ${}_4F_3$ series (setting $\sigma_i = 0$ and $a = 1$).

5. EXPANSION OF R_α IN TERMS OF AN ORTHONORMAL BASIS

The elements of the set $\{R_\alpha : |\alpha| = n, \alpha \in \mathbb{N}_0^{d+1}\}$ are not linearly independent, since the number of elements in the set is greater than the dimension of $\mathcal{H}_n^{d+1}(h_\kappa^2)$. It contains a basis as shown in Proposition 2.5. The basis is not orthonormal, however, since its elements are orthogonal to lower degree polynomials but not among themselves. On the other hand, an orthonormal basis for $\mathcal{H}_n^{d+1}(h_\kappa^2)$ can be given explicitly in terms of the generalized Gegenbauer polynomials $C_n^{(\lambda, \mu)}$. We first state this basis then derive the expansion of R_α in terms of it.

For $d \geq 1$, $\kappa \in \mathbb{R}^{d+1}$ and $\alpha \in \mathbb{N}_0^d$, we introduce the notation:

$$(5.1) \quad \alpha^j = (\alpha_j, \dots, \alpha_d) \quad \text{and} \quad \kappa^j = (\kappa_j, \dots, \kappa_{d+1}), \quad 1 \leq j \leq d+1.$$

Since κ^{d+1} consists of only the last element of κ , write $\kappa^{d+1} = \kappa_{d+1}$. These we treat as elements in \mathbb{N}_0^{d-j+1} and \mathbb{R}^{d-j+2} , respectively, so that the quantities $|\alpha^j|$ and $|\kappa^j|$ are defined as before. Note $|\alpha^{d+1}| = 0$. We also introduce the notation

$$(5.2) \quad a_j := a_j(\alpha, \kappa) = |\alpha^{j+1}| + |\kappa^{j+1}| + \frac{d-j}{2}, \quad 1 \leq j \leq d.$$

Note that for $\alpha \in \mathbb{N}_0^d$ and $\kappa \in \mathbb{R}_+^{d+1}$, $a_d = |\kappa^{d+1}| = \kappa_{d+1}$. Finally, for $x \in \mathbb{R}^{d+1}$, let $r = \|x\|$ and define $r_j = (x_j^2 + \dots + x_{d+1}^2)^{1/2}$ for $1 \leq j \leq d+1$. Notice that $r_1 = r$.

Proposition 5.1. *An orthonormal basis of $\mathcal{H}_n^{d+1}(h_\kappa^2)$ is given by*

$$\tilde{Y}_\alpha(x) = [A_{\alpha,\kappa}]^{-1}Y_\alpha(x;\kappa), \quad \tilde{Y}'_\alpha(x) = [A'_{\alpha,\kappa}]^{-1}Y'_\alpha(x;\kappa),$$

where $\alpha \in \mathbb{N}_0^d$ and $|\alpha| = n$,

$$Y_\alpha(x;\kappa) = \prod_{j=1}^d r_j^{\alpha_j} C_{\alpha_j}^{(a_j, \kappa_j)}(x_j/r_j), \quad Y'_\alpha(x;\kappa) = x_{d+1} Y_{\alpha-e_d}(x;\kappa + e_{d+1}),$$

in which $A'_{\alpha,\kappa} = ((\kappa_{d+1} + 1/2)/(|\kappa| + (d+1)/2))^{1/2} A_{\alpha-e_d, \kappa+e_{d+1}}$ and

$$[A_{\alpha,\kappa}]^2 = \frac{1}{(|\kappa| + \frac{d+1}{2})_n} \prod_{j=1}^d (a_j + \kappa_j)_{\alpha_j} C_{\alpha_j}^{(a_j, \kappa_j)}(1).$$

The formulae given above are a reformulation of the basis given in [16] (also [7, p. 198]), where they are given in spherical coordinates which corresponds to $x_j/r_j = \cos \theta_{d+1-j}$, $1 \leq j \leq d$. The formulae there are given in terms of the normalized generalized Gegenbauer polynomials $\tilde{C}_n^{(\lambda, \mu)}(t) = (1/h_n) C_n^{(\lambda, \mu)}(t)$. The normalization constant h_n is given by $h_n^2 = C_n^{(\lambda, \mu)}(1)(\lambda + \mu)/(n + \lambda + \mu)$ (cf. [7, p. 27]), which is used to rewrite the formulae in [16] to the above form. The fact $a_j(\kappa + e_{d+1}, \alpha - e_d) = a_j(\kappa, \alpha)$, $1 \leq j \leq d-1$, and $a_d(\kappa + e_{d+1}, \alpha - e_d) = a_d(\kappa, \alpha) + 1$ is useful for writing down $A'_{\alpha,\kappa}$.

Since $R_\alpha \in \mathcal{H}_n^{d+1}(h_\kappa^2)$, it can be expanded in terms of the orthonormal basis of Y_α and Y'_α . Below we give this expansion explicitly. To do so, we need the following formula:

Proposition 5.2. *Let $r_j^2 = b_j^2 + \dots + b_{d+1}^2$ and let $\rho = |\kappa| + (d-1)/2$. Then*

$$\begin{aligned} \sum_{|\alpha|=n} b^\alpha \tilde{R}_\alpha(x) &= \sum_{|\nu|=n} \frac{(\rho)_n}{\prod_{j=1}^d (\kappa_j + a_j)_{\nu_j}} Y_\nu(x;\kappa) \prod_{j=1}^d r_j^{\nu_j} \frac{C_{\nu_j}^{(a_j, \kappa_j)}(b_j/r_j)}{C_{\nu_j}^{(a_j, \kappa_j)}(1)} \\ &+ x_{d+1} b_{d+1}^{\kappa_d + \kappa_{d+1}} \sum_{|\nu|=n} \frac{(\rho)_n}{\prod_{j=1}^d (\kappa_j + a_j)_{\nu_j}} Y_{\tilde{\nu}}(x, \tilde{\kappa}) \prod_{j=1}^d r_j^{\tilde{\nu}_j} \frac{C_{\tilde{\nu}_j}^{(\tilde{a}_j, \tilde{\kappa}_j)}(b_j/r_j)}{C_{\tilde{\nu}_j}^{(\tilde{a}_j, \tilde{\kappa}_j)}(1)}, \end{aligned}$$

where $\tilde{\nu} = \nu - e_d$, $a_j = a_j(\nu, \kappa)$ and $\tilde{a}_j = a_j(\nu - e_d, \kappa + e_{d+1})$.

Proof. By the definition of the reproducing kernel, we can write

$$P_n(h_\kappa^2; x, y) = \sum_{|\nu|=n} \left(\tilde{Y}_\nu(x;\kappa) \tilde{Y}_\nu(y;\kappa) + \tilde{Y}'_\nu(x;\kappa) \tilde{Y}'_\nu(y;\kappa) \right).$$

Hence, the second part of Proposition 2.2 shows that

$$\sum_{|\alpha|=n} b^\alpha \tilde{R}_\alpha(x) = \frac{\rho}{n + \rho} \sum_{|\nu|=n} \left(\tilde{Y}_\nu(x;\kappa) \tilde{Y}_\nu(y;\kappa) + \tilde{Y}'_\nu(x;\kappa) \tilde{Y}'_\nu(y;\kappa) \right).$$

Hence, the stated results follows from the explicit formula of $Y_\nu(x;\kappa)$ and $Y'_\nu(x;\kappa)$, $(\rho + 1)_n = (\rho)_n(n + \rho)/\rho$, and checking the constants. \square

This proposition shows that to expand R_α in terms of $Y_\alpha(x;\kappa)$ we essentially have to work out the expansion of $\prod_{j=1}^d r_j^{\nu_j} C_{\nu_j}^{(a_j, \kappa_j)}(b_j/r_j)$ in power of b . Furthermore, the relation in (2.5) shows that

$$C_{2n+1}^{(\lambda, \mu)}(x)/C_{2n+1}^{(\lambda, \mu)}(1) = x C_{2n}^{(\lambda, \mu+1)}(x)/C_{2n}^{(\lambda, \mu+1)}(1).$$

Hence, introducing the notation $\varepsilon(\alpha) = \alpha - 2[\alpha/2]$, or equivalently,

$$\varepsilon(\alpha) = (\varepsilon_1(\alpha), \dots, \varepsilon_{d+1}(\alpha)) \quad \text{with} \quad \varepsilon_i(\alpha) = \begin{cases} 0 & \text{if } \alpha_i \text{ is even} \\ 1 & \text{if } \alpha_i \text{ is odd} \end{cases},$$

we can write for $\nu \in \mathbb{N}_0^d$ and $b \in \mathbb{R}^{d+1}$,

$$(5.3) \quad \prod_{j=1}^d r_j^{\nu_j} \frac{C_{\nu_j}^{(a_j, \kappa_j)}(b_j/r_j)}{C_{\nu_j}^{(a_j, \kappa_j)}(1)} = b^{\varepsilon(\nu^*)} \prod_{j=1}^d r_j^{2[\nu_j/2]} \frac{C_{2[\nu_j/2]}^{(a_j, \kappa_j + \varepsilon_j(\nu))}(b_j/r_j)}{C_{2[\nu_j/2]}^{(a_j, \kappa_j + \varepsilon_j(\nu))}(1)} \\ = b^{\varepsilon(\nu^*)} \prod_{j=1}^d r_j^{2[\nu_j/2]} \frac{P_{[\nu_j/2]}^{(a_j - \frac{1}{2}, \kappa_j + \varepsilon_j(\nu) - \frac{1}{2})}(2b_j^2/r_j^2 - 1)}{P_{[\nu_j/2]}^{(a_j - \frac{1}{2}, \kappa_j + \varepsilon_j(\nu) - \frac{1}{2})}(1)},$$

where $\nu^* = (\nu, 0) \in \mathbb{N}_0^{d+1}$ and $r_j^2 = b_j^2 + \dots + b_{d+1}^2$. Consequently, the problem reduces to find the power expansion of the product of the Jacobi polynomials.

The expansion can be derived using the Hahn polynomials of several variables studied by Karlin and McGregor in [13]. For one variable, the Hahn polynomial $Q(x; a, b, N)$ is defined using the ${}_3F_2$ series by

$$(5.4) \quad Q_n(x; a, b, N) := {}_3\tilde{F}_2 \left(\begin{matrix} -n, n+a+b+1, -x \\ a+1, -N \end{matrix}; 1 \right), \quad n = 0, 1, \dots, N,$$

where ${}_3\tilde{F}_2$ is defined as the usual ${}_3F_2$ with the summation terminating at N . These polynomials are the discrete orthogonal polynomials defined on the set $\{0, 1, \dots, N\}$, which are orthogonal with respect to the binomial distribution, i.e.,

$$\sum_{x=0}^N \frac{(a+1)_x (b+1)_{N-x}}{x!(N-x)!} Q_n(x; a, b, N) Q_m(x; a, b, N) \\ = \frac{(-1)^n n! (b+1)_n (n+a+b+1)_{N+1}}{N! (2n+a+b+1) (-N)_n (a+1)_n} \delta_{n,m}, \quad n, m \leq N.$$

A generating function for the Hahn polynomials of one variable is ([12])

$$(5.5) \quad (1+t)^N \frac{P_j^{(a,b)}(\frac{1-t}{1+t})}{P_j^{(a,b)}(1)} = \sum_{n=0}^N \binom{N}{n} Q_j(n; a, b, N) t^n.$$

For several variables we denote the Hahn polynomials by $\phi_\nu(\alpha; \sigma, N)$. These are discrete orthogonal polynomials indexed by $\nu \in \mathbb{N}_0^d$, $|\nu| \leq N$ which are defined on the set $\{\alpha \in \mathbb{N}_0^{d+1} : |\alpha| = N\}$ and are orthogonal with respect to the binomial distribution given by the parameter $\sigma = (\sigma_1, \dots, \sigma_{d+1})$. They are defined by the following generating function,

Definition 5.3. Suppose $\sigma \in \mathbb{R}^{d+1}$ with $\sigma_i > -1$, and $N \in \mathbb{N}$. For $\nu \in \mathbb{N}_0^d$, $|\nu| \leq N$ define the Hahn polynomials $\phi_\nu(\alpha; \sigma, N)$ at $\alpha \in \mathbb{N}_0^{d+1}$, $|\alpha| = N$ by

$$|y|^{N-|\nu|} \prod_{j=1}^d |y^j|^{\nu_j} \frac{P_{\nu_j}^{(b_j, \sigma_j)}(2y_j/|y^j| - 1)}{P_{\nu_j}^{(b_j, \sigma_j)}(1)} = \sum_{|\alpha|=N} \frac{N!}{\alpha!} \phi_\nu(\alpha; \sigma, N) y^\alpha, \quad y \in \mathbb{R}^{d+1},$$

where $|y^j| = y_j + \dots + y_{d+1}$ and $b_j = a_j(2\nu, \sigma + 1/2) - 1/2$ with a_j as in (5.2).

Setting $d = 1$, $y_1 = t$, $y_2 = 1$ in the above, the left hand side becomes

$$(1+t)^{N-\nu_1}(1+t)^{\nu_1} \frac{P_{\nu_1}^{(\sigma_2, \sigma_1)}(2t/(t+1)-1)}{P_{\nu_1}^{(\sigma_2, \sigma_1)}(1)} = (1+t)^N \frac{(-1)^{\nu_1} P_{\nu_j}^{(\sigma_2, \sigma_1)}(\frac{1-t}{1+t})}{P_{\nu_1}^{(\sigma_2, \sigma_1)}(1)}$$

and the right hand side becomes

$$\begin{aligned} & \sum_{\alpha_1=0}^N \frac{N!}{\alpha_1!(N-\alpha_1)!} \phi_\nu(\alpha_1, N-\alpha_1; \sigma_1, \sigma_2, N) x_1^{\alpha_1} x_2^{N-\alpha_1} \\ &= \sum_{\alpha_1=0}^N \binom{N}{\alpha_1} \phi_\nu(\alpha_1, N-\alpha_1; \sigma_1, \sigma_2, N) t^{\alpha_1}. \end{aligned}$$

Hence, $\phi_{\nu_1}(\alpha_1, N-\alpha_1; \sigma_1, \sigma_2, N) = (-1)^{\nu_1} Q_{\nu_1}(\alpha_1; \sigma_2, \sigma_1, N)$.

Let us indicate how our definition agrees with that given in [13]. There the generating function is denoted by $G_{r,N}(\frac{\bar{w}}{\bar{\alpha}}|\bar{\nu})$, which is defined by an inductive formula (see [13, (5.7), p. 278] and the first equation on page 279). We make the following substitutions: $r = d+1$, $\bar{\alpha} = (\sigma_{d+1}, \sigma_d, \dots, \sigma_1)$, $\bar{w} = (y_{d+1}, y_d, \dots, y_1)$, $\bar{\nu} = (\nu_d, \nu_{d-1}, \dots, \nu_1)$, and work out the generating function explicitly to obtain the form presented in Definition 5.3. Although we will not use the explicit formulae or the orthogonal relation of $\phi_\nu(\alpha; \sigma, N)$, we state them below for completeness and for future reference. Both are stated in [13] by inductive formulae, from which the explicit formulae can be worked out using the aforementioned substitutions. Further simplification leads to the formula for $\phi_\nu(\alpha; \sigma, N)$ presented below.

Proposition 5.4. For $\alpha \in \mathbb{N}_0^{d+1}$, $|\alpha| = N$ and $\nu \in \mathbb{N}_0^d$, $|\nu| \leq N$,

$$\phi_\nu(\alpha; \sigma, N) = \frac{(-1)^{|\nu|}}{(-|\alpha|)_{|\nu|}} \prod_{j=1}^d \frac{(\sigma_j+1)_{\nu_j}}{(a_j+1)_{\nu_j}} (-|\alpha^j| + |\nu^{j+1}|)_{\nu_j} Q_{\nu_j}(\alpha_j; \sigma_j, a_j, |\alpha^j| - |\nu^{j+1}|).$$

The proof that the $\phi_\nu(\alpha; \sigma, N)$ are orthogonal with respect to the binomial distribution is given in [13] and the constant B_ν below is given by inductive formulae (5.13), (5.14), (5.18) in [13]. The verification (using the substitution that we mentioned earlier) is straightforward.

Proposition 5.5. For $\nu, \mu \in \mathbb{N}_0^d$ with $|\nu|, |\mu| \leq N$,

$$\sum_{|\alpha|=N} \frac{(\sigma+1)_\alpha}{\alpha!} \phi_\nu(\alpha; \sigma, N) \phi_\mu(\alpha; \sigma, N) = B_\nu \delta_{\nu, \mu},$$

where B_ν is given by

$$B_\nu := \frac{(-1)^{|\nu|} (|\sigma| + d + 1)_{N+|\nu|}}{(-N)_{|\nu|} N! (|\sigma| + d + 1)_{2|\nu|}} \prod_{j=1}^d \frac{(\sigma_j + b_j + 1)_{2\nu_j} (\sigma_j + 1)_{\nu_j} \nu_j!}{(\sigma_j + b_j + 1)_{\nu_j} (b_j + 1)_{\nu_j}}.$$

For other properties of these polynomials, such as recurrence relations, see [13].

Using Proposition (5.4) and (5.3), we can now derive the expansion of R_α in terms of the orthonormal basis Y_ν . Recall that $\varepsilon(\alpha) = \alpha - 2[\alpha/2]$.

Proposition 5.6. For $\nu \in \mathbb{N}_0^d$ let $\rho = |\kappa| + (d-1)/2$ and let $\nu^* = (\nu, 0) \in \mathbb{N}_0^{d+1}$. Let $\alpha \in \mathbb{N}_0^{d+1}$. If α_{d+1} is an even integer, then

$$R_\alpha(x) = \left(\kappa + \frac{1}{2}\right)_{[\frac{\alpha+1}{2}]} \sum_{\substack{|\nu|=|\alpha| \\ \varepsilon(\nu^*)=\varepsilon(\alpha)}} \frac{(|[\nu/2]|)!}{\prod_{i=1}^d (\kappa_i + a_i)_{\nu_i}} \\ \times \phi_{[\frac{\nu}{2}]} \left(\left[\frac{\alpha}{2} \right], \kappa - \frac{1}{2} + \varepsilon(\nu^*), \left| \left[\frac{\nu}{2} \right] \right| \right) Y_\nu(x; \kappa);$$

and if α_{d+1} is an odd integer, then

$$R_\alpha(x) = \left(\kappa + \frac{1}{2}\right)_{[\frac{\alpha+1}{2}]} \frac{\kappa_d + \kappa_{d+1}}{\kappa_{d+1} + \frac{1}{2}} \sum_{\substack{|\nu|=|\alpha| \\ \varepsilon(\tilde{\nu}^*)=\varepsilon(\tilde{\alpha})}} \frac{(|[\tilde{\nu}/2]|)!}{\prod_{i=1}^d (\kappa_i + a_i)_{\nu_i}} \\ \times \phi_{[\frac{\tilde{\nu}}{2}]} \left(\left[\frac{\alpha}{2} \right], \tilde{\kappa} - \frac{1}{2} + \varepsilon(\tilde{\nu}^*), \left| \left[\frac{\tilde{\nu}}{2} \right] \right| \right) Y_\nu(x; \kappa),$$

where $\tilde{\nu} = \nu - e_d$, $\tilde{\kappa} = \kappa + e_{d+1}$ and $\tilde{\alpha} = \alpha + e_{d+1}$.

Proof. Using (5.3) and the Definition 5.3 we can expand the right hand side of the formula in Proposition 5.2 in powers of b . There are two terms, the first one contains only even powers of b_{d+1} and the second contains only odd powers. Hence, we need to consider the two cases separately. For example, setting $\sigma_i = \kappa_i - 1/2$ and $y_j = b_j^2$ so that $|y^j| = r_j^2$, the Definition 5.3 and (5.3) gives

$$\prod_{j=1}^d r_j^{\nu_j} \frac{C_{\nu_j}^{(a_j, \kappa_j)}(b_j/r_j)}{C_{\nu_j}^{(a_j, \kappa_j)}(1)} = \sum_{|\beta|=|\nu/2|} \frac{(|[\nu/2]|)!}{\beta!} \phi_{[\frac{\nu}{2}]} \left(\beta, \kappa - \frac{1}{2} + \varepsilon(\nu^*), \left| \left[\frac{\nu}{2} \right] \right| \right) b^{2\beta + \varepsilon(\nu^*)}.$$

which gives the expansion of the first term in the right hand side of the formula in Proposition 5.2. That is, for α_{d+1} being even,

$$\sum_{\substack{|\alpha|=n \\ \alpha_{d+1}=\text{even}}} b^\alpha \tilde{R}_\alpha(x) = \sum_{|\nu|=n} \sum_{|\beta|=|\nu/2|} \frac{(\rho)_{|\nu|}}{\prod_{i=1}^d (\kappa_i + a_i)_{\nu_i}} \frac{(|[\nu/2]|)!}{\beta!} \\ \times \phi_{[\frac{\nu}{2}]} \left(\beta, \kappa - \frac{1}{2} + \varepsilon(\nu^*), \left| \left[\frac{\nu}{2} \right] \right| \right) Y_\nu(x; \kappa) b^{2\beta + \varepsilon(\nu^*)}.$$

To derive the formula of R_α , we set $2\beta + \varepsilon(\nu^*) = \alpha = 2[\alpha/2] + \varepsilon(\alpha)$. This gives $\beta = [\alpha/2]$ and $\varepsilon(\nu^*) = \varepsilon(\alpha)$, so that

$$\tilde{R}_\alpha(x) = \sum_{\substack{|\nu|=|\alpha| \\ \varepsilon(\nu^*)=\varepsilon(\alpha)}} \frac{(\rho)_{|\nu|}}{\prod_{i=1}^d (\kappa_i + a_i)_{\nu_i}} \frac{(|[\nu/2]|)!}{(|[\alpha/2]|)!} \\ \times \phi_{[\frac{\nu}{2}]} \left(\left[\frac{\alpha}{2} \right], \kappa - \frac{1}{2} + \varepsilon(\nu^*), \left| \left[\frac{\nu}{2} \right] \right| \right) Y_\nu(x; \kappa).$$

Then we use the relation in Proposition 2.3 to replace \tilde{R}_α by R_α . The constant is simplified by the fact that $\alpha! = 2^{|\alpha|} (1/2)_\beta (\alpha/2)!$, $\beta = [\frac{\alpha+1}{2}]$. The case of α_{d+1} is odd is proved similarly. \square

The expansion of R_α^B or R_α^T in terms of an explicit orthonormal basis can be derived from the above proposition. We give the result for R_α^T below. First we state an orthonormal basis with respect to W_κ^T .

Definition 5.7. For $\nu \in \mathbb{N}_0^d$ and $x \in \mathbb{R}^d$,

$$P_\nu(x) := \prod_{j=1}^d (1 - |\mathbf{x}_{j-1}|)^{\nu_j} P_{\nu_j}^{(a_j-1/2, \kappa_j-1/2)} \left(\frac{2x_j}{1 - |\mathbf{x}_{j-1}|} - 1 \right),$$

where $|\mathbf{x}_j| = x_1 + \dots + x_j$ for $1 \leq j \leq d$ and $\mathbf{x}_0 := 0$.

The set $\{P_\nu : |\nu| = n\}$ is a basis for orthogonal polynomials of degree n with respect to W_κ^T and the elements in the set are mutually orthogonal; see, for example, [7, p. 47]. The L^2 norm of P_ν is given by

$$w_\kappa^T \int_{T^d} |P_\nu(x)|^2 W_\kappa^T(x) dx = \frac{1}{(|\kappa| + \frac{d+1}{2})_{2|\nu|}} \prod_{j=1}^d \frac{(\kappa_j + a_j)_{2\nu_j} (a_j + 1/2)_{\nu_j} (\kappa_j + 1/2)_{\nu_j}}{(\kappa_j + a_j)_{\nu_j} \nu_j!}.$$

Under the correspondence (2.9), the polynomial P_ν is related to $Y_{2\nu}(x, \kappa)$ in Proposition 5.1. In fact, for $x = (x_1, \dots, x_d, x_{d+1}) \in S^d$, the relation (2.5) gives

$$Y_{2\nu}(x, \kappa) = \prod_{i=1}^d \frac{(\kappa_i + a_i)_{\nu_i}}{(\kappa_i + 1/2)_{\nu_i}} P_\nu(x_1^2, \dots, x_d^2),$$

where we have used the fact that $r_j^2 = 1 - x_1^2 - \dots - x_{j-1}^2$ if $\|x\| = 1$. Hence,

Corollary 5.8. Let $\rho = |\kappa| + (d-1)/2$. For $\alpha \in \mathbb{N}_0^{d+1}$, $n = |\alpha|$,

$$R_\alpha^T(x) = n! \left(\kappa + \frac{1}{2} \right)_\alpha \sum_{|\nu|=n} \prod_{i=1}^d \frac{(\kappa_i + a_i)_{\nu_i}}{(\kappa_i + a_i)_{2\nu_i} (\kappa_i + \frac{1}{2})_{\nu_i}} \phi_\nu \left(\alpha, \kappa - \frac{1}{2}, n \right) P_\nu(x).$$

Proof. Using the fact $R_{2\alpha}(x) = R_\alpha^T(x_1^2, \dots, x_{d+1}^2)$, the formula comes from $R_{2\alpha}$ in the Proposition 5.6. Note that $\varepsilon(\nu^*) = \varepsilon(2\alpha)$ implies that ν_i is even for every i . \square

Since polynomials Y_ν are mutually orthogonal, the expansion in Proposition 5.6 can be used to compute the inner product of Y_ν and R_α . Similarly, the Corollary 5.8 can be used to compute the inner product of R_α^T and P_ν .

6. FURTHER RESULTS

The definition of R_α in Definition 2.1 makes sense for h -harmonics associated with other reflection groups. For background on the theory of h -harmonics in general, see [4, 5, 7] and the references therein. Although a formula of the intertwining operator V_κ is unknown in general, it is known that V_κ is a bounded operator in the following sense ([5]). Let $\|p\|_\infty := \sup_{B^{d+1}} |p(x)|$ for any polynomial p . For formal sums $f(x) = \sum_{n=0}^\infty f_n(x)$ with $f_n \in \mathcal{P}_n^{d+1}$, let $\|f\|_A := \sum_{n=0}^\infty \|f_n\|_\infty$ and let $A := \{f : \|f\|_A < \infty\}$. Then for $f \in A$, $|Vf(x)| \leq \|f\|_A$ for $x \in B^{d+1}$. This fact can be used to justify the definition of R_α in Definition 2.1 for other reflection groups.

We will not discuss R_α associated with general reflection groups any further, but merely point out that Proposition 2.2 holds in the general setting and prove one more such result which gives the expansion of Vx^α in terms of R_β . Recall that Proposition 2.2 shows

$$R_\alpha(x) = \sum_\gamma \frac{(-\alpha/2)_\gamma ((-\alpha+1)/2)_\gamma}{(-|\alpha| - \rho + 1)_{|\gamma|} \gamma!} \|x\|^{2|\gamma|} V_\kappa x^{\alpha-2\gamma}.$$

The following proposition states that the above expansion can be reversed.

Proposition 6.1. *Let $\alpha \in \mathbb{N}_0^{d+1}$. Then*

$$V_\kappa x^\alpha = \sum_{2\beta \leq \alpha} \frac{(-1)^{|\beta|} (-\alpha/2)_\beta ((-\alpha+1)/2)_\beta}{(-|\alpha| - \rho + 1)_{2|\beta|} \beta!} \\ \times (2(-|\alpha| - \rho + 1)_{|\beta|} - (-|\alpha| - \rho)_{|\beta|}) \|x\|^{2|\beta|} R_{\alpha-2\beta}(x).$$

Proof. We show that there exist a_β such that $a_0 = 1$ and

$$V_\kappa x^\alpha = \sum_{2\beta \leq \alpha} a_\beta \|x\|^{2|\beta|} R_{\alpha-2\beta}(x),$$

the values of a_β will be uniquely determined as the stated value. Using the formula $R_\alpha(x) = \sum_\gamma c_{\alpha,\gamma} \|x\|^{2|\gamma|} V_\kappa x^{\alpha-2\gamma}$, it follows that

$$\begin{aligned} \sum_{2\beta \leq \alpha} a_\beta \|x\|^{2|\beta|} R_{\alpha-2\beta}(x) &= \sum_{2\beta \leq \alpha} a_\beta \sum_{2\gamma \leq \alpha-2\beta} c_{\alpha-2\beta,\gamma} \|x\|^{2|\beta|+2|\gamma|} V_\kappa x^{\alpha-2\beta-2\gamma} \\ &= \sum_{2\gamma \leq \alpha} V_\kappa x^{\alpha-2\gamma} \|x\|^{2|\gamma|} \sum_{\beta \leq \gamma} a_\beta c_{\alpha-2\beta,\gamma-\beta}. \end{aligned}$$

Since $(a+m)_{n-m} = (a)_n / (a)_m$, we have

$$c_{\alpha-2\beta,\gamma-\beta} = \frac{(-|\alpha| - \rho + 1)_{2|\beta|}}{(-\alpha/2)_\beta ((-\alpha+1)/2)_\beta (-|\alpha| - \rho + 1)_{|\gamma|+|\beta|} (\gamma - \beta)!}$$

so that we need to show that there exist a_β such that $a_0 = 1$ and

$$\Sigma_\gamma := \sum_{\beta \leq \gamma} \frac{a_\beta^*}{(-|\alpha| - \rho + 1)_{|\gamma|+|\beta|} (\gamma - \beta)! \beta!} = 0, \quad a_\beta^* = \frac{(-|\alpha| - \rho + 1)_{2|\beta|} \beta!}{(-\alpha/2)_\beta ((-\alpha+1)/2)_\beta} a_\beta,$$

for $\gamma \neq 0$. For each Σ_γ , a_γ^* has the dominating subindex among all a_β^* in Σ_γ . Consequently, one can solve for a_γ^* recursively from the equations $\Sigma_\gamma = 0$. The fact that $a_0^* = a_0 = 1$ shows then that the solution is unique. Hence, to complete the proof, we only have to show that $a_\beta^* = (-1)^\beta (2(-|\alpha| - \rho + 1)_{|\beta|} - (-|\alpha| - \rho)_{|\beta|})$ is a solution. To do so, we need to recall the definition of another Lauricella function, the function of type D , defined by

$$F_D(a, \alpha; c; x) = \sum_{\beta} \frac{(a)_{|\beta|} (\alpha)_\beta}{(c)_{|\beta|} \beta!} x^\beta, \quad a, c \in \mathbb{R}, \quad \alpha \in \mathbb{N}_0^{d+1}, \quad \max_{1 \leq i \leq d+1} |x_i| < 1.$$

Then, with a_β^* so chosen, using the fact that $(\gamma - \beta)! = (-1)^{|\beta|} \gamma! / (-\gamma)_\beta$ and $(a)_{n+m} = (a+n)_m (a)_n$, we obtain

$$\begin{aligned} \Sigma_\gamma &= \sum_{\beta \leq \gamma} \frac{(-1)^\beta (2(-|\alpha| - \rho + 1)_{|\beta|} - (-|\alpha| - \rho)_{|\beta|})}{(-|\alpha| - \rho + 1)_{|\gamma|+|\beta|} (\gamma - \beta)! \beta!} \\ &= \frac{1}{\gamma! (-|\alpha| - \rho + 1)_{|\gamma|}} \sum_{\beta \leq \gamma} \frac{(-\gamma)_\beta (2(-|\alpha| - \rho + 1)_{|\beta|} - (-|\alpha| - \rho)_{|\beta|})}{(-|\alpha| - \rho + 1 + |\gamma|)_{|\beta|} \beta!} \\ &= \frac{1}{\gamma! (-|\alpha| - \rho + 1)_{|\gamma|}} [2F_D(-|\alpha| - \rho + 1, -\gamma; -|\alpha| - \rho + 1 + |\gamma|; \mathbf{1}) \\ &\quad - F_D(-|\alpha| - \rho, -\gamma; -|\alpha| - \rho + 1 + |\gamma|; \mathbf{1})]. \end{aligned}$$

Using Lauricella's identity [2, p. 116] and the Chu-Vandermonde identity

$$F_D(a, \alpha; c; \mathbf{1}) = {}_2F_1(a, |\alpha|; c; 1) \quad \text{and} \quad {}_2F_1(-n, b; c; 1) = \frac{(c-b)_n}{(c)_n},$$

we can then conclude that $\Sigma_\gamma = 0$. \square

For the case of h_κ^2 in (1.1), the explicit formula of the intertwining operator V_κ gives that

$$V_\kappa x^\alpha = \frac{(\frac{1}{2})_\beta}{(\kappa + \frac{1}{2})_\beta} x^\alpha, \quad \beta = \frac{\alpha + \mathbf{1}}{2}.$$

which gives the following corollary:

Corollary 6.2. *Let $\alpha \in \mathbb{N}_0^{d+1}$. For h_κ^2 in (1.1) and $\rho = |\kappa| + (d-1)/2$,*

$$x^\alpha = \frac{(\kappa + \frac{1}{2})_{[\frac{\alpha+1}{2}]}}{(\frac{1}{2})_{[\frac{\alpha+1}{2}]}} \sum_{2\beta \leq \alpha} \frac{(-1)^{|\beta|} (-\alpha/2)_\beta ((-\alpha + \mathbf{1})/2)_\beta}{(-|\alpha| - \rho + 1)_{2|\beta|} \beta!} \\ \times (2(|-\alpha| - \rho + 1)_{|\beta|} - (-|\alpha| - \rho)_{|\beta|}) \|x\|^{2|\beta|} R_{\alpha-2\beta}(x).$$

For orthogonal polynomials with respect to W_κ^B on B^d and W_κ^T on T^d , we can also derive the explicit formula of the expansion of x^α in terms of monomial orthogonal basis. For example, we have

Corollary 6.3. *Let $\alpha \in \mathbb{N}_0^d$. For W_κ^T in (1.3) and $\rho = |\kappa| + (d-1)/2$,*

$$x^\alpha = \frac{(\kappa + \frac{1}{2})_\alpha}{(\frac{1}{2})_\alpha} \sum_{\beta \leq \alpha} \frac{(-1)^{|\beta|} (-\alpha)_\beta (-\alpha + \mathbf{1}/2)_\beta}{(2|\alpha| + \rho + 1)_{2|\beta|} \beta!} \\ \times (2(2|\alpha| + \rho + 1)_{|\beta|} - (2|\alpha| + \rho)_{|\beta|}) R_{\alpha-\beta}^T(x).$$

Let us point out that the Proposition 6.1 holds for other reflection groups, since it is a formal inverse of the definition of R_α . Note that for other reflection groups, $V_\kappa x^\alpha$ is not a constant multiple of x^α in general and neither is R_α an orthogonal projection of x^α . One interesting aspect of Proposition 6.1 lies in the fact that R_α can be computed explicitly if an orthonormal basis is known, since such a basis will give a formula for the reproducing kernel of $\mathcal{H}_n^{d+1}(h_\kappa^2)$ so that Proposition 2.2 can be used to produce a formula of R_α . Once the formula of R_α is known, the formula in Proposition 6.1 gives an explicit formula of $V_\kappa x^\alpha$, which is of interest since an explicit formula of V_κ is not known for general reflection groups. For example, in the case of dihedral group I_{2k} for which

$$h_\kappa(x) = |\cos m\theta|^{\kappa_1} |\sin m\theta|^{\kappa_2}, \quad x = (\cos \theta, \sin \theta),$$

an orthonormal basis of $\mathcal{H}_n^2(h_\kappa^2)$ is known [4]. Hence, the above outline can be carried out to give an explicit formula of $V_\kappa x^\alpha$. However, the formula is complicated and it does not seem to give any indication of the explicit formula of V_κ . We shall not present them.

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