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SUMMABILITY OF PRODUCT JACOBI EXPANSIONS

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Abstract. Orthogonal expansions in product Jacobi polynomials with respect to the weight function $W_{\alpha,\beta}(\mathbf{x}) = \prod_{j=1}^d (1-x_j)^{\alpha_j}(1+x_j)^{\beta_j}$ on $[-1,1]^d$ are studied. For $\alpha_j, \beta_j > -1$ and $\alpha_j + \beta_j \ge -1$, the Cesàro (C,δ) means of the product Jacobi expansion converge in the norm of $L^p(W_{\alpha,\beta}, [-1,1]^d)$, $1 \le p < \infty$, and $C([-1,1]^d)$ if

$$\delta > \sum_{j=1}^d \max\{\alpha_j, \beta_j\} + \frac{d}{2} + \max\Big\{0, -\sum_{j=1}^d \min\{\alpha_j,\beta_j\} - \frac{d+2}{2}\Big\}.$$

Moreover, for $\alpha_j, \beta_j \ge -1/2$, the (C,δ) means define a positive linear operator if and only if $\delta \ge \sum_{i=1}^d (\alpha_i + \beta_i) + 3d - 1$.

1. Introduction and main results

For $\alpha, \beta > -1$, we denote by $P_n^{(\alpha,\beta)}$ the usual Jacobi polynomials of degree n, which is orthogonal on $[-1,1]$ with respect to the weight function $w_{\alpha,\beta}$ defined by

$$w_{\alpha,\beta}(x) = c_{\alpha,\beta}(1-x)^\alpha(1+x)^\beta, \qquad \text{where} \qquad c_{\alpha,\beta} = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)2^{\alpha+\beta+1}}.$$

The weight function $w_{\alpha,\beta}$ is normalized to have the unit integral on $[-1,1]$. We denote the orthonormal Jacobi polynomial by $p_n^{(\alpha,\beta)}$, which differs from $P_n^{(\alpha,\beta)}$ by a normalization constant ([20, p. 68]). We study the Fourier orthogonal expansion in product Jacobi polynomials. Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $\beta = (\beta_1, \ldots, \beta_d)$ with $\alpha_i, \beta_i > -1$ for $1 \le i \le d$. The product Jacobi weight function is defined by

$$W_{\alpha,\beta}(\mathbf{x}) = \prod_{i=1}^d w_{\alpha_i,\beta_i}(x_i) = \prod_{i=1}^d c_{\alpha_i,\beta_i}(1-x_i)^{\alpha_i}(1+x_i)^{\beta_i},$$

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where $\mathbf{x} = (x_1, \dots, x_d) \in [-1,1]^d$. Let $\mathbb{N}_0^d$ denote the set of nonnegative integers. Then an orthonormal basis of polynomials in $L^2(W_{\alpha,\beta}, [-1,1]^d)$ is given by

$$P_{\mathbf{k}}^{(\alpha,\beta)}(\mathbf{x}) = \prod_{i=1}^{d} p_{k_i}^{(\alpha,\beta)}(x_i), \qquad \mathbf{k} = (k_1, \dots, k_d) \in \mathbb{N}_0^d.$$

These polynomials are called the product Jacobi polynomials. Let f be a Lebesgue integrable function with respect to $W_{\alpha,\beta}$ on $[-1,1]^d$. Its Fourier orthogonal expansion with respect to $W_{\alpha,\beta}$, called the product Jacobi expansion of f, is defined by

$$f \sim \sum_{m=0}^{\infty} \sum_{|\mathbf{k}|=m} a_{\mathbf{k}}(f) P_{\mathbf{k}}^{(\alpha,\beta)}, \qquad a_{\mathbf{k}}(f) = \int_{[-1,1]^d} f(\mathbf{y}) P_{\mathbf{k}}^{(\alpha,\beta)}(\mathbf{y}) W_{\alpha,\beta}(\mathbf{y}) d\mathbf{y}.$$

The n-th partial sum of this expansion is defined by

$$S_{n,d}(W_{\alpha,\beta}; f, \mathbf{x}) := \sum_{m=0}^{n} \sum_{|\mathbf{k}|=m} a_{\mathbf{k}}(f) P_{\mathbf{k}}^{(\alpha,\beta)}(\mathbf{x}). \tag{1.1}$$

For $\delta > 0$, the Cesàro (C, δ) means of the product Jacobi expansion is defined by

$$S_{n,d}^{\delta}(W_{\alpha,\beta}; f) = \binom{n+\delta}{n}^{-1} \sum_{k=0}^{n} \binom{n-k+\delta-1}{n-k} S_{k,d}(W_{\alpha,\beta}; f). \tag{1.2}$$

Throughout this paper we denote by $L^p(W_{\alpha,\beta}; [-1,1]^d)$, $1 \le p < \infty$, the space of Lebesgue measurable functions with finite norm

$$\|f\|_p := \Big(\int_{[-1,1]^d} |f(\mathbf{y})|^p W_{\alpha,\beta}(\mathbf{y}) d\mathbf{y} \Big)^{1/p},$$

and we denote by $C([-1,1]^d)$ the space of continuous functions on $[-1,1]^d$ with supremum norm $\|\cdot\|_\infty$. The Jacobi expansion in one variable has been studied extensively by many authors. We refer to [2, 7, 8, 11, 13-18, 20] and the references there. In particular, using a convolution structure of the Jacobi polynomials ([2, 10]), it follows from the boundedness of $S_{n,1}^{\delta}(w_{\alpha,\beta}; f)$ at $x = 1$ ([20, Theorem 9.1.3, p. 246]) that $\|S_{n,1}^{\delta}(w_{\alpha,\beta}; f)\|_\infty$ is uniformly bounded provided that $\delta > \max\{\alpha, \beta\} + 1/2$ and $\alpha + \beta \ge -1$. Moreover, it was proved in [11] (see also [1]) that the $(C, \alpha+\beta+2)$ means $S_{n,1}^{\alpha+\beta+2}(w_{\alpha,\beta}; f)$ define a positive linear operator provided $\alpha, \beta \ge -1/2$. The purpose of the present paper is to prove the analogous of these results for the product Jacobi expansions. Here are the main results.

Theorem 1.1. *Let $\alpha_j > -1$, $\beta_j > -1$ and $\alpha_j + \beta_j \ge -1$ for $1 \le j \le d$. Then the Cesàro (C, δ) means $S_{n,d}^{\delta}(W_{\alpha,\beta}; f)$ of the product Jacobi expansion are uniformly bounded in the norm of $L^p(W_{\alpha,\beta}; [-1,1]^d)$, $1 \le p < \infty$, and the norm of $C([-1,1]^d)$ provided*

$$\delta > \delta_0 := \sum_{j=1}^{d} \max\{\alpha_j, \beta_j\} + \frac{d}{2} + \max\Big\{0, -\sum_{j=1}^{d} \min\{\alpha_j, \beta_j\} - \frac{d+2}{2}\Big\}. \tag{1.3}$$

Moreover, if $-\sum_{j=1}^{d} \min\{\alpha_j, \beta_j\} - \frac{d}{2} - 1 > 0$ and $\alpha_i = \beta_i = -1/2$ does not hold for any i, then the above conclusion holds for $\delta \ge \delta_0$. In particular, for f in $L^p(W_{\alpha,\beta}; [-1,1]^d)$, $1 \le p < \infty$, or in $C([-1,1]^d)$, $S_n^{\delta}(W_{\alpha,\beta}; f)$ converges to f in norm as $n \to \infty$.

Theorem 1.2. *Let $\alpha_j \geq -1/2$ and $\beta_j \geq -1/2$ for $1 \leq j \leq d$. Then the Cesàro (C,δ) means $S_{n,d}^{\delta}(W_{\alpha,\beta}; f)$ of the product Jacobi expansion define a positive linear approximation identity on $C([-1,1])^d$ if $\delta \geq \sum_{i=1}^{d}(\alpha_i+\beta_i)+3d-1$; moreover, the order of summability is best possible in the sense that (C,δ) means are not positive for $0<\delta<\sum_{i=1}^{d}(\alpha_i+\beta_i)+3d-1$.*

Some remarks are in order. Concerning the statement of Theorem 1.1, we note that

$$\delta_0 = \sum_{j=1}^{d} \max\{\alpha_j,\beta_j\} + \frac{d}{2}, \quad \text{if} \quad \alpha_j,\beta_j \geq -1/2, \quad \text{or} \quad d=1,2. \tag{1.4}$$

In particular, for $d=1$, the Theorem 1.1 agrees with the result of Jacobi expansion in one variable. It should be mentioned that our proof is different from that in [20].

For $d \geq 3$, the last term of (1.3) may be needed. For example, consider $\alpha_j = -1/2 + 1/m$ and $\beta_j = -1/2 - 1/m$, where $d > m > 2$. Then $-\sum_{i=1}^{d}\min\{\alpha_j,\beta_j\} - d/2 - 1 = (d-m)/m > 0$.

Since $\alpha_j + \beta_j \geq -1$ implies that $\max\{\alpha_j,\beta_j\} \geq -1/2$, it follows that $\delta_0 \geq 0$ for δ_0 in (1.3). If $\alpha_j = \beta_j = -1/2$, then $\delta_0 = 0$ and the condition $\delta > \delta_0$ becomes simply $\delta > 0$. This extremal case corresponds to the ℓ-1 summability of the multiple Fourier series or integral (see [5,6]), which is in sharp contrast with the usual radial (that is, ℓ-2) means of the multiple Fourier series or integrals (cf. [19]). The case of $\alpha = \beta = 2m - 1/2$, $m \in \mathbb{N}$, was discussed in [23], but the proof there depends on a representation of the kernel in terms of finite difference, which cannot be used in the general setting. The proof of Theorem 1.1 uses a new representation of the kernel, derived from taking Fourier transform of the generating function of the product Jacobi polynomials.

These results are likely only the first step in studying the product Jacobi expansions. One important question is to show that the index δ_0 is indeed the *critical index* for the Cesàro means in L^1 norm or C norm; that is, to show that the convergence in the norm of $L^1(W_{\alpha,\beta},[-1,1]^d)$ or $C([-1,1]^d)$ fails for $\delta \leq \delta_0$. In one variable setting, the weak boundedness of the maximal Cesàro operators at the critical index of L^p norm was proved in [7]. Various convergence criteria of $S_n^{\delta}(w_{\alpha,\beta}, f)$ at the L^1 critical index $\delta = \max\{\alpha,\beta\}+1/2$ are derived in [13], and the results are extended to conjugate Jacobi expansions in [14, 15]. More generally, one may study the L^p critical index and almost everywhere convergence of the product Jacobi expansions. Since our proof relies on the product convolution structure, it does not give results in these directions. However, we should mention that in the one variable setting, the L^p critical index is obtained from the L^1 boundedness of the Cesàro means ([2,10]) and the L^p boundedness of the partial sum operators ($\delta = 0$) ([16,17,18]) by the interpolation theorem of analytic families of operators ([19. p. 239]); see, for example, [8]. Hence, one nature next step is to get a sharp result for the L^p boundedness of the partial sums $S_{n,d}(W_{\alpha,\beta}; f)$.

2. Preliminaries and Representation of the Kernel

In this section we recall various facts about Jacobi polynomials, including the convolution structure in [10], which leads to a product convolution structure for the product

Jacobi expansions and allows us to reduce the proof of our theorems on $[-1,1]^d$ to essentially at one point. We also derive a representation of the Cesàro kernel for the product Jacobi expansion. The estimate of the kernel and the proof of the theorems are given in the next section.

Let $K_{n,d}(W_{\alpha,\beta};\mathbf{x},\mathbf{y})$ be the n-th reproducing kernel of the space of polynomials of degree n in $L^2(W_{\alpha,\beta},[-1,1]^d)$, which is defined by

$$K_{n,d}(W_{\alpha,\beta};\mathbf{x},\mathbf{y}) = \sum_{m=0}^{n}\sum_{|\mathbf{k}|=m} P_{\mathbf{k}}^{(\alpha,\beta)}(\mathbf{x})P_{\mathbf{k}}^{(\alpha,\beta)}(\mathbf{y}). \tag{2.1}$$

Then the partial sum operator $S_{n,d}^{\delta}(W_{\alpha,\beta};f)$ can be written as

$$S_{n,d}(W_{\alpha,\beta};f,\mathbf{x}) = \int_{[-1,1]^d} f(\mathbf{y})K_{n,d}(W_{\alpha,\beta};\mathbf{x},\mathbf{y})W_{\alpha,\beta}(\mathbf{y})d\mathbf{y}.$$

Moreover, if we denote the Cesàro (C,δ) means of the reproducing kernel by

$$K_{n,d}^{\delta}(W_{\alpha,\beta};\mathbf{x},\mathbf{y}) = \binom{n+\delta}{n}^{-1}\sum_{k=0}^{n}\binom{n-k+\delta-1}{n-k}K_{k,d}(W_{\alpha,\beta};\mathbf{x},\mathbf{y}), \tag{2.2}$$

then the (C,δ) means of the product Jacobi expansion can be written as

$$S_{n,d}^{\delta}(W_{\alpha,\beta};f,\mathbf{x}) = \int_{[-1,1]^d} f(\mathbf{y})K_{n,d}^{\delta}(W_{\alpha,\beta};\mathbf{x},\mathbf{y})W_{\alpha,\beta}(\mathbf{y})d\mathbf{y}. \tag{2.3}$$

First we show that a product convolution structure for the Jacobi expansions allows us to write the kernel $K_{n,d}^{\delta}(W_{\alpha,\beta};\mathbf{x},\mathbf{y})$ in terms of $K_{n,d}^{\delta}(W_{\alpha,\beta};\mathbf{e},\mathbf{y})$, where $\mathbf{e}=(1,1,\ldots,1)$. For this purpose we recall the following result due to Gasper [10, p. 262].

Lemma 2.1. *Let $\alpha,\beta>-1$. There is an integral representation of the form*

$$p_n^{(\alpha,\beta)}(x)p_n^{(\alpha,\beta)}(y) = p_n^{(\alpha,\beta)}(1)\int_{-1}^{1} p_n^{(\alpha,\beta)}(t)d\mu_{x,y}^{(\alpha,\beta)}(t), \qquad n\ge 0, \tag{2.4}$$

with the real Borel measures $d\mu_{x,y}^{(\alpha,\beta)}$ on $[-1,1]$ satisfying

$$\int_{-1}^{1}|d\mu_{x,y}^{(\alpha,\beta)}(t)|\le M, \qquad -1<x,y<1,$$

for some constant M independent of x,y if and only if $\alpha\ge\beta$ and $\alpha+\beta\ge-1$. Moreover, the measures are nonnegative, i.e., $d\mu_{x,y}^{(\alpha,\beta)}(t)\ge 0$, if and only if $\beta\ge-1/2$ or $\alpha+\beta\ge 0$.

For further properties of the measures $\mu_{x,y}^{(\alpha,\beta)}$, see [10, p. 262]. Weaker results were obtained earlier in [2]. In the case of $\alpha\ge\beta\ge-1/2$, an explicit product formula for the Jacobi polynomials was discovered by Koornwinder, see [12], which gives the measures explicitly. Using this formula, the summability of the Jacobi expansions for continuous functions on the interval $[-1,1]$ follows from the summability at the point $x=1$. From the formula (2.4) we obtain the following formula for the product Jacobi polynomials,

$$P_{\mathbf{k}}^{(\alpha,\beta)}(\mathbf{x})P_{\mathbf{k}}^{(\alpha,\beta)}(\mathbf{y}) = P_{\mathbf{k}}^{(\alpha,\beta)}(\mathbf{e})\int_{[-1,1]^d} P_{\mathbf{k}}^{(\alpha,\beta)}(\mathbf{t})d\mu_{\mathbf{x},\mathbf{y}}^{(\alpha,\beta)}(\mathbf{t}), \qquad n\ge 0, \tag{2.5}$$

where $d\mu_{\mathbf{x},\mathbf{y}}^{(\alpha,\beta)}(\mathbf{t}) = \prod_{j=1}^{d} d\mu_{x_j,y_j}^{(\alpha_j,\beta_j)}(t_j)$ provided $\alpha_j\ge\beta_j>-1$ and $\alpha_j+\beta_j\ge-1$. As in the one variable setting, this formula allows us to prove the following result.

Lemma 2.2. *In order to prove Theorem 1.1 it is sufficient to prove that, for* $\alpha_j \ge \beta_j > -1$, $\alpha_j + \beta_j > -1$,

$$\int_{[-1,1]^d} |K_{n,d}^\delta(W_{\alpha,\beta}; \mathbf{e}, \mathbf{y})| W_{\alpha,\beta}(\mathbf{y}) d\mathbf{y} \le c, \qquad \mathbf{e} = (1, 1, \ldots, 1), \tag{2.6}$$

where c *is a constant independent of* n*, under the condition (1.3). In order to prove Theorem 1.2, it is sufficient to prove that, for* $\alpha_j \ge \beta_j \ge -1/2$, $K_{n,d}^\delta(W_{\alpha,\beta}; \mathbf{e}, \mathbf{y}) \ge 0$ *on* $[-1,1]^d$ *if and only if* $\delta \ge \sum_{j=1}^d (\alpha_j + \beta_j) + 3d - 1$.

Proof. The reason that we can assume $\alpha_j \ge \beta_j$ lies in the formula of Jacobi polynomials, $p_n^{(\alpha,\beta)}(x) = (-1)^n p_n^{(\beta,\alpha)}(-x)$. To prove the norm boundedness of $S_n^\delta(W_{\alpha,\beta}, f)$ in $L^1(W_{\alpha,\beta}, [-1,1]^d)$ and $C([-1,1]^d)$, a standard argument shows that it suffices to prove

$$\int_{[-1,1]^d} |K_{n,d}^\delta(W_{\alpha,\beta}; \mathbf{x}, \mathbf{y})| W_{\alpha,\beta}(\mathbf{y}) d\mathbf{y} \le c, \qquad \mathbf{x} \in [-1,1]^d, \quad n \ge 0. \tag{2.7}$$

From (2.1), (2.2) and (2.5), it follows that

$$K_{n,d}^\delta(W_{\alpha,\beta}; \mathbf{x}, \mathbf{y}) = \int_{[-1,1]^d} K_{n,d}^\delta(W_{\alpha,\beta}; \mathbf{t}, \mathbf{e}) d\mu_{\mathbf{x},\mathbf{y}}^{(\alpha,\beta)}(\mathbf{t}). \tag{2.8}$$

Hence, this leads to a convolution structure as in the case of one variable, which implies that (2.7) holds if (2.6) holds; see [10, p. 264]. The case of the norm boundedness in $L^p(W_{\alpha,\beta}, [-1,1]^d)$, $1 < p < \infty$, follows from the usual interpolation argument. The statement about Theorem 1.2 follows from (2.3), (2.8) and Lemma 2.1. $\square$

For $d = 1$, the inequality (2.6) is proved in [20] using an elementary identity [20, p. 256, (9.4.1)] which, however, has no analogy in several variables. Moreover, (2.6) cannot be reduced to an inequality of one variable despite the fact that it comes from a product setting. For the extreme case $\alpha_j = \beta_j = -1/2$ of the ℓ-1 sum of the multiple Fourier series, a compact formula of $K_n^\delta(W_{\alpha,\beta}; f, \mathbf{x}, \mathbf{e})$ is given in terms of divided differences [22]; later a related formula in the ℓ-1 Fourier integral is given in terms of the Poisson integral by the first author and used in [6]. Here we derive a compact formula in the general setting from the Poisson formula, or the generating function of the Jacobi polynomials, which is given by (see [3, p. 102, Ex. 19] or [1, p. 21])

$$\begin{aligned} G^{(\alpha,\beta)}(r; x) &:= \sum_{k=0}^\infty p_k^{(\alpha,\beta)}(1) p_k^{(\alpha,\beta)}(x) r^n \\ &= \frac{1-r}{(1+r)^{\alpha+\beta+2}} \, {}_2F_1\left(\begin{matrix} \frac{\alpha+\beta+2}{2}, \frac{\alpha+\beta+3}{2} \\ \beta+1 \end{matrix} ; \frac{2r(1+x)}{(1+r)^2} \right), \qquad 0 \le r < 1, \end{aligned} \tag{2.9}$$

where ${}_2F_1$ is the Gauss hypergeometric function. Using the transformation formula

$${}_2F_1(a, b; c; z) = (1-z)^{c-a-b} {}_2F_1(c-a, c-b; c, z), \qquad |z| < 1,$$

(cf. [9, Vol. 1, p. 64, 2.1.4 (23)]), we can write this generating function as

$$G^{(\alpha,\beta)}(r;x) = \frac{(1-r)(1+r)^{\alpha-\beta+1}}{(1-2rx+r^2)^{\alpha+3/2}}\, {}_2F_1\left(\begin{matrix} \frac{\beta-\alpha}{2}, \frac{\beta-\alpha-1}{2} \\ \beta+1 \end{matrix}; \frac{2r(1+x)}{(1+r)^2}\right), \quad 0 \le r < 1, \tag{2.10}$$

where for $\alpha = \beta$, the ${}_2F_1$ part is taken to be 1 and we end up with the generating formula for the Gegenbauer polynomials. This form of the generating formula has been used to study summability in [8] and [14]. From (2.9) we get the generating function for the product Jacobi polynomials. Let again $\alpha = (\alpha_1, \dots, \alpha_d)$ and $\beta = (\beta_1, \dots, \beta_d)$. Then

$$\sum_{n=0}^{\infty} r^n \sum_{|\mathbf{k}|=n} P_{\mathbf{k}}^{(\alpha,\beta)}(\mathbf{x}) P_{\mathbf{k}}^{(\alpha,\beta)}(\mathbf{e}) = \prod_{i=1}^{d} G^{(\alpha_i,\beta_i)}(r;x_i) := G_d^{(\alpha,\beta)}(r;\mathbf{x}). \tag{2.11}$$

Multiplying this formula by $(1-r)^{-\delta-1} = \sum_{n=0}^{\infty} \binom{n+\delta}{n} r^n$ and using (2.1) and (2.2), we conclude that

$$\sum_{n=0}^{\infty} \binom{n+\delta}{n} K_{n,d}^{\delta}(W_{\alpha,\beta};\mathbf{x},\mathbf{e}) r^n = (1-r)^{-\delta-1} G_d^{(\alpha,\beta)}(r;\mathbf{x}). \tag{2.12}$$

Since both sides are analytic functions of r for $|r| < 1$, the above formula holds for r being complex numbers. Replacing r by $re^{i\theta}$, we get

$$\sum_{n=0}^{\infty} \binom{n+\delta}{n} K_{n,d}^{\delta}(W_{\alpha,\beta};\mathbf{x},\mathbf{e}) r^n e^{in\theta} = (1-re^{i\theta})^{-\delta-1} G_d^{(\alpha,\beta)}(re^{i\theta};\mathbf{x}).$$

Hence, we see that $\binom{n+\delta}{n} K_{n,d}^{\delta}(W_{\alpha,\beta};\mathbf{x},\mathbf{e}) r^n$ is the n-th Fourier coefficient of the function (of θ) in the right hand side. Thus, we conclude the following.

Lemma 2.3. *For $d \ge 1$ and $0 \le r < 1$,*

$$K_{n,d}^{\delta}(W_{\alpha,\beta};\mathbf{x},\mathbf{e}) = \binom{n+\delta}{n}^{-1} \frac{1}{\pi r^n} \int_{-\pi}^{\pi} (1-re^{i\theta})^{-\delta-1} G_d^{(\alpha,\beta)}(re^{i\theta};\mathbf{x}) e^{-in\theta} d\theta. \tag{2.13}$$

This representation of the Cesàro means is the key in the proof of Theorem 1.1 in the following section, where we will use it to derive a sharp estimate of $K_{n,d}^{\delta}(W_{\alpha,\beta};\mathbf{x},\mathbf{e})$. Since the left hand side is independent of r, it is tempting to let $r \to 1$ and using the result of the singular oscillatory integral to deal with the right hand side. Such an approach was used, for example, in [21] for studying the Riesz kernel of the multiple Hermite expansions. However, in contrast to the Hermite case, the singularities in the product Jacobi expansions are not isolated. Indeed, the integral in the right hand of (2.13) with $r = 1$ has singularity whenever $\theta = \phi_i$, where $x = \cos\phi_i$ for $1 \le i \le d$, and the order of singularity increases whenever two or more x_j are equal (see, for example, Lemma 3.1). The prototype of such singularities appears already in the case of $\alpha_j = \beta_j = -1/2$, that is, the ℓ-1 Cesàro (or Reisz) means of the Fourier series (or integral) ([4,5,6]), which is characteristically different from that of radial (or ℓ-2) Riesz means. In this respect, it is interesting to note that the summability of the orthogonal expansion on the unit ball and on the simplex in $\mathbb{R}^d$ is closely related to the ℓ-2 summability ([24,25]). In the present case, instead of appealing to the theory of singular integrals, we will use (2.13) to derive a sharp estimate of the kernel $K_{n,d}^{\delta}(W_{\alpha,\beta};\mathbf{x},\mathbf{e})$ by choosing $r = 1 - n^{-1}$.

3. Proof of the theorems

Throughout this section, we denote by c a generic constant independent of n, its value may be different at different occurrence. We start with the following result.

Lemma 3.1. *For $x_j = \cos\phi_j$, where $0 \le \phi_j \le \pi$ and $1 \le j \le d$,*

$$\begin{aligned} (3.1) \qquad & |K^\delta_{n,d}(W_{\alpha,\beta}, \mathbf{x}, \mathbf{e})| \\ & \le \frac{c}{n^\delta} \int_0^\pi \frac{(\sin\frac{\theta}{2} + n^{-1})^{d-\delta-1} \prod_{j=1}^d (\cos\frac{\theta}{2} + \cos\frac{\phi_j}{2} + n^{-1})^{\alpha_j-\beta_j+1}}{\prod_{j=1}^d (|\sin\frac{\theta-\phi_j}{2}| + n^{-1})^{\alpha_j+3/2} (|\sin\frac{\theta+\phi_j}{2}| + n^{-1})^{\alpha_j+3/2}} d\theta, \end{aligned}$$

where if $\alpha_j = \beta_j$ for some j, the term $\cos\frac{\theta}{2} + \cos\frac{\phi_j}{2}$ is replaced by $\cos\frac{\theta}{2}$.

Proof. First we derive an estimate for $G^{(\alpha,\beta)}(r;x)$ defined in (2.10). From [9, Vol. 1, p. 76, (9)], we have that for $\alpha, \beta > -1$ and $\alpha \ne \beta$,

$$|{}_2F_1((\beta-\alpha)/2,\ (\beta-\alpha-1)/2; \beta+1; z)| \le c(1+|z|)^{(\alpha-\beta+1)/2}, \qquad z \in \mathbb{C} \setminus (1, +\infty).$$

Therefore, by (2.10), we see that the following estimate holds,

$$(3.2) \quad |G^{(\alpha,\beta)}(re^{i\theta};x)| \le c\, \frac{|1-re^{i\theta}|}{|1-2re^{i\theta}x + r^2e^{2i\theta}|^{\alpha+3/2}} \left[|1+re^{i\theta}|^2 + 2r(1+x)\right]^{(\alpha-\beta+1)/2}.$$

Write $x = \cos\phi$, $0 \le \phi \le \pi$, and set $r = 1 - n^{-1}$. Then we have

$$|1-re^{i\theta}| \sim \sin(\theta/2) + n^{-1} \quad \text{and} \quad |1+re^{i\theta}| \sim \cos(\theta/2) + n^{-1},$$

where $A \sim B$ means that $c_1 \le |A/B| \le c_2$; for example, the first relation follows from $|1-re^{i\theta}|^2 = n^{-2} + 4(1-n^{-1})\sin^2(\theta/2)$. The second relation also leads to

$$|1+re^{i\theta}|^2 + 2r(1+x) \sim (\cos(\theta/2) + \cos(\phi/2) + n^{-1})^2,$$

using the fact that $1 + x = 2\cos^2(\phi/2)$. Moreover, we also have

$$\begin{aligned} |1-2re^{i\theta}x + r^2e^{2i\theta}| =& |1-re^{i(\theta-\phi)}| \cdot |1-re^{i(\theta+\phi)}| \\ \sim& (|\sin(\theta-\phi)/2| + n^{-1})(|\sin(\theta+\phi)/2| + n^{-1}). \end{aligned}$$

Together, these relations yield that for $\alpha \ne \beta$

$$|G^{(\alpha,\beta)}(re^{i\theta};x)| \le c\, \frac{(\sin\frac{\theta}{2} + n^{-1})(\cos\frac{\theta}{2} + \cos\frac{\phi}{2} + n^{-1})^{\alpha-\beta+1}}{(|\sin\frac{\theta-\phi}{2}| + n^{-1})^{\alpha+3/2}(|\sin\frac{\theta+\phi}{2}| + n^{-1})^{\alpha+3/2}}.$$

If $\alpha = \beta$, then the ${}_2F_1$ part in (2.10) is 1 and we do not have $2r(1+x)$ term in the right hand of (3.2). From this inequality and the fact that $\binom{n+\delta}{n} \sim n^\delta$, (3.1) follows from the definition of $G_d^{(\alpha,\beta)}(r;\mathbf{x})$ in (2.11) and the representation of $K^\delta_{n,d}(W_{\alpha,\beta}, \mathbf{x}, \mathbf{e})$ in (2.13). $\square$

For the proof of Theorem 1.1, we need to estimate the integral $I^{(\alpha,\beta)}(\theta)$ defined by

$$I^{(\alpha,\beta)}(\theta) := \int_0^\pi \frac{(\cos\frac{\theta}{2} + \cos\frac{\phi}{2} + n^{-1})^{\alpha-\beta+1}}{(|\sin\frac{\theta-\phi}{2}| + n^{-1})^{\alpha+3/2}(|\sin\frac{\theta+\phi}{2}| + n^{-1})^{\alpha+3/2}} \sin^{2\alpha+1}\frac{\phi}{2} \cos^{2\beta+1}\frac{\phi}{2} d\phi$$

if $\alpha \ne \beta$, and

$$I^{(\alpha,\alpha)}(\theta) := \int_0^\pi \frac{\cos\frac{\theta}{2} + n^{-1}}{(|\sin\frac{\theta-\phi}{2}| + n^{-1})^{\alpha+3/2}(|\sin\frac{\theta+\phi}{2}| + n^{-1})^{\alpha+3/2}} \sin^{2\alpha+1}\frac{\phi}{2} \cos^{2\alpha+1}\frac{\phi}{2} d\phi.$$

Lemma 3.2. *Let $\alpha \ge \beta > -1$, $\alpha+\beta \ge -1$ and $0 \le \theta \le \pi$. If $\alpha \ne \beta$ or $\alpha = \beta > -1/2$,*

$$I^{(\alpha,\beta)}(\theta) \le c\, n^{\alpha+1/2}(\sin(\theta/2)+n^{-1})^{\alpha-1/2}(\cos(\theta/2)+n^{-1})^{\beta+1/2}\ ; \tag{3.3}$$

and if $\alpha=\beta=-1/2$, then

$$I^{(\alpha,\beta)}(\theta) \le c\,(\sin(\theta/2)+n^{-1})^{-1}\log(2+n\theta))\ . \tag{3.4}$$

Proof. From $\alpha \ge \beta$ and $\alpha+\beta \ge -1$, it follows that $\alpha \ge -1/2$. Moreover, $\alpha=-1/2$ only if $\beta=-1/2$. First we consider the case $\alpha > \beta$, which implies that $\alpha > -1/2$. Several estimates below uses the elementary relation $\sin\xi \sim \xi$ for $0 \le \xi \le \pi/2$.

Case 1. $0 \le \theta \le \pi/2$. In this case $\cos\frac{\theta}{2}+\cos\frac{\phi}{2}+n^{-1} \sim \cos\frac{\theta}{2} \sim 1$. We have

$$I^{(\alpha,\beta)}(\theta) \le c\int_0^\pi \frac{\sin^{2\alpha+1}\frac{\phi}{2}\cos^{2\beta+1}\frac{\phi}{2}}{(|\sin\frac{\theta-\phi}{2}|+n^{-1})^{\alpha+3/2}(|\sin\frac{\theta+\phi}{2}|+n^{-1})^{\alpha+3/2}}d\phi.$$

We split the integral over $[0,\pi]$ into two integrals over $[0,3\theta/2]$ and over $[3\theta/2,\pi]$, respectively. On the first interval we have $\cos\phi/2 \sim 1$. For $\phi \in [0,3\theta/2]$, we have $\sin\phi/2 \le c\sin\theta/2$ and $\sin(\theta+\phi)/2 \sim \sin\theta/2$, since $\theta/2 \le (\theta+\phi)/2 \le 5\theta/4$. Hence, it follows from $\sin(\theta-\phi)/2 \sim (\theta-\phi)/2$ that

$$\begin{aligned}\int_0^{3\theta/2} &\le c\,\frac{\sin^{2\alpha+1}\frac{\theta}{2}}{(\sin\frac{\theta}{2}+n^{-1})^{\alpha+3/2}}\int_0^{3\theta/2}\frac{d\phi}{(|\theta-\phi|+n^{-1})^{\alpha+3/2}}\\ &\le c\, n^{\alpha+1/2}(\sin(\theta/2)+n^{-1})^{\alpha-1/2},\end{aligned}$$

in which the last integral can be evaluated exactly. For later use, we notice that if $\alpha=-1/2$, then the evaluation of the integral yields a factor $\log(2+n\theta)$. For $\phi \ge 3\theta/2$, we have $\sin(\theta+\phi)/2 \sim \sin\phi/2$ and $\sin(\phi-\theta)/2 \sim \sin\phi/2$, since $\phi/2 \le (\theta+\phi)/2 \le 5\phi/6$ and $\phi/2 \ge (\phi-\theta)/2 \ge \phi/6$. Hence,

$$\begin{aligned}\int_{3\theta/2}^{\pi} &\le c\int_{3\theta/2}^{3\pi/4}\frac{d\phi}{(\phi+n^{-1})^2}+\int_{3\pi/4}^{\pi}\cos^{2\beta+1}\phi\, d\phi\\ &\le c\,(1+(\theta+n^{-1})^{-1}) \le c\, n^{\alpha+1/2}(\sin(\theta/2)+n^{-1})^{\alpha-1/2}.\end{aligned}$$

Together, these estimates show that $I^{(\alpha,\beta)}(\theta)$ has the desired estimate for $0 \le \theta \le \pi/2$.

Case 2. $\pi/2 \le \theta \le \pi$. Let $\theta' = \pi-\theta$. Then $0 \le \theta' \le \pi/2$. Upon change variable $\phi \mapsto \pi-\phi$ in the integral, we conclude that

$$I^{(\alpha,\beta)}(\theta) = \int_0^\pi \frac{(\sin\frac{\theta'}{2}+\sin\frac{\phi}{2}+n^{-1})^{\alpha-\beta+1}}{(|\sin\frac{\theta'-\phi}{2}|+n^{-1})^{\alpha+3/2}(|\sin\frac{\theta'+\phi}{2}|+n^{-1})^{\alpha+3/2}}\cos^{2\alpha+1}\frac{\phi}{2}\sin^{2\beta+1}\frac{\phi}{2}d\phi.$$

If $0 \le \theta' \le n^{-1}$, then θ' can be dropped in all sine functions in the integral, and we have

$$I^{(\alpha,\beta)}(\theta) \le \ c \int_0^{\pi/2} \frac{\sin^{2\beta+1}\frac{\phi}{2}}{(\sin\frac{\phi}{2}+n^{-1})^{\alpha+\beta+2}} d\phi + \ c \int_{\pi/2}^{\pi} \cos^{2\alpha+1}\frac{\phi}{2} d\phi \le \ c\, n^{\alpha-\beta},$$

where if $\beta < -1/2$, the first integral is estimated by splitting it to two integrals over $[0, n^{-1}]$ and $[n^{-1}, \pi/2]$, respectively. Since $0 \le \theta' \le n^{-1}$, $\cos(\theta/2) = \sin(\theta'/2) \le n^{-1}$, so that $c\, n^{\alpha-\beta}$ gives the desired bound. For $n^{-1} \le \theta' \le \pi/2$, we can estimate the integral as in Case 1. On the interval $[0, 3\theta'/2]$, in addition to the relations used before, we have $\sin\theta'/2 + \sin\phi/2 + n^{-1} \sim \theta' + n^{-1}$. Hence,

$$\begin{aligned}\int_0^{3\theta'/2} &\le c\,(\theta'+n^{-1})^{-\beta-1/2} \int_0^{3\theta'/2} \frac{\phi^{2\beta+1}}{(|\theta'-\phi|+n^{-1})^{\alpha+3/2}} d\phi \\ &\le c\, n^{\alpha+1/2}(\theta'+n^{-1})^{\beta+1/2},\end{aligned}$$

where the estimate is straightforward for $\beta \ge -1/2$, and for $\beta < -1/2$, we split the last integral as two integrals over $[0, \theta'/2 + n^{-1}]$ and $[\theta'/2 + n^{-1}, 3\theta'/2]$, respectively, and use $(|\theta'-\phi|+n^{-1})^{\alpha+3/2} \ge c(\theta'+n^{-1})^{\alpha+3/2}$ on the first interval and $\phi^{2\beta+1} \le c(\theta'+n^{-1})^{2\beta+1}$ on the second interval. Since $\cos\theta/2 = \sin\theta'/2 \sim \theta'$, we have shown that the integral over $[0, 3\theta'/2]$ has the desired bound. On the interval $[3\theta'/2, \pi]$, in addition to the elementary relations used in Case 1, we have $\sin\theta'/2 + \sin\phi/2 \sim \phi$. Hence,

$$\begin{aligned}\int_{3\theta'/2}^{\pi} &\le c \int_{3\theta'/2}^{3\pi/4} \frac{\phi^{2\beta+1}}{(\phi+n^{-1})^{\alpha+\beta+2}} d\phi + c \int_{3\pi/4}^{\pi} (\cos\phi/2)^{2\alpha+1} d\phi \\ &\le c\big[(\theta'+n^{-1})^{\beta-\alpha}+1\big] \ \le \ c\, n^{\alpha+1/2}(\theta'+n^{-1})^{\beta+1/2},\end{aligned}$$

where we use the fact that $(\phi+n^{-1})/2 \le \phi \le (\phi+n^{-1})$, since $\phi \ge 3\theta'/2 \ge n^{-1}$, to estimate the integral over $[3\theta'/2, 3\pi/4]$.

We still have to consider the case $\alpha = \beta$. If $\alpha > -1/2$, then we can estimate the integral just as before. In fact, the case $0 \le \theta \le \pi/2$ is identical to the estimate in the Case 1 of $\alpha \ne \beta$ and the case $\pi/2 \le \theta \le \pi$ follows from the identity

$$I^{(\alpha,\alpha)}(\theta)/(\cos(\theta/2)+n^{-1}) = I^{(\alpha,\alpha)}(\pi-\theta)/(\sin\theta/2+n^{-1}).$$

Finally, in the case $\alpha = \beta = -1/2$, we have an additional $\log(2+n\theta)$ factor as mentioned in the Case 1 of $\alpha \ne \beta$. $\square$

We are now in position to prove Theorem 1.1.

Proof of Theorem 1.1. By Lemma 2.2, it suffices to establish (2.6) under the condition (1.3) and $\alpha_j \ge \beta_j$, $1 \le j \le d$. Moreover, the proof of Lemma 2.2 shows that we can assume that $\alpha_j \ge \beta_j$ for $1 \le j \le d$. Let m be the number of pairs $\{\alpha_j, \beta_j\}$ such that

$\alpha_j = \beta_j = -1/2$. Throughout the rest of the section, we write $\rho(\alpha) = \sum_{j=1}^d \alpha_j$. Then, it follows from Lemma 3.1 and Lemma 3.2 that

$$\int_{[-1,1]^d} |K_n^\delta(W_{\alpha,\beta}; \mathbf{x}, \mathbf{e})| W_{\alpha,\beta}(\mathbf{x}) d\mathbf{x} \le \frac{c}{n^\delta} \int_0^\pi (\sin(\theta/2) + n^{-1})^{d-\delta-1} \prod_{j=1}^d I^{(\alpha_j,\beta_j)}(\theta) d\theta$$
$$\le c\, n^{\rho(\alpha)+d/2-\delta} \int_0^\pi (\sin(\theta/2) + n^{-1})^{\rho(\alpha)+d/2-\delta-1}$$
$$\times (\cos(\theta/2) + n^{-1})^{\rho(\beta)+d/2} \log^m(2 + n\theta)\, d\theta,$$

which, upon splitting the integral to two integrals over $[0, \pi/2]$ and $[\pi/2, \pi]$, respectively, and changing variable in the integral over $[0, \pi/2]$, becomes

$$\le c \int_0^{n\pi/2} (u + 1)^{\rho(\alpha)+d/2-\delta-1} \log^m(1 + u) du$$
$$+ c\, n^{\rho(\alpha)+d/2-\delta} \log^m n \int_{\pi/2}^\pi (\cos(\theta/2) + n^{-1})^{\rho(\beta)+d/2} d\theta.$$

The first integral is finite if and only if $\rho(\delta) + d/2 - \delta < 0$, or $\delta > \rho(\alpha) + d/2$. Under this condition, the second integral is also bounded if $\rho(\beta) + d/2 \ge -1$. However, if $\rho(\beta) + d/2 < -1$, then the second integral is bounded by

$$c\, n^{\rho(\alpha)+d/2-\delta} n^{-\rho(\beta)-d/2-1} \log^m n = c\, n^{\rho(\alpha)-\rho(\beta)-\delta-1} \log^m n,$$

which is bounded if $\delta > \rho(\alpha) - \rho(\beta) - 1$, or $\delta \ge \rho(\alpha) - \rho(\beta) - 1$ when $m = 0$. The condition $\rho(\beta) + d/2 < -1$ implies that $\rho(\alpha) - \rho(\beta) - 1 > \rho(\alpha) + d/2$. $\square$

It is worthwhile to mention that the above proof gives an alternative proof for the Theorem 9.1.3 of [20, p. 246].

Proof of Theorem 1.2. By Lemma 2.2, we only need to prove that, for $\alpha_j \ge \beta_j$, the kernel $K_{n,d}^\delta(W_{\alpha,\beta}; \mathbf{x}, \mathbf{e}) \ge 0$ if and only if $\delta \ge \rho(\alpha) + \rho(\beta) + 3d - 1$. For $d = 1$, it follows from [11] that the $(C, \alpha + \beta + 2)$ means $K_{n,1}^{\alpha+\beta+2}(w_{\alpha,\beta}; x, 1)$ is nonnegative for $-1 \le x \le 1$. Hence, by (2.12) with $d = 1$, the function $(1 - r)^{-\alpha-\beta-3} G^{(\alpha,\beta)}(r; x)$ is a *completely monotone function* of r; that is, a function whose power series has all nonnegative coefficients. Since multiplication is closed in the space of complete monotonic functions, it follows that

$$(1 - r)^{-\rho(\alpha)-\rho(\beta)-3d} G_d^{(\alpha,\beta)}(r; \mathbf{x}) = \prod_{j=1}^d (1 - r)^{-\alpha_j-\beta_j-3} G_d^{(\alpha_j,\beta_j)}(r; x_j)$$

is a complete monotone function. Consequently, by (2.12), we conclude that the means $K_{n,d}^{\rho(\alpha)+\rho(\beta)+3d-1}(W_{\alpha,\beta}; \mathbf{x}, \mathbf{e}) \ge 0$. We now prove that the order of summation can not be improved. If (C, δ_0) means are positive, then (C, δ) means are positive for $\delta \ge \delta_0$. Hence, we only need to show that the $(C, \rho(\alpha) + \rho(\beta) + 3d - 1 - \varepsilon)$ means of the kernel is not positive for $0 < \varepsilon < 1$. From (2.9) and the fact that ${}_2F_1(a,b;c;0) = 1$, we conclude that for $\delta = \rho(\alpha) + \rho(\beta) + 3d - 2$,

$$(1-r)^{-\delta-1} G_d^{(\alpha,\beta)}(r;-\mathbf{e}) = (1-r)(1-r^2)^{-\rho(\alpha)-\rho(\beta)-2d}$$
$$= \sum_{k=0}^{\infty} \binom{\rho(\alpha)+\rho(\beta)+2d+k-1}{k} r^{2k} - \sum_{k=0}^{\infty} \binom{\rho(\alpha)+\rho(\beta)+2d+k-1}{k} r^{2k+1}.$$

Therefore, setting $A_k = \binom{k+\rho(\alpha)+\rho(\beta)+3d-2}{k} K_{k,d}^{\rho(\alpha)+\rho(\beta)+3d-2}(W_{\alpha,\beta}; -\mathbf{e}, \mathbf{e})$ and comparing with (2.12), we conclude that

$$A_{2k} = -A_{2k+1} = \binom{\rho(\alpha)+\rho(\beta)+2d+k-1}{k} \geq 0$$

Therefore, it follows that

$$\binom{2n+1+\rho(\alpha)+\rho(\beta)+3d-1-\varepsilon}{2n-1} K_{2n+1,d}^{\rho(\alpha)+\rho(\beta)+3d-1-\varepsilon}(W_{\alpha,\beta}; -\mathbf{e}, \mathbf{e})$$
$$= \sum_{k=0}^{2n+1} \binom{2n-k-\varepsilon}{2n+1-k} A_k = -\varepsilon \sum_{k=0}^{n} \frac{1}{2n-2k+1} \binom{2n-2k-\varepsilon}{2n-2k} A_{2k}.$$

Since $0 < \varepsilon < 1$, we conclude that the $(C, \rho(\alpha) + \rho(\beta) + 3d - 1 - \varepsilon)$ means are not positive. ☐

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