SUMMABILITY OF ORTHOGONAL EXPANSIONS OF SEVERAL VARIABLES

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ABSTRACT. Summability of spherical h-harmonic expansions with respect to the weight function $\prod_{j=1}^{d}|x_j|^{2\kappa_j}$ $(\kappa_j \ge 0)$ on the unit sphere S^{d-1} is studied. The main result characterizes the critical index of summability of the Cesàro (C,δ) means of the h-harmonic expansion; it is proved that the (C,δ) means of any continuous function converge uniformly in the norm of $C(S^{d-1})$ if and only if $\delta > (d-2)/2 + \sum_{j=1}^{d}\kappa_j - \min_{1\le j\le d}\kappa_j$. Moreover, it is shown that for each point not on the great circles defined by the intersection of the coordinate planes and S^{d-1}, the (C,δ) means of the h-harmonic expansion of a continuous function f converges pointwisely to f if $\delta > (d-2)/2$. Similar results are established for the orthogonal expansions with respect to the weight functions $\prod_{j=1}^{d}|x_j|^{2\kappa_j}(1-|\mathbf{x}|^2)^{\mu-1/2}$ on the unit ball B^d and $\prod_{j=1}^{d}x_j^{\kappa_j-1/2}(1-|\mathbf{x}|_1)^{\mu-1/2}$ on the simplex T^d. As a related result, the Cesàro summability of the generalized Gegenbauer expansions associated to the weight function $|t|^{2\mu}(1-t^2)^{\lambda-1/2}$ on $[-1,1]$ is studied, which is of interest in itself.

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1. Introduction

Ordinary spherical harmonics are homogeneous polynomials orthogonal with respect to the Lebesgue measure on the unit sphere S^{d-1} of $\mathbb{R}^d$, which is the unique measure on S^{d-1} invariant under the orthogonal group. The fact that the spherical harmonics are associated with the orthogonal group plays an essential role in studying the summability of spherical harmonic expansions; it allows us to invoke the techniques in classical Fourier analysis; see, for example, [1, 11].

The objective of the present paper is to study the Fourier expansions in h-harmonics, which are homogeneous polynomials orthogonal with respect to the measure $h_\kappa^2 d\omega$ on S^{d-1}, where

$$h_\kappa(\mathbf{x}) = |x_1|^{\kappa_1} \cdots |x_d|^{\kappa_d}, \qquad \kappa_i \ge 0, \tag{1.1}$$

and $d\omega$ denotes the ordinary surface measure on S^{d-1}. The measure is invariant under the group $\mathbb{Z}_2^d$, a subgroup of the orthogonal group. The theory of h-harmonics is developed by Dunkl ([4, 5, 6] and references therein) for measures invariant under finite reflection groups. The abelian group $\mathbb{Z}_2^d$ is the simplest example. In the general setting, let G be a finite reflection group with positive roots R_+. For $\mathbf{v} \in \mathbb{R}^d$, let $\sigma_{\mathbf{v}}\mathbf{x} = \mathbf{x} - 2(\langle \mathbf{x}, \mathbf{v}\rangle / \langle \mathbf{v}, \mathbf{v}\rangle)\mathbf{v}$ denote the reflection with respect to the hyperplane perpendicular to $\mathbf{v}$, where $\langle \mathbf{x}, \mathbf{v}\rangle$ is the Euclidean inner product of $\mathbb{R}^d$. Let κ be a nonnegative multiplicative function $\mathbf{v} \mapsto \kappa_{\mathbf{v}}$ defined on R_+ with the property that $\kappa_{\mathbf{v}} = \kappa_{\mathbf{u}}$ whenever $\sigma_{\mathbf{v}}$ is conjugate to $\sigma_{\mathbf{u}}$ in G, that is, when there exists $g \in G$ such that $\mathbf{u}g = \mathbf{v}$. Then the measure h_κ invariant under G is defined by

$$h_\kappa(\mathbf{x}) = \prod_{\mathbf{v} \in R_+} |\langle \mathbf{x}, \mathbf{v}\rangle|^{\kappa_{\mathbf{v}}}. \tag{1.2}$$

If $G = \mathbb{Z}_2^d$, (1.2) becomes (1.1). The theory of h-harmonics is in many ways comparable to the theory of the ordinary harmonics. There is a family of commuting operators, $\mathcal{D}_i$ (Dunkl's operators), defined by

$$\mathcal{D}_i f(\mathbf{x}) := \partial_i f(\mathbf{x}) + \sum_{\mathbf{v} \in R_+} \kappa_{\mathbf{v}} \frac{f(\mathbf{x}) - f(\mathbf{x}\sigma_{\mathbf{v}})}{\langle \mathbf{x}, \mathbf{v}\rangle} \langle \mathbf{v}, \mathbf{e}_i\rangle, \qquad 1 \le i \le d,$$

where ∂_i is ordinary partial derivative with respect to x_i and $\mathbf{e}_1, \ldots, \mathbf{e}_d$ are the standard unit vectors of $\mathbb{R}^d$. The h-harmonics are homogeneous polynomials satisfying the equation $\Delta_h p = 0$, where $\Delta_h = \mathcal{D}_1^2 + \ldots + \mathcal{D}_d^2$ is the analogous of the usual Laplace operator. Let $\mathcal{P}_n^d$ denote the space of homogeneous polynomials of degree n in d variables, and let $\mathcal{H}_n^d(h_\kappa^2) \subset \mathcal{P}_n^d$ denote the space of h-harmonic polynomials of degree n. If all $\kappa_i = 0$, then $\mathcal{H}_n^d(h_\kappa^2)$ is the space of ordinary harmonics. It is known that

$$\int_{S^{d-1}} p q h_\kappa^2 d\omega = 0, \qquad p \in \mathcal{H}_n^d(h_\kappa^2), \quad q \in \Pi_{n-1}^d,$$

where Π_m^d denotes the set of polynomials of degree at most m in d variables. Moreover, we have

$$\dim \mathcal{P}_n^d = \binom{n+d-1}{d} \quad \text{and} \quad \dim \mathcal{H}_n^d(h_\kappa^2) = \dim \mathcal{P}_n^d - \dim \mathcal{P}_{n-2}^d.$$

Let $\{Y_{n,i}\}$ denote an orthonormal basis of $\mathcal{H}_n^d(h_\kappa^2)$. The reproducing kernel of the space $\mathcal{H}_n^d(h_\kappa^2)$ is defined by

$$P_n(h_\kappa^2;\mathbf{x},\mathbf{y}) = \sum_{i=1}^{N_n} Y_{n,i}(\mathbf{x})Y_{n,i}(\mathbf{y}), \qquad N_n = \dim \mathcal{H}_n^d(h_\kappa^2).$$

For $f \in L^2(h_\kappa^2; S^{d-1})$, we denote its projection to $\mathcal{H}_n^d(h_\kappa^2)$ by $P_n(h_\kappa^2; f)$. It follows that

$$P_n(h_\kappa^2; f, \mathbf{x}) = \int_{S^{d-1}} f(\mathbf{y})P_n(h_\kappa^2;\mathbf{x},\mathbf{y})h_\kappa^2(\mathbf{y})d\omega(\mathbf{y}). \tag{1.3}$$

Although orthonormal bases of $\mathcal{H}_n^d(h_\kappa^2)$ are not unique, the projection operator is independent of the bases and is uniquely defined; moreover, the reproducing kernel $P_n(h_\kappa^2;\mathbf{x},\mathbf{y})$ is also unique. For $f \in L^2(h_\kappa^2; S^{d-1})$, its h-harmonic expansion is defined uniquely by

$$f(\mathbf{x}) = \sum_{n=0}^{\infty} P_n(h_\kappa^2; f, \mathbf{x}).$$

We note that this expansion is independent of the choice of orthonormal bases. For the ordinary harmonics, such an expansion is called the Laplace series (cf. [7, Chapt. 12]). As in the case of ordinary harmonics, if f is merely continuous, the partial sums of the expansion do not converge uniformly in general and we need to consider the summability method such as the Cesàro (C,δ) means.

For $\delta > 0$, the Cesàro (C,δ) means, s_n^δ, of a sequence $\{s_n\}$ are defined by

$$s_n^\delta = \frac{1}{\binom{n+\delta}{n}} \sum_{k=0}^{n} \binom{n-k+\delta-1}{n-k} s_k = \frac{1}{\binom{n+\delta}{n}} \sum_{k=0}^{n} \binom{n-k+\delta}{n-k} c_k,$$

where the second equality holds if s_n is the n-th partial sum of the series $\sum_{k=0}^{\infty} c_k$. We say that $\{s_n\}$ is Cesàro (C,δ) summable to s if s_n^δ converges to s as $n \to \infty$.

In order to study the summability of the orthogonal expansion, we need to have knowledge of the reproducing kernel. The compact formula of the reproducing kernel for h-harmonics associated to h_κ^2 for any finite reflection group is known to be ([15])

$$P_n(h_\kappa^2;\mathbf{x},\mathbf{y}) = \frac{n + |\kappa|_1 + (d-2)/2}{|\kappa|_1 + (d-2)/2} V[C_n^{(|\kappa|_1+(d-2)/2)}(\langle \cdot, \mathbf{y}\rangle)](\mathbf{x}), \tag{1.4}$$

where $V : \Pi^d \mapsto \Pi^d$ is the so-called intertwining operator between the commutative algebras generated by the partial derivatives and that generated by the Dunkl operators, $C_n^{(\lambda)}$ denotes the Gegenbauer polynomial of degree n with index λ, and $|\kappa|_1 = \sum_{\mathbf{v}\in R_+} \kappa_{\mathbf{v}}$. However, an explicit formula of V is known only in the case of symmetric group S_3 and in the case of $\mathbb{Z}_2^d$. In the latter case, the formula of V ([14]) leads to an explicit formula for the reproducing kernel; more precisely, for the weight function h_κ^2 in (1.1), the n-th reproducing kernel function satisfies the following formula ([14])

$$\begin{aligned} P_n(h_\kappa^2;\mathbf{x},\mathbf{y}) &= c_\kappa \frac{n + |\kappa|_1 + (d-2)/2}{|\kappa|_1 + (d-2)/2} \\ &\times \int_{[-1,1]^d} C_n^{(|\kappa|_1+(d-2)/2)}(x_1y_1t_1 + \ldots + x_dy_dt_d) \prod_{i=1}^{d} (1+t_i)(1-t_i^2)^{\kappa_i - 1} d\mathbf{t}, \end{aligned} \tag{1.5}$$

where $|\kappa|_1 = \sum_{j=1}^d \kappa_j$ and c_κ denotes the constant

$$c_\kappa = c_{\kappa_1} \dots c_{\kappa_d}, \qquad \text{where} \quad c_r = \Big(\int_{-1}^1 (1-t^2)^{r-1} dt \Big)^{-1}.$$

If some $\kappa_i = 0$, then the formula holds under the limit relation

$$\lim_{\lambda \to 0} c_\lambda \int_{-1}^1 f(t)(1-t)^{\lambda-1} dt = [f(1) + f(-1)]/2. \tag{1.6}$$

When all $\kappa_i = 0$, we have $V = id$ and (1.5) reduces to the usual zonal polynomial for the ordinary spherical harmonics

$$P_n(\mathbf{x}, \mathbf{y}) = \frac{n + (d-2)/2}{(d-2)/2} C_n^{((d-2)/2)}(\langle \mathbf{x}, \mathbf{y} \rangle).$$

For the Cesàro summability, it is proved in [15] that for h_κ^2 in (1.2) and any finite reflection group, the h-harmonic expansion of a continuous function on S^{d-1} is uniformly (C, δ) summable if

$$\delta > \frac{d-2}{2} + \sum_{\mathbf{v} \in R_+} \kappa_{\mathbf{v}}. \tag{1.7}$$

The proof uses the formula (1.4) and an integral formula of the intertwining operator V, which reduces the problem to the summability of Gegenbauer expansion in one variable, just as in the case of ordinary harmonics; that is, the proof reduces the convergence over S^{d-1} to convergence at just one point (say, north pole). However, reducing to one point is reasonable for the ordinary spherical harmonic expansion, since the orthogonal group acts transitively on S^{d-1}; but it is not as natural for the h-harmonic expansion, since the subgroup $\mathbb{Z}_2^d$ no longer acts transitively on S^{d-1}. Moroever, for d=2, the h-harmonic expansion on S^1 corresponds to the orthogonal expansion with respect to the Jacobi weight function $(1-t)^{\kappa_1}(1+t)^{\kappa_2}$ on (-1,1). The result in [12] and [9] shows that the critical index of (C, δ) summability is $\max\{\kappa_1, \kappa_2\}$, which is smaller than $\kappa_1 + \kappa_2$ in (1.7). The main result of the present paper is to characterize the critical index of the (C, δ) means for h_κ associated with $\mathbb{Z}_2^d$: the h-harmonic expansion of every continuous function on S^{d-1} with respect to (1.1) is uniformly (C, δ) summable if and only if

$$\delta > \frac{d-2}{2} + \sum_{i=1}^d \kappa_i - \min_{1 \le i \le d} \kappa_i$$

(see Theorem 2.1 in the following section). The proof is based on an accurate estimate of the Cesàro means of the reproducing kernel. The integrals in the formula (1.5) of $P_n(h_\kappa^2)$ make this possible, but the task is much more difficult comparing with the case of ordinary harmonics; the hard estimate is the major technical part of the paper. Our investigation also uncovers other interesting phenomena that do not appear in the study of ordinary harmonics; for example, we shall show that the space S^{d-1} equipped with the measure $h_\kappa^2 d\omega$ has a boundary consisting of the great circles defined by the intersection of S^{d-1} with the coordinate planes (that is, the zero set of h_κ in (1.1)), and the pointwise summability on the boundary is worse than that in the interior.

There is another reason that we pay special attention to the weight function h_κ in (1.1). Recently in [16, 17] we have shown that orthogonal polynomials on the sphere S^d and those on the unit ball

$$B^d = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| \le 1\}$$

and on the simplex

$$T^d = \{\mathbf{x} \in \mathbb{R}^d : x_1 \ge 0, \ldots, x_d \ge 0, 1 - x_1 - \ldots - x_d \ge 0\}$$

are closely related. In particular, the h-harmonics associated with h_κ in (1.1) in $d+1$ variables are related to the orthogonal polynomials with respect to the weight function

$$W^B_{\kappa,\mu}(\mathbf{x}) = \prod_{i=1}^d |x_i|^{2\kappa_i}(1-|\mathbf{x}|^2)^{\mu-1/2} \tag{1.8}$$

on B^d, where $|\mathbf{x}|^2 = x_1^2 + \ldots + x_d^2$, and those with respect to the weight function

$$W^T_{\kappa,\mu}(\mathbf{x}) = \prod_{i=1}^d x_i^{\kappa_i-1/2}(1-|\mathbf{x}|_1)^{\mu-1/2} \tag{1.9}$$

on the simplex T^d, where $|\mathbf{x}|_1 = x_1 + \ldots + x_d$ for $\mathbf{x} \in T^d$. The orthogonal polynomials with respect to the weight function $W^B_\mu(\mathbf{x}) = (1-|\mathbf{x}|^2)^{\mu-1/2}$ (the case $\kappa = 0$ on $W^B_{\kappa,\mu}$) and $W^T_{\kappa,\mu}$ are the classical orthogonal polynomials, since they are eigenfunctions of a second order differential operators (see [7, Chapt. 12]).

Let $\mathcal{V}^d_n(W^\Omega_{\kappa,\mu})$ denote the space of polynomials of degree n that are orthogonal to polynomials of lower degrees with respect to $W^\Omega_{\kappa,\mu}$ on Ω, where $\Omega = B^d$ or $\Omega = T^d$. It is known that $\dim \mathcal{V}^d_n(W^\Omega_{\kappa,\mu}) = \dim \mathcal{P}^d_n$. Let $\{P^n_\alpha\}_{|\alpha|=n}$ denote an orthonormal basis of $\mathcal{V}^d_n(W^\Omega_{\kappa,\mu})$, where $\alpha \in \mathbb{N}^d_0$. Then the reproducing kernel of $\mathcal{V}^d_n(W^\Omega_{\kappa,\mu})$ is defined by

$$\mathbf{P}_n(W^\Omega_{\kappa,\mu}; \mathbf{x}, \mathbf{y}) = \sum_{|\alpha|=n} P^n_\alpha(\mathbf{x}) P^n_\alpha(\mathbf{y}).$$

For $f \in L^2(W^\Omega_{\kappa,\mu})$, we denote its projection to $\mathcal{V}^d_n(W^\Omega_{\kappa,\mu})$ by $\mathbf{P}_n(W^\Omega_{\kappa,\mu}; f)$. It follows that

$$\mathbf{P}_n(W^\Omega_{\kappa,\mu}; f, \mathbf{x}) = \int_\Omega f(\mathbf{y}) \mathbf{P}_n(W^\Omega_{\kappa,\mu}; \mathbf{x}, \mathbf{y}) W^\Omega_{\kappa,\mu}(\mathbf{y}) d\mathbf{y}. \tag{1.10}$$

Although orthonormal bases of $\mathcal{V}^d_n(W^\Omega_{\kappa,\mu})$ are not unique, the projection operator is independent of the bases and is uniquely defined; moreover, the reproducing kernel $\mathbf{P}_n(W^\Omega_{\kappa,\mu})$ is also unique. For $f \in L^2(W^\Omega_{\kappa,\mu})$, its Fourier expansion in terms of the associated orthogonal polynomials is defined uniquely by

$$f(\mathbf{x}) = \sum_{n=0}^\infty \mathbf{P}_n(W^\Omega_{\kappa,\mu}; f, \mathbf{x}),$$

which, in turn, is independent of the choice of orthonormal bases.

The study of the Cesàro (C, δ) summability of the classical orthogonal expansion for W^B_μ on B^d begain with the work of Chen, Koschmieder and others (see [7, Chapt. 12]), but the necessary and sufficient condition was found only recently in [19]. For the summability of the classical orthogonal expansion on the simplex T^d, it is proved in [18] that the uniform (C, δ) summability holds if $\delta > |\kappa|_1 + (d-1)/2$ which, however, is not sharp. In the present paper, among other results, we shall give necessary and sufficient conditions for the uniform summability for both $W^B_{\kappa,\mu}$ on B^d and $W^T_{\kappa,\mu}$ on T^d.

This is possible since recently in [16, 17] we have shown that orthogonal polynomials on the sphere S^d and those on the unit ball and on the simplex are closely related. In particular, the h-harmonics associated with h_κ in (1.1) in $d+1$ variables are related to the orthogonal polynomials with respect to the weight function $W^B_{\kappa,\mu}$ and $W^T_{\kappa,\mu}$. In the case of B^d, it is shown in [20] that

$$\begin{aligned}(1.11)\qquad \mathbf{P}_n(W^B_{\kappa,\mu};\mathbf{x},\mathbf{y}) &= c_\kappa c_\mu \frac{n+|\kappa|_1+\mu+(d-1)/2}{|\kappa|_1+\mu+(d-1)/2} \\ &\times \int_{-1}^1\int_{[-1,1]^d} C_n^{(|\kappa|_1+\mu+\frac{d-1}{2})}(x_1y_1\,t_1+\ldots+x_dy_dt_d+s\sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}) \\ &\qquad\times\prod_{i=1}^d(1+t_i)(1-t_i^2)^{\kappa_i-1}d\mathbf{t}(1-s^2)^{\mu-1}ds.\end{aligned}$$

In fact, the summability of the orthogonal expansion on B^d follows from that of the corresponding orthogonal expansion on S^d (see Section 3 below and [20]). Consequently, our study of the h-harmonic expansion for h_κ^2 will yield results for the orthogonal expansion with respect to the weight function $W^B_{\kappa,\mu}$ on B^d. This will give the sufficient part of our necessary and sufficient result on the uniform convergence of the (C,δ) means on B^d (Theorem 2.4). On the other hand, the necessary part of the theorem goes the other way: the necessity of the condition for h-harmonic expansions will follow from that for the orthogonal expansion with respect to $W^B_{\kappa,\mu}$.

The result turns out to be new even in the case of $d=1$. Indeed, for $d=1$, we are dealing with orthogonal expansion associated with the weight function

$$(1.12)\qquad w_{\lambda,\mu}(t)=\frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda+1/2)\Gamma(\mu+1/2)}|t|^{2\mu}(1-t^2)^{\lambda-1/2}$$

on $[-1,1]$. The orthogonal polynomials associated to $w_{\lambda,\mu}$ are called the generalized Gegenbauer polynomials; they are related to the Jacobi polynomials $P_n^{(\alpha,\beta)}(t)$ associated with the weight function $(1-t)^\alpha(1+t)^\beta$ as follows:

$$(1.13)\qquad \begin{aligned} C_{2n}^{(\lambda,\mu)}(x) &= \frac{(\lambda+\mu)_n}{(\mu+\frac12)_n}P_n^{(\lambda-1/2,\mu-1/2)}(2x^2-1),\\ C_{2n+1}^{(\lambda,\mu)}(x) &= \frac{(\lambda+\mu)_{n+1}}{(\mu+\frac12)_{n+1}}xP_n^{(\lambda-1/2,\mu+1/2)}(2x^2-1).\end{aligned}$$

As it is shown in [14], an orthonormal basis of the h-spherical harmonics associated with $h_{\lambda,\mu}(\mathbf{x})=|x_1|^\lambda|x_2|^\mu$ can be given in terms of the generalized Gegenbauer polynomials. Moreover, let $\tilde C_n^{(\lambda,\mu)}$ denote the normalized generalized Gegenbauer polynomials; then the formula (1.11) reduces to

$$\begin{aligned}(1.14)\qquad \tilde C_n^{(\lambda,\mu)}(x)\tilde C_n^{(\lambda,\mu)}(y) &= \frac{n+\lambda+\mu}{\lambda+\mu}c_\lambda c_\mu \\ \times\int_{-1}^1\int_{-1}^1 C_n^{(\lambda+\mu)}&(txy+s\sqrt{1-x^2}\sqrt{1-y^2})(1+t)(1-t^2)^{\mu-1}(1-s^2)^{\lambda-1}dtds\end{aligned}$$

(see [14]). Although the generalized Gegenbauer polynomials are related to the Jacobi polynomials, the (C,δ) summability of the generalized Gegenbauer expansion and that of the Jacobi expansion do not follow from each other, since (C,δ) means of a sequence s_n are in general not related to the (C,δ) means of s_{2n}. Our result gives the necessary and sufficient condition for the (C,δ) summability of the generalized Gegenbauer expansion. It is used, in turn, in the proof of the necessity in the case of S^{d-1} and B^d (see Section 3). The summability of the generalized Gegenbauer expansions was studied from the point of view of positivity in [8], where a positive convolution structure was defined. For the most part, such a structure can be derived from the explicit formula (1.14) and it does not give the sharp result for the (C,δ) means.

For the weight function $W^T_{\kappa,\mu}$, the relation between the h-harmonics and the orthogonal polynomials with respect to $W^T_{\kappa,\mu}$ leads to a compact formula for the reproducing kernel $\mathbf{P}_n(W^T_{\kappa,\mu};\mathbf{x},\mathbf{y})$ ([18])

$$\mathbf{P}_n(W^T_{\kappa,\mu};\mathbf{x},\mathbf{y}) = c_\kappa c_\mu \frac{2n+|\kappa|_1+\mu+(d-1)/2}{|\kappa|_1+\mu+(d-1)/2} \tag{1.15}$$
$$\times \int_{-1}^{1}\int_{[-1,1]^d} C_{2n}^{(|\kappa|_1+\mu+\frac{d-1}{2})}(\sqrt{x_1y_1}\,t_1+\ldots+\sqrt{x_dy_d}t_d+s\sqrt{1-|\mathbf{x}|_1}\sqrt{1-|\mathbf{y}|_1})$$
$$\times \prod_{i=1}^{d}(1-t_i^2)^{\kappa_i-1}d\mathbf{t}(1-s^2)^{\mu-1}ds.$$

However, the case of the orthogonal expansion with respect to $W^T_{\kappa,\mu}$ on T^d is more complicated; its summability does not follow directly from that of h-harmonics. Consequently, it is necessary to derive accurate estimates for the kernel function. There are clearly similarities between (1.5) and (1.15), so that the (C,δ) means of the kernel $\mathbf{P}_n(W^T_{\kappa,\mu};\mathbf{x},\mathbf{y})$ can be estimated using a similar technique as in the case of h-harmonics. There are, however, additional difficulties that lead to a certain restriction on the parameters in our results for $W^T_{\kappa,\mu}$.

The paper is organized as follows. In the next section we state and discuss the main results of the paper. The proof of these results are given in Section 3, assuming a result on the generalized Gegenbauer polynomials that is needed for the necessary part of Theorem 2.4 and the estimate of the (C,δ) kernel in the case of $W^T_{\kappa,\mu}$. The result on the general Gegenbauer polynomials is proved in Section 4, which amounts to prove a lower bound for a double integral of the Jacobi polynomials. The estimate of the (C,δ) kernel of the of h-harmonic expansion is given in Section 5 and the estimate of the kernel in the case of $W^T_{\kappa,\mu}$ is given in Section 6.

2. Main Resultsk

2.1. **h-harmonic expansion.** Let $S_n^\delta(h_\kappa^2;f)$ denote the Cesàro (C,δ) means of the Fourier series of f in h-harmonics. It follows from (1.3) that we can write

$$S_n^\delta(h_\kappa^2;f,\mathbf{x}) = \int_{S^{d-1}} f(\mathbf{y})K_n^\delta(h_\kappa^2;\mathbf{x},\mathbf{y})h_\kappa^2(\mathbf{y})d\omega(\mathbf{y}),$$

where $K_n^\delta(h_\kappa^2;\mathbf{x},\mathbf{y})$ denote the Cesàro (C,δ) means of the kernel $P_n(h_\kappa^2;\mathbf{x},\mathbf{y})$,

$$K_n^\delta(h_\kappa^2;\mathbf{x},\mathbf{y}) = \frac{1}{\binom{n+\delta}{n}}\sum_{k=0}^{n}\binom{n-k+\delta}{n-k}P_k(h_\kappa^2;\mathbf{x},\mathbf{y}).$$

Our first result is about uniform summability of the h-harmonic expansion.

Theorem 2.1. *The (C,δ) means of the h-harmonic expansion of every continuous function f with respect to h_κ^2 in (1.1) converge uniformly to f on S^{d-1} if and only* *if*

$$\delta > (d-2)/2 + |\kappa|_1 - \min_{1\le i\le d} \kappa_i. \tag{2.1}$$

Let $L^p(h_\kappa^2; S^{d-1})$, $1 \le p < \infty$, denote the weighted L^p space. An immediate corollary of Theorem 2.1 is the following result.

Corollary 2.2. *The (C,δ) means of the h-harmonic expansion of a function f for h_κ^2 in (1.1) converge to f in the $L^p(h_\kappa^2; S^{d-1})$ norm, $1 \le p < \infty$, or $C(S^d)$ norm for $p = \infty$, provided (2.1) holds. For $p = 1$ and $p = \infty$, the condition (2.1) is also necessary.*

The proof of Theorem 2.1 amounts to show that the Lebesgue functions

$$I_n(\mathbf{x}) := \int_{S^{d-1}} |K_n^\delta(h_\kappa^2; \mathbf{x}, \mathbf{y})| h_\kappa^2(\mathbf{y}) d\omega$$

are uniformly bounded in n for all $\mathbf{x} \in S^{d-1}$ if and only if (2.1) holds. If one of the κ_i is zero, then the simple proof [15] applies, which reduces the proof to that of Gegenbauer expansion in one variable, similar to the case of ordinary spherical harmonics. However, as discussed in the introduction, if none of the κ_i is zero, then the proof of the sharp result in Theorem 2.1 can no longer be reduced to that of one variable. Indeed, the sufficient part of the proof is based on an accurate estimate of the kernel $K_n^\delta(h_\kappa^2; \mathbf{x}, \mathbf{y})$, proved using the explicit formula (1.5) of the reproducing kernel. The proof of the necessity follows from evaluating $I_n(\mathbf{x})$ at the points of intersection of certain great circles, which are defined by the intersection of S^{d-1} and the coordinate planes. In fact, these great circles are like boundaries on S^{d-1} and the proof of necessity shows that the $I_n(\mathbf{x})$ attains its maximum on this boundary. Let us define

$$S_{int}^{d-1} := S^{d-1} \setminus \bigcup_{i=1}^{d} \{\mathbf{x} \in S^{d-1} : x_i = 0\},$$

which is the interior region bounded by these boundaries on S^{d-1}. We note that the points on the planes $\{\mathbf{x} : x_i = 0\}$ are exactly where the weight function h_κ^2 in (1.1) has singularity. We have the following result.

Theorem 2.3. *Let f be continuous on S^{d-1}. If $\delta > (d-2)/2$, then the (C,δ) means of the h-harmonic expansion of f for h_κ^2 in (1.1) converge to f for every $\mathbf{x} \in S_{int}^{d-1}$. Moreover, the convergence is uniform over each compact set contained inside S_{int}^{d-1}.*

In other words, for the pointwise convergence away from the singularity of h_κ, the convergence holds if $\delta > (d-2)/2$, independent of the value κ, which is the same as the critical index for the ordinary harmonics. This phenomenon does not show up when we deal with the ordinary harmonics, for which there is no difference in critical index between uniform and pointwise convergence. According to this theorem, S^{d-1} equipped with $h_\kappa^2 d\omega$ possesses boundaries. This fact can be better understood when we consider the connection between h-harmonics and orthogonal polynomials on B^d and those on T^d.

2.2. **Orthogonal expansion on the ball.** The connection between h-harmonics and orthogonal polynomials on B^d is described in [16] for a large class of weight functions on S^{d-1}. We shall restrict ourself to h_κ in (1.1) and $W^B_{\kappa,\mu}$ in (1.8). We need to emphasis that the connection is between orthogonal polynomials on B^d and on S^d (not S^{d-1}); the weight function $W^B_{\kappa,\mu}$ is related to the weight function $h_{\kappa,\mu}$ defined on S^d by

$$h_{\kappa,\mu}(\mathbf{x}) = \prod_{i=1}^{d} |x_i|^{\kappa_i} |x_{d+1}|^{\mu}, \quad \kappa_i \ge 0, \quad \mu \ge 0 \quad \mathbf{x} = (x_1, \ldots, x_{d+1}) \in S^d. \tag{2.2}$$

Let us denote by $\{P^n_\alpha\}_{|\alpha|=n}$ an orthonormal basis of $\mathcal{V}^d_n(W^B_{\kappa,\mu})$, where $\alpha \in \mathbb{N}^d_0$. Since $W^B_{\kappa,\mu}$ is an even function in each of its variables, P^n_α can be chosen as even functions when n is even and odd functions when n is odd. Define functions Y^n_α on $\mathbb{R}^{d+1}$ by

$$Y^n_\alpha(\mathbf{y}) = r^n P^n_\alpha(\mathbf{x}), \quad \text{where} \quad \mathbf{y} = r(\mathbf{x}, x_{d+1}), \quad r = |\mathbf{y}|.$$

Note that $\mathbf{x} = (x_1, \ldots, x_d) \in B^d$ under the above change of variables. It turns out that Y^n_α are homogeneous polynomials of degree n in $\mathbf{y}$, and they form an orthonormal basis for $\mathcal{H}^{d+1}_n(h^2_\kappa; \mathbb{Z}_2)$ which consists of h-harmonics of degree n that are invariant under sign changes of the last component; that is,

$$\mathcal{H}^{d+1}_n(h^2_\kappa; \mathbb{Z}_2) = \{Y \in \mathcal{H}^{d+1}_n(h^2_\kappa) : Y(\mathbf{x}, x_{d+1}) = Y(\mathbf{x}, -x_{d+1})\}.$$

In particular, the great circle $x_{d+1} = 0$ on S^d becomes the boundary of B^d, since $x^2_{d+1} = 1 - |\mathbf{x}|^2$. As a consequence of this connection, we can derive a compact formula for the reproducing kernel $\mathbf{P}_n(W^B_{\kappa,\mu}; \mathbf{x}, \mathbf{y})$ from the relation

$$\begin{aligned} \mathbf{P}_n(W^B_{\kappa,\mu}; \mathbf{x}, \mathbf{y}) = \\ [P_n(h^2_{\kappa,\mu}; (\mathbf{x}, x_{d+1}), (\mathbf{y}, y_{d+1})) + P_n(h^2_{\kappa,\mu}; (\mathbf{x}, x_{d+1}), (\mathbf{y}, -y_{d+1}))]/2. \end{aligned} \tag{2.3}$$

Combining (2.3) and (1.5) we derive the explicit formula (1.11) for $\mathbf{P}_n(W^B_{\kappa,\mu}; \mathbf{x}, \mathbf{y})$.

Let $S^\delta_n(W^B_{\kappa,\mu}; f)$ denote the Cesàro (C, δ) means of the orthogonal expansion of f with respect to $W^B_{\kappa,\mu}$. It follows from (1.10) that we can write

$$S^\delta_n(W^B_{\kappa,\mu}; f, \mathbf{x}) = \int_{B^d} f(\mathbf{y}) \mathbf{K}^\delta_n(W^B_{\kappa,\mu}; \mathbf{x}, \mathbf{y}) W^B_{\kappa,\mu}(\mathbf{y}) d\mathbf{y},$$

where $\mathbf{K}^\delta_n(W^B_{\kappa,\mu})$ denote the Cesàro (C, δ) means of the reproducing kernel $\mathbf{P}_n(W^B_{\kappa,\mu})$,

$$\mathbf{K}^\delta_n(W^B_{\kappa,\mu}; \mathbf{x}, \mathbf{y}) = \frac{1}{\binom{n+\delta}{n}} \sum_{k=0}^{n} \binom{n-k+\delta}{n-k} \mathbf{P}_n(W^B_{\kappa,\mu}; \mathbf{x}, \mathbf{y}).$$

We state the results concerning $W^B_{\kappa,\mu}$ as follows.

Theorem 2.4. *The (C, δ) means of the orthogonal expansion of every continuous function f with respect to $W^B_{\kappa,\mu}$ in (1.8) converge uniformly to f on B^d if and only if*

$$\delta > (d-1)/2 + |\kappa|_1 + \mu - \min\{\kappa_1, \cdots, \kappa_d, \mu\}. \tag{2.4}$$

Corollary 2.5. *The (C, δ) means of the orthogonal expansion of f with respect to $W^B_{\kappa,\mu}$ in (1.8) converge to f in the $L^p(W^B_{\kappa,\mu}; B^d)$ norm, $1 \le p < \infty$, or $C(B^d)$ norm for $p = \infty$, if (2.4) holds. Moreover, the condition (2.4) is necessary for $p = 1$ and $p = \infty$.*

For the classical orthogonal expansions with respect to W_μ^B on B^d, some partial results were obtained early in the literature, see [7, Chapt. 12], with the restriction $\mu=(d-1)/2$; the sufficient and necessary condition in Theorem 2.4 was first proved in [19].

To state the pointwise convergence, we need to define the following set

$$B^d_{int} := \{\mathbf{x} \in B^d : |\mathbf{x}| < 1 \quad \text{and} \quad x_i \ne 0, 1 \le i \le d\},$$

which is the interior region bounded by the boundary of B^d and by the hyperplanes $\{\mathbf{x} : x_i = 0\}$ for $1 \le i \le d$. Then we have

Theorem 2.6. *Let f be continuous on B^d. If $\delta > (d-1)/2$, then the (C,δ) means of the orthogonal expansion of f with respect to $W^B_{\kappa,\mu}$ in (1.8) converge to f for every $\mathbf{x} \in B^d_{int}$. Moreover, the convergence is uniform over each compact set contained inside B^d_{int}.*

In other words, for the pointwise convergence away from the singularity of $W^B_{\kappa,\mu}$, the convergence holds if $\delta > (d-1)/2$, independent of the values of κ and μ. The result appears to be new even for the classical weight function $W^B_\mu(\mathbf{x})$. To illustrate the result, we note that for the Lebesgue measure ($W_\mu(x) = 1$ or $\mu = 1/2$) the uniform convergence on B^d holds if and only if $\delta > d/2$, while the pointwise convergence holds in the interior of B^d whenever $\delta > (d-1)/2$.

2.3. **Orthogonal expansion for $w_{\lambda,\mu}$ on $[-1,1]$.** The necessity of Theorem 2.1 and Theorem 2.4 will be proved by choosing some special points on the boundary, which reduces the problem to the case of one variable; in fact, it reduces the problem essentially to the Cesàro summability of the generalized Gegenbauer expansion associated with the weight function $w_{\lambda,\mu}$ on $[-1,1]$ in (1.12).

Denote the Cesàro means of the generalized Gegenbauer expansion by $s_n^\delta(w_{\lambda,\mu}; f)$. It can be written as an integral operator

$$s_n^\delta(w_{\lambda,\mu}; f, x) = \int_{-1}^1 K_n^\delta(w_{\lambda,\mu}; x, y) f(y) w_{\lambda,\mu}(y) dy,$$

where the kernel $K_n^\delta(w_{\lambda,\mu})$ can be written in terms of the kernel $K_n^\delta(w_{\lambda+\mu})$ of the Cesàro means of the Gegenbauer polynomials,

$$\begin{aligned} (2.5) \qquad & K_n^\delta(w_{\lambda,\mu}; x, y) = c_\lambda c_\mu \\ \times & \int_{-1}^1 \int_{-1}^1 K_n^\delta(w_{\lambda+\mu}; 1, txy + s\sqrt{1-x^2}\sqrt{1-y^2})(1+t)(1-t^2)^{\mu-1}(1-s^2)^{\lambda-1} dt ds. \end{aligned}$$

This formula follows from taking the (C,δ) means of the product type formula (1.14) of the generalized Gegenbauer polynomials. As we mentioned in the introduction, the (C,δ) summability of the generalized Gegenbauer expansion does not follow from that of the Jacobi expansion, even though the orthogonal polynomials are related. As far as we know, the following theorem is new. It should be compared with the Theorem 9.1.3 in [12, p. 246].

Theorem 2.7. *The (C,δ) means of the generalized Gegenbauer expansion of every continuous function f converge uniformly to f on $[-1,1]$ if and only if $\delta > \max\{\lambda, \mu\}$.*

To prove this result, a standard argument shows that it suffices to prove that

$$T_n^\delta(w_{\lambda,\mu};x) := \int_{-1}^{1} |K_n^\delta(w_{\lambda,\mu},x,y)| w_{\lambda,\mu}(y)dy \tag{2.6}$$

is uniformly bounded if and only if $\delta > \max\{\lambda,\mu\}$. The sufficient part follows from taking $d=1$ in Theorem 2.4. The necessary part is the consequence of the Proposition 2.8 below.

Proposition 2.8. *If $\lambda \ge \mu$, then $T_n^\lambda(w_{\lambda,\mu};1) \ge c\log n$; if $\mu > \lambda$, then $T_n^\mu(w_{\lambda,\mu};0) \ge c\log n$.*

This proposition will be proved in Section 4, which essentially comes down to prove a lower bound for a double integral of the Jacobi polynomials (Proposition 4.2) that is of interest in itself.

2.4. **Orthogonal expansion on the simplex.** The connection between the h-harmonics with respect to (2.2) and orthogonal polynomials with respect to $W_{\kappa,\mu}^T$ is established in [17]. Let us denote by $\{Q_\alpha^n\}_{|\alpha|=n}$ an orthonormal basis of $\mathcal{V}_n^d(W_{\kappa,\mu}^T)$. We define functions Y_α^{2n} on $\mathbb{R}^{d+1}$ by

$$Y_\alpha^{2n}(\mathbf{y}) = r^{2n} Q_\alpha^n(x_1^2,\ldots,x_d^2), \quad \text{where} \quad \mathbf{y} = r(\mathbf{x},x_{d+1}), \quad r = |\mathbf{y}|.$$

Note that $(x_1^2,\ldots,x_d^2) \in T^d$ and we define x_{d+1} by $|\mathbf{x}|^2 + x_{d+1}^2 = 1$. It turns out that Y_α^{2n} are homogeneous polynomials of degree $2n$ in $\mathbf{y}$, and they form an orthonormal basis for $\mathcal{H}_{2n}^{d+1}(h_\kappa^2;\mathbb{Z}_2^{d+1})$ which consists of h-harmonics of degree $2n$ that are invariant under $\mathbb{Z}_2^{d+1}$ (cf. [17, Theorem 3.2]); that is,

$$\mathcal{H}_{2n}^{d+1}(h_\kappa^2;\mathbb{Z}_2^{d+1}) = \{Y \in \mathcal{H}_{2n}^{d+1}(h_\kappa^2) : Y(\pm y_1,\ldots,\pm y_{d+1}) = Y(y_1,\ldots y_{d+1})\}.$$

In particular, the great circles $x_i = 0$, $1 \le i \le d+1$, on S^d become the boundary of T^d. The formula (1.15) of the reproducing kernel $\mathbf{P}_n(W_{\kappa,\mu}^T;\mathbf{x},\mathbf{y})$ is obtained from (1.5) using this connection (cf. [18, Theorem 2.2]).

Let the Cesàro (C,δ) means $S_n^\delta(W_{\kappa,\mu}^T;f)$ and the kernel $K_n^\delta(W_{\kappa,\mu}^T;\mathbf{x},\mathbf{y})$ be defined similarly as in the case of $W_{\kappa,\mu}^B$ on B^d. If one of κ_i is zero, then the (C,δ) summability has been studied in [18], while the proof essentially reduces to that of Jacobi expansion on $[-1,1]$. However, in the general case of all κ_i nonzero, we need to derive sharp estimates of the kernel $K_n^\delta(W_{\kappa,\mu}^T;\mathbf{x},\mathbf{y})$, as in the case of the proof of Theorem 2.1. Although there is similarity between the explicit formulae (1.5) and (1.15), there is also a significant difference between their (C,δ) means. In fact, we are able to derive the estimate for $K_n^\delta(W_{\kappa,\mu}^T;\mathbf{x},\mathbf{y})$ only under the following additional assumption on κ,

$$\sum_{i=1}^{d+1} \big(2\kappa_i - [\kappa_i]\big) \ge 1 + \min_{1\le i\le d+1} \kappa_i \qquad \text{with} \quad \mu = \kappa_{d+1}, \tag{2.7}$$

where $[x]$ stands for the largest integer part of x. Consequently, our result on summability holds also under this assumption. We should like to point out that the assumption (2.7) excludes only a small range of the parameters. Indeed, if one of the parameter, say κ_1 or μ, is $1/2$, or if one of the parameter is ≥ 1, then (2.7) holds. In particular, it holds for the unit weight function ($\kappa_1 = \ldots = \kappa_{d+1} = 1/2$). Our result is as follows.

Theorem 2.9. *Suppose the parameters of $W^T_{\kappa,\mu}$ satisfy* (2.7)*. Then the* (C,δ) *means of the orthogonal expansion of every continuous function with respect to $W^T_{\kappa,\mu}$ converge uniformly to f on T^d if and only if* (2.4) *holds.*

Corollary 2.10. *Suppose the parameters of $W^T_{\kappa,\mu}$ satisfy* (2.7)*. The* (C,δ) *means of the orthogonal expansion of f for $W^T_{\kappa,\mu}$ converge to f in the $L^p(W^T_{\kappa,\mu};T^d)$ norm, $1 \le p < \infty$, or $C(T^d)$ norm for $p=\infty$, if* (2.4) *holds. Moreover, the condition* (2.4) *is necessary for $p=1$ and $p=\infty$.*

The proof will show that the necessary part of the theorem holds without the condition (2.7). Naturally, we expect that the sufficient part also holds for all $\kappa_i \ge 0$ without the condition (2.7). This is indeed the case if at least one $\kappa_i = 0$, as proved in [18]. The sufficient part of the theorem is not proved only for some of the cases in the range $0 < \kappa_i < 1$, $1 \le i \le d+1$, and $2\sum_{i=1}^{d+1}\kappa_i < 1 + \min_{1\le i\le d+1}\kappa_i$. In the case that all κ_i are the same, this could happen only if $0 < \kappa_i < 1/(2d+1)$.

For the pointwise convergence, the result similar to Theorem 2.6 holds with the boundary becoming the natural boundary of the simplex. However, a stronger condition on the parameters is needed in this case, which is

$$\sum_{i=1}^{d+1}(\kappa_i - [\kappa_i]) \ge 1 \qquad \text{with} \quad \mu = \kappa_{d+1}. \tag{2.8}$$

We note that this condition is satisfied if two or more of the parameters lie in the interval $[1/2,1)$, which include the case of unit weight function.

Theorem 2.11. *Suppose the parameters of $W^T_{\kappa,\mu}$ satisfy* (2.7)*. Let f be continuous on T^d. If $\delta > (d-1)/2$, then the* (C,δ) *means of the orthogonal expansion of f with respect to $W^T_{\kappa,\mu}$ converge to f for every point in the interior of T^d. Moreover, the convergence is uniform over each compact set contained in the interior of T^d.*

Again we expect that this theorem holds without the condition (2.7). To illustrate the results, We state the case of unit weight function $W(\mathbf{x}) = 1$ as the following corollary.

Corollary 2.12. *The* (C,δ) *means of the orthogonal expansion of every continuous function f with respect to the unit weight function $W(\mathbf{x}) = 1$ on T^d converge uniformly to f if and only if $\delta > d - 1/2$. Furthermore, the* (C,δ) *means converge to f uniformly over each compact set contained in the interior of T^d if $\delta > (d-1)/2$.*

This is the case of $\kappa_i = 1/2$ for $1 \le i \le d$ and $\mu = 1/2$ in the theorems.

Much of the difference between orthogonal expansions on T^d and those on B^d can be seen already in the case $d=1$. For $W^B_{\kappa,\mu}$, the case $d=1$ is the generalized Gegenbauer weight (1.12). For $W^T_{\kappa,\mu}$, the case $d=1$ is the Jacobi weight function

$$w^{(\alpha,\beta)}(t) = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}(1-t)^\alpha(1+t)^\beta$$

defined on $[-1,1]$. The corresponding Jacobi polynomials are customarily denoted by $P_n^{(\alpha,\beta)}(t)$. Let $p_n^{(\alpha,\beta)}$ denote the orthonormal Jacobi polynomial which differs from $P_n^{(\alpha,\beta)}$ by a constant (cf. [12, p. 68, (4.3.4)]). Denote the Cesàro means of the Jacobi polynomial expansion by $s_n^\delta(w^{(\alpha,\beta)};f)$. It can be written as an integral operator

$$s_n^\delta(w^{(\alpha,\beta)}; f) = \int_{-1}^1 f(y) K_n^\delta(w^{(\alpha,\beta)}; x, y) w^{(\alpha,\beta)}(y) dy,$$

where the kernel $K_n^\delta(w^{(\alpha,\beta)}; x, y)$ is the (C,δ) means of the n-th reproducing kernel $K_n(w^{(\alpha,\beta)}; x, y) = \sum_{k=0}^n p_k^{(\alpha,\beta)}(x) p_k^{(\alpha,\beta)}(y)$. The necessary part of Theorem 2.9 will be proved using the (C,δ) summability of the Jacobi expansion.

2.5. **Further comments.** As we mentioned in the introduction, the (C,δ) means of the h-harmonic expansion for h_κ in (1.2) and any finite reflection group converge uniformly if (1.7) holds. Restricted to the group $\mathbb{Z}_2^d$, the condition (1.7) becomes $\delta > (d-2)/2 + |\kappa|_1$. Hence, Theorem 2.1 shows that the condition (1.7) is not sharp in the case of $\mathbb{Z}_2^d$. This naturally suggests that the condition (1.7) is not sharp for other reflection groups. Although this is likely the case, the lack of the explicit formula for the intertwining operator V makes this problem inaccessible at the moment.

We have found the critical index for these classical type weight functions on S^{d-1}, B^d and T^d. There are many other questions that one may consider; for examples, summability below the critical index, almost everywhere convergence, various multiplier type theorems. For further study, however, more delicate estimates of the kernel functions are likely to be necessary and considerable difficulties will have to be resolved. In this respect, the estimate for the Jacobi expansion in [2] may be helpful.

Our results show that the summability of orthogonal expansions on B^d and that on T^d have similar behavior, and the critical index in both cases look to be the same. This, however, should not leave the false impression that the behavior is a typical one for other family of weight functions on $\mathbb{R}^d$. The other cases that have been studied so far show significant differences in both results and proof. We refer to [10] for the (C,δ) summability of multiple Jacobi expansion on $[-1,1]^d$, and to the monograph [13] for the multiple Hermite and multiple Laguerre expansions.

3. Proof of the main theorems

Throughout the rest of this paper, we denote by c a generic positive constant whose value may vary in different occurrences. We will also use the convention $A \sim B$ which means that there exist positive constants c_1 and c_2 such that $c_1 \le |A/B| \le c_2$.

For the proof of the sufficient part of Theorem 2.1 the following estimate is fundamental.

Theorem 3.1. *Let $\bar{\mathbf{x}} = (|x_1|, \ldots, |x_d|)$. For $\mathbf{x}, \mathbf{y} \in S^{d-1}$ and $\delta > (d-2)/2$,*

$$|K_n^\delta(h_\kappa^2; \mathbf{x}, \mathbf{y})| \le c \Big[\frac{\prod_{j=1}^d (|x_j y_j| + n^{-1}|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-2})^{-\kappa_j}}{n^{\delta-(d-2)/2}(|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^{\delta+(d/2)}} + \frac{\prod_{j=1}^d (|x_j y_j| + |\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2 + n^{-2})^{-\kappa_j}}{n(|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^d} \Big].$$

The proof of this estimate is rather involved and long, we delay it to Section 5. Below we use it to prove the sufficient part of Theorem 2.1. Note that the estimate is invariant under the action of $\mathbb{Z}_2^d$. We denote by S_+^d the positive quadrant of S^d; that is, $S_+^d = \{\mathbf{x} \in S^d : x_1 \ge 0, \ldots, x_d \ge 0\}$.

Proof of Theorem 2.1 (sufficient part). The proof amounts to showing that

$$I_n(S^{d-1};\mathbf{x}) := \int_{S^{d-1}} |K_n^\delta(h_\kappa^2;\mathbf{x},\mathbf{y})| h_\kappa^2(\mathbf{y}) d\omega(\mathbf{y}) \le M,$$

where M is a constant, for all $\mathbf{x}\in S^{d-1}$. Using the estimate in Theorem 3.1 and the fact that the estimate is $\mathbb{Z}_2^d$ invariant, we only need to establish the inequality for $\mathbf{x}\in S_+^{d-1}$ and we can restrict the integral of $I_n(S^{d-1})$ to S_+^{d-1}. Let $\mathbf{x} = (x_1,\ldots,x_d)\in S_+^{d-1}$ be fixed. Then by Theorem 3.1,

$$\begin{aligned} I_n(S^{d-1};\mathbf{x}) \le & \frac{c}{n^{\delta-(d-2)/2}} \int_{S_+^{d-1}} \frac{\prod_{j=1}^d (x_j y_j + n^{-1}|\mathbf{x}-\mathbf{y}| + n^{-2})^{-\kappa_j}}{(|\mathbf{x}-\mathbf{y}|+n^{-1})^{\delta+(d/2)}} \prod_{j=1}^d |y_j|^{2\kappa_j} d\omega \\ & + \frac{c}{n}\int_{S_+^{d-1}} \frac{\prod_{j=1}^d (x_j y_j + |\mathbf{x}-\mathbf{y}|^2 + n^{-2})^{-\kappa_j}}{(|\mathbf{x}-\mathbf{y}|+n^{-1})^{d}} \prod_{j=1}^d |y_j|^{2\kappa_j} d\omega \end{aligned}$$

In the following, we use the notation $I_n(E;\mathbf{x})$ when the integral region is restricted to a subset E of S_+^{d-1}. Let l be the index for which x_l is the largest element in $\mathbf{x}$. Then $x_l \ge |\mathbf{x}|/\sqrt{d} = 1/\sqrt{d}$. Without loss of generality, let us assume $l = d$. Let $S_{\{y_d\le\sigma\}} := \{\mathbf{y}\in S_+^{d-1} : y_d \le \sigma\}$ and define $S_{\{y_d\ge\sigma\}}$ similarly. If $0 < y_d < 1/2\sqrt{d}$, then $|\mathbf{x}-\mathbf{y}| \ge |x_d - y_d| \ge 1/2\sqrt{d}$, and the second integral is bounded by a constant. Hence, we have

$$I_n(S_{\{y_d\le 1/2\sqrt{d}\}};\mathbf{x}) \le cn^{-\delta+(d-2)/2}\int_{S_+^{d-1}} n^{|\kappa|_1-\kappa_d}\prod_{j=1}^{d-1} y_j^{2\kappa_j} y_d^{\kappa_d} d\omega(\mathbf{y}) + cn^{-1} \le c.$$

If $y_d \ge 1/2\sqrt{d}$, then $|x_d y_d| \ge 1/2d$ so that the term involving $x_d y_d$ is bounded by a constant. Let $\mathbf{x}' = (x_1,\ldots,x_{d-1})$. We note that $|\mathbf{x}'|^2 = 1 - x_d^2 \le 1 - 1/d$. Using the elementary inequalities $|\mathbf{x}-\mathbf{y}| \ge |\mathbf{x}'-\mathbf{y}'|$, and $|\mathbf{y}'| \le 1 - 1/4d$ for $\mathbf{y}\in S_{\{y_d\ge 1/2\sqrt{d}\}}$, and changing the integral over $S^{d-1}_{\{y_d\ge 1/2\sqrt{2}\}}$ to an integral over $\mathbb{R}_+^{d-1}$, we conclude that

$$\begin{aligned} & I_n(S_{\{y_d\ge 1/2\sqrt{d}\}};\mathbf{x}) \\ & \le \frac{c}{n^{\delta-(d-2)/2}} \int_B \frac{\prod_{j=1}^{d-1} (x_j u_j + n^{-1}|\mathbf{x}'-\mathbf{u}| + n^{-2})^{-\kappa_j}}{(|\mathbf{x}'-\mathbf{u}|+n^{-1})^{\delta+(d/2)}} \prod_{j=1}^{d-1} |u_j|^{2\kappa_j} d\mathbf{u} \\ & + \frac{c}{n}\int_B \frac{\prod_{j=1}^{d-1} (x_j u_j + |\mathbf{x}'-\mathbf{u}|^2 + n^{-2})^{-\kappa_j}}{(|\mathbf{x}'-\mathbf{u}|+n^{-1})^{d}} \prod_{j=1}^{d-1} |u_j|^{2\kappa_j} d\mathbf{u}, \end{aligned}$$

where $B = \{\mathbf{u}\in\mathbb{R}_+^{d-1} : |\mathbf{u}| \le \sqrt{1-1/4d}\}$. We denote the two integrals by $I_1(\mathbf{x})$ and $I_2(\mathbf{x})$, respectively, and estimate them separately.

In order to estimate $I_1(\mathbf{x})$, first we claim that for $\mathbf{u},\mathbf{x}'\in\mathbb{R}_+^{d-1}$,

$$(|\mathbf{x}'-\mathbf{u}| + n^{-1})(x_j u_j + n^{-1}|\mathbf{x}'-\mathbf{u}|) \ge n^{-1}|u_j|^2/4.$$

Indeed, if $x_j \ge u_j/2$, then simply drop $|\mathbf{x}'-\mathbf{u}|$ terms; if $x_j \le u_j/2$, then use the inequality $|\mathbf{x}'-\mathbf{u}| \ge |x_j - u_j| \ge u_j/2$ and drop the $x_j u_j$ term. Using this inequality we see that

$$I_1(\mathbf{x}) \le cn^{-\delta+(d-2)/2+|\kappa|_1-\kappa_d}\int_B \frac{1}{(|\mathbf{x}'-\mathbf{u}|+n^{-1})^{\delta+(d/2)-|\kappa|_1+\kappa_d}} d\mathbf{u}.$$

Let $\sigma = \delta - (d-2)/2 - (|\kappa|_1 - \kappa_d)$. Enlarging the integral domain from B to $\{\mathbf{y} : |\mathbf{x}' - \mathbf{u}| \le 1\}$, we conclude that

$$I_1(\mathbf{x}) \le cn^{-\sigma} \int_{|\mathbf{x}'-\mathbf{u}|\le 1} \frac{1}{(|\mathbf{x}' - \mathbf{u}| + n^{-1})^{\sigma+d-1}} d\mathbf{u} = cn^{-\sigma} \int_{B^{d-1}} \frac{1}{(|\mathbf{u}| + n^{-1})^{\sigma+d-1}} d\mathbf{u}$$
$$= cn^{-\sigma} \int_0^1 r^{d-2} \frac{1}{(r + n^{-1})^{\sigma+d-1}} dr \le c \int_0^n \frac{dt}{(1+t)^{1+\sigma}},$$

which is bounded uniformly in n if $\sigma > 0$. This gives the desired result since (2.1) implies $\sigma > 0$.

To estimate $I_2(\mathbf{x})$, we use the elementary inequality that for $\mathbf{x}', \mathbf{u} \in \mathbb{R}_+^{d-1}$,

$$x_j u_j + |\mathbf{x}' - \mathbf{u}|^2 \ge x_j u_j + |x_j - u_j|^2/2 = (x_j^2 + u_j^2)/2 \ge u_j^2/2.$$

Using this inequality and enlarging the integral domain as before, we conclude that

$$I_2(\mathbf{x}) \le cn^{-1} \int_B \frac{1}{(|\mathbf{x}' - \mathbf{u}| + n^{-1})^d} d\mathbf{u}$$
$$\le cn^{-1} \int_0^1 r^{d-2} \frac{1}{(r + n^{-1})^d} dr \le c \int_0^n \frac{1}{(1+t)^2} dt \le c.$$

Putting these estimates together, we see that $I_n(\mathbf{x})$ is bounded uniformly in n under the condition (2.1). This completes the proof of the sufficient part of Theorem 2.1. □

For the proof of Theorem 2.3, a different estimate is needed for $\mathbf{x}$ not on the great circles defined by the intersection of S^{d-1} and the coordinate planes $\{\mathbf{x} : x_i = 0\}$. Recall that the interior region bounded by the great circles is denoted by S_{int}^{d-1}. We use the notation $c(\mathbf{x})$ to denote a generic function that depends on $\mathbf{x}$ only, whose value may vary in different occurrences.

Theorem 3.2. *Let $\mathbf{x} \in S_{int}^{d-1}$. Then for $(d-2)/2 < \delta \le d/2$ and $\mathbf{y} \in S^{d-1}$,*
(i) if $|y_j| \le |x_j|/2$ for some j, then

$$|K_n^\delta(h_\kappa^2; \mathbf{x}, \mathbf{y})| \le c(\mathbf{x}) n^{-\delta+(d-2)/2} \prod_{i=1}^d (|y_i| + n^{-1})^{-\kappa_i};$$

(ii) if for all $1 \le j \le d$, $|y_j| > |x_j|/2$ and $x_j y_j > 0$, then

$$|K_n^\delta(h_\kappa^2; \mathbf{x}, \mathbf{y})| \le c(\mathbf{x}) n^{-\delta+(d-2)/2} (|\mathbf{x} - \mathbf{y}| + n^{-1})^{-\delta-d/2};$$

(iii) if for all $1 \le j \le d$, $|y_j| > |x_j|/2$ and $x_i y_i < 0$ for some i, then there exists a constant $\eta > 0$ (independent of $\mathbf{y}$), such that

$$|K_n^\delta(h_\kappa^2; \mathbf{x}, \mathbf{y})| \le c(\mathbf{x}) n^{-\delta+(d-2)/2} (|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^{-\delta-d/2+\eta}.$$

Again we delay the proof of the estimate to Section 5 and go on with the proof of the main theorem.

Proof of Theorem 2.3. Let $\delta > (d-2)/2$. We may assume that $\delta \le d/2$, since if (C, δ_0) means converge, then (C, δ) means converge for all $\delta > \delta_0$. Let $\mathbf{x} \in S_{int}^{d-1}$ be fixed. For each $\rho > 0$, we define

$$E_\rho := E_\rho(\mathbf{x}) := \{\mathbf{y} \in S^{d-1} : |\mathbf{y} - \mathbf{x}| > \rho\}.$$

A standard argument shows that, to prove the theorem, it is sufficient to prove that for $\delta > (d-2)/2$ and $\rho > 0$

$$I_n(E_\rho;\mathbf{x}) := \int_{E_\rho(\mathbf{x})} |K_n^\delta(h_\kappa^2;\mathbf{x},\mathbf{y})| h_\kappa^2(\mathbf{y})d\omega \to 0 \qquad \text{as} \quad n\to\infty, \tag{3.1}$$

and

$$I_n(S^{d-1};\mathbf{x}) := \int_{\mathbf{y}\in S^{d-1}} |K_n^\delta(h_\kappa^2;\mathbf{x},\mathbf{y})| h_\kappa^2(\mathbf{y})d\omega \le c(\mathbf{x}) < \infty. \tag{3.2}$$

To prove (3.1), we define three sets corresponding to the cases in Theorem 3.2:

$$\begin{aligned}
F_1 &:= \{\mathbf{y}\in S^{d-1} : |y_j| \le |x_j|/2 \text{ for some } j\,\},\\
F_2 &:= \{\mathbf{y}\in S^{d-1} : |y_i| > |x_i|/2 \text{ and } x_iy_i > 0,\ 1\le i\le d\,\},\\
F_3 &:= \{\mathbf{y}\in S^{d-1} : |y_i| > |x_i|/2,\ 1\le i\le d, \text{ and } x_jy_j < 0 \text{ for some } j\,\}.
\end{aligned}$$

Recall that $h_\kappa^2(\mathbf{y}) = \prod_{i=1}^d |y_i|^{2\kappa_i}$. By (i) of Theorem 3.2 we have

$$I_n(E_\rho\cap F_1;\mathbf{x}) \le c(\mathbf{x})n^{-\delta+(d-2)/2}\int_{S^{d-1}}\prod_{i=1}^d(|y_i|+n^{-1})^{-\kappa_i}|y_i|^{2\kappa_i}d\omega \le c(\mathbf{x})n^{-\delta+(d-2)/2},$$

which goes to 0 as $n\to\infty$, since $\delta > (d-2)/2$. By (ii) of Theorem 3.2 and the fact that $|\mathbf{x}-\mathbf{y}| \ge \rho$ for $\mathbf{y}\in E_\rho$, we have

$$I_n(E_\rho\cap F_2;\mathbf{x}) \le c(\mathbf{x})n^{-\delta+(d-2)/2}\int_{S^{d-1}} h_\kappa^2(\mathbf{y})d\omega,$$

which again goes to 0 as $n\to\infty$. Finally, by part (iii) of Theorem 3.2 and the fact that $h_\kappa^2(\mathbf{y}) \le 1$, we have

$$\begin{aligned}
I_n(E_\rho\cap F_3;\mathbf{x}) &\le c(\mathbf{x})n^{-\delta+(d-2)/2}\int_{S^{d-1}}(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|+n^{-1})^{-\delta-d/2+\eta}h_\kappa^2(\mathbf{y})d\omega\\
&\le c(\mathbf{x})n^{-\delta+(d-2)/2}\int_{S_+^{d-1}}(|\bar{\mathbf{x}}-\mathbf{y}|^2+n^{-2})^{-(\delta+d/2-\eta)/2}d\omega,
\end{aligned}$$

while passing to S_+^{d-1} allows us to replace $\bar{\mathbf{y}}$ by $\mathbf{y}$. We then enlarge the integral domain from S_+^{d-1} to S^{d-1} and use the well-known formula for $f:\mathbb{R}\to\mathbb{R}$,

$$\int_{S^{d-1}} f(\langle\mathbf{x},\mathbf{y}\rangle)d\omega(\mathbf{y}) = \omega_{d-2}\int_{-1}^1 f(s)(1-s^2)^{(d-3)/2}ds, \qquad \mathbf{x}\in S^{d-1}, \tag{3.3}$$

and the fact that $|\bar{\mathbf{x}}-\mathbf{y}|^2 = 2(1-\langle\bar{\mathbf{x}},\mathbf{y}\rangle)$ to conclude that for $0<\eta<\delta+d/2$,

$$\begin{aligned}
\int_{S_+^{d-1}}(|\bar{\mathbf{x}}-\mathbf{y}|^2+n^{-2})^{-(\delta+d/2-\eta)/2}d\omega &\le \omega_{d-2}\int_{-1}^1\frac{(1-s^2)^{(d-3)/2}}{(1-s+n^{-2})^{(\delta+d/2-\eta)/2}}ds\\
&\le c\int_0^1\frac{(1-s)^{(d-3)/2}}{(1-s+n^{-2})^{(\delta+d/2-\eta)/2}}ds\\
&\le c\begin{cases} n^{\delta-(d-2)/2-\eta}, & \text{if } \delta-(d-2)/2-\eta > 0,\\ \log n, & \text{if } \delta-(d-2)/2-\eta = 0,\\ 1, & \text{if } \delta-(d-2)/2-\eta < 0.\end{cases}
\end{aligned}$$

Consequently, we conclude that

$$I_n(E_\rho \cap F_3; \mathbf{x}) \le c(\mathbf{x}) \begin{cases} n^{-\eta}, & \text{if } \delta - (d-2)/2 - \eta > 0, \\ n^{-\delta+(d-2)/2} \log n, & \text{if } \delta - (d-2)/2 - \eta = 0, \\ n^{-\delta+(d-2)/2}, & \text{if } \delta - (d-2)/2 - \eta < 0, \end{cases}$$

which goes to 0 as $n \to \infty$. Putting these estimates together and noticing that $S^{d-1} = F_1 \cup F_2 \cup F_3$, we have proved (3.1).

To prove (3.2), we again use the estimate of the kernel in Theorem 3.2. Since in the above estimate of $I_n(E_\rho \cap F_1; \mathbf{x})$ and $I_n(E_\rho \cap F_3; \mathbf{x})$ we did not use the fact that $\mathbf{y} \in E_\rho$; these estimates hold for $I_n(F_1; \mathbf{x})$ and $I_n(F_3; \mathbf{x})$ as well. In particular, these two terms are bounded. Hence, we are left with the case $I_n(F_2; \mathbf{x})$. From (ii) in Theorem 3.2, it follows that

$$I_n(F_2; \mathbf{x}) \le c(\mathbf{x}) n^{-\delta+(d-2)/2} \int_{S^{d-1}} (|\mathbf{x}-\mathbf{y}| + n^{-1})^{-\delta-d/2} h_\kappa^2(\mathbf{y}) d\omega$$

Estimating the integral by the use of (3.3) as we did in the case of $I_n(E_\rho \cap F_3; \mathbf{x})$, we obtain

$$\begin{aligned} I_n(F_2; \mathbf{x}) &\le c(\mathbf{x}) n^{-\delta+(d-2)/2} \int_{-1}^{1} \frac{(1-s^2)^{(d-3)/2}}{(1-s+n^{-2})^{(\delta+d/2)/2}} ds \\ &\le c(\mathbf{x}) n^{-\delta+(d-2)/2} \cdot n^{\delta-(d-2)/2} = c(\mathbf{x}). \end{aligned}$$

Consequently, (3.2) is proved. The proof of Theorem 2.3 is complete. □

Proof of Theorem 2.4 (sufficient part). This will follow from that of Theorem 2.1, and a more general result has been proved in [20]. We shall be brief. Let $\mathbf{K}_n^\delta(W_{\kappa,\mu}^B; \mathbf{x}, \mathbf{y})$ denote the (C, δ) means of the reproducing kernel associated with $W_{\kappa,\mu}^B$. Using the integral formula (see [16, (2.5)])

$$\int_{S^d} g(\mathbf{z}) d\omega_d = \int_{B^d} \left[g(\mathbf{y}, \sqrt{1-|\mathbf{y}|^2}) + g(\mathbf{y}, -\sqrt{1-|\mathbf{y}|^2}) \right] \frac{d\mathbf{y}}{\sqrt{1-|\mathbf{y}|^2}}$$

with $g(\mathbf{y}, y_{d+1}) = |\mathbf{K}_n^\delta(W_{\kappa,\mu}^B; \mathbf{x}, \mathbf{y})| \prod_{i=1}^d |y_i|^{2\kappa_i} |y_{d+1}|^{2\mu}$ we conclude that

$$\int_{B^d} |\mathbf{K}_n^\delta(W_{\kappa,\mu}^B; \mathbf{x}, \mathbf{y})| W_{\kappa,\mu}^B(\mathbf{y}) d\mathbf{y} = \frac{1}{2} \int_{S^d} |K_n^\delta(W_{\kappa,\mu}^B; \mathbf{x}, \mathbf{y})| h_{\kappa,\mu}^2(\mathbf{z}) d\mathbf{z}$$

where $\mathbf{z} = (\mathbf{y}, y_{d+1})$. On the other hand, taking the (C, δ) means of (2.3) gives

$$\mathbf{K}_n^\delta(W_{\kappa,\mu}^B; \mathbf{x}, \mathbf{y}) = [K_n^\delta(h_{\kappa,\mu}^2; (\mathbf{x}, x_{d+1}), (\mathbf{y}, y_{d+1})) + K_n^\delta(h_{\kappa,\mu}^2; (\mathbf{x}, x_{d+1}), (\mathbf{y}, -y_{d+1}))]/2$$

with $x_{d+1} = \sqrt{1-|\mathbf{x}|^2}$ for $h_{\kappa,\mu}$ in (2.2). Hence,

$$\int_{B^d} |\mathbf{K}_n^\delta(W_{\kappa,\mu}^B; \mathbf{x}, \mathbf{y})| W_{\kappa,\mu}^B(\mathbf{y}) d\mathbf{y} \le \frac{1}{2} \int_{S^d} |K_n^\delta(h_{\kappa,\mu}^2; (\mathbf{x}, x_{d+1}), \mathbf{z})| h_{\kappa,\mu}^2(\mathbf{z}) d\mathbf{z},$$

from which we conclude that the sufficiency of Theorem 2.4 follows from that of Theorem 2.1. □

Evidently, the same argument also shows that Theorem 2.6 is the consequence of Theorem 2.3.

Proof of Theorems 2.1 and 2.4 (necessary part). The case $d = 1$ reduces to the generalized Gegenbauer expansion. So we assume $d \ge 2$. Since the proof of the sufficient part of Theorem 2.4 follows from the sufficient part of Theorem 2.1, the necessary part of Theorem 2.4 will imply the necessary part of Theorem 2.1. Thus, it suffices to prove the necessity of (2.4) in Theorem 2.4.

Let $\kappa_{d+1} = \mu$. We first work with the case that $\kappa_j = \min_{1\le i\le d+1} \kappa_i$, where j is an index between 1 and d. We may assume $j = 1$. Setting $\mathbf{y} = \mathbf{e}_1 := (1, 0, \ldots, 0)$ in the reproducing kernel (1.11) and using the formula (1.14), we obtain

$$\begin{aligned} \mathbf{P}_n(W^B_{\kappa,\mu}; \mathbf{x}, \mathbf{e}_1) &= c_{\kappa_1} \frac{n+\gamma}{\gamma} \int_{-1}^{1} C_n^{(\gamma)}(x_1 t_1)(1+t_1)(1-t_1^2)^{\kappa_1 - 1} dt_1 \\ &= \tilde{C}_n^{(\gamma-\kappa_1,\kappa_1)}(1) \tilde{C}_n^{(\gamma-\kappa_1,\kappa_1)}(x_1) \end{aligned} \tag{3.4}$$

where $\gamma = \sum_{i=1}^{d+1} \kappa_i + (d-1)/2$ ($\mu = \kappa_{d+1}$). Consequently, recall the notation $w_{\lambda,\mu}$ in (1.12),

$$\mathbf{K}_n^\delta(W^B_{\kappa,\mu}; \mathbf{x}, \mathbf{e}_1) = K_n^\delta(w_{\gamma-\kappa_1,\kappa_1}; 1, x_1).$$

Hence, if $\delta = \gamma - \kappa_1$, then changing variables $x_i = u_i\sqrt{1-x_1^2}$ for $i = 2, \ldots, d$, so that the integral on B^d reduces to an integral of one variable, we get

$$\begin{aligned} &\int_{B^d} \left|\mathbf{K}_n^\delta(W^B_{\kappa,\mu}; \mathbf{x}, \mathbf{e}_1)\right| W^B_{\kappa,\mu}(\mathbf{x}) d\mathbf{x} \\ &\qquad = \int_{B^d} \left|K_n^\delta(w_{\gamma-\kappa_1,\kappa_1}; 1, x_1)\right| \prod_{i=1}^{d} |x_i|^{2\kappa_i} (1-|\mathbf{x}|^2)^{\mu-1/2} d\mathbf{x} \\ &\qquad = c \int_{-1}^{1} \left|K_n^\delta(w_{\gamma-\kappa_1,\kappa_1}; 1, x_1)\right| w_{\gamma-\kappa_1,\kappa_1}(x_1) dx_1 = c T_n^\delta(w_{\gamma-\kappa_1,\kappa_1}; 1), \end{aligned}$$

where the value of the constant c can be determined by setting $n = 0$. Hence, since $\delta = \gamma - \kappa_1 \ge \kappa_1$, the desired result follows from Proposition 2.8.

We are left with the case that $\mu = \kappa_{d+1} = \min_{1\le i\le d+1} \kappa_i$. In this case, taking $\mathbf{y} = 0$ in (1.11) and using (1.14), we obtain

$$\begin{aligned} \mathbf{P}_n(W^B_{\kappa,\mu}; \mathbf{x}, 0) &= c_\mu \frac{n+\gamma}{\gamma} \int_{-1}^{1} C_n^{(\gamma)}(s\sqrt{1-|\mathbf{x}|^2})(1-s^2)^{\mu_1 - 1} ds \\ &= \tilde{C}_n^{(\mu,\gamma-\mu)}(0) \tilde{C}_n^{(\mu,\gamma-\mu)}(|\mathbf{x}|). \end{aligned}$$

Consequently, we conclude that

$$\mathbf{K}_n^\delta(W^B_{\kappa,\mu}; \mathbf{x}, 0) = K_n^\delta(w_{\mu,\gamma-\mu}; |\mathbf{x}|, 0)$$

Hence, if $\delta = \gamma - \mu$, then using the polar coordinates we obtain

$$\begin{aligned} \int_{B^d} \left|\mathbf{K}_n^\delta(W^B_{\kappa,\mu}; \mathbf{x}, 0)\right| W^B_{\kappa,\mu}(\mathbf{x}) d\mathbf{x} &= c \int_0^1 \left|K_n^\delta(w_{\mu,\gamma-\mu}; 0, r)\right| w_{\mu,\gamma-\mu}(r) dr \\ &= c T_n^\delta(w_{\mu,\gamma-\mu}; 0). \end{aligned}$$

Hence, since $\delta = \gamma - \mu \ge \mu$, we can use Proposition 2.8 to finish the proof. □

Corollary 2.5 follows from the fact that the $L^1(W^B_{\kappa,\mu}; B^d)$ norm of $S_n^\delta(h_\kappa^2; f)$ is the same as the $C(B^d)$ norm and the standard argument of the Riesz interpolation. The proof of Corollary 2.12 follows similarly.

The proof of Theorem 2.7 and Proposition 2.8 are given in the next section.

In order to prove the theorems on the simplex T^d, we need sharp estimates of the kernel $\mathbf{K}_n^\delta(W^T_{\kappa,\mu}; \mathbf{x}, \mathbf{y})$. Recall that $[x]$ denotes the integer part of x. For $\mathbf{x}, \mathbf{y} \in T^d$, define $\xi = (\sqrt{x_1}, \ldots, \sqrt{x_d}, \sqrt{1-|\mathbf{x}|_1})$ and $\zeta = (\sqrt{y_1}, \ldots, \sqrt{y_d}, \sqrt{1-|\mathbf{y}|_1})$. .

Theorem 3.3. *Assume* $\delta + \sum_{i=1}^{d+1}(\kappa_i - [\kappa_i]) \ge (d+1)/2$ *with* $\kappa_{d+1} = \mu$. *For* $\mathbf{x},\mathbf{y} \in T^d$,

$$|\mathbf{K}_n^\delta(W_{\kappa,\mu}^T;\mathbf{x},\mathbf{y})| \le c\Big[\frac{\prod_{j=1}^{d+1}(\sqrt{x_jy_j}+n^{-1}|\xi-\zeta|+n^{-2})^{-\kappa_j}}{n^{\delta-(d-1)/2}(|\xi-\zeta|+n^{-1})^{\delta+(d+1)/2}} + \frac{\prod_{j=1}^{d+1}(\sqrt{x_jy_j}+|\xi-\zeta|^2+n^{-2})^{-\kappa_j}}{n(|\xi-\zeta|+n^{-1})^{d+1}}\Big].$$

The proof of this estimate is being delayed to Section 5. Here we use it to prove Theorem 2.9.

Proof of Theorem 2.9. To prove the sufficient part, we fix a δ satisfying (2.4); that is, $\delta > (d-1)/2+|\kappa|_1-\min_{1\le i\le d+1}\kappa_i$ with $\mu=\kappa_{d+1}$. Then the condition (2.7) implies that $\delta + \sum_{i=1}^{d+1}(\kappa_i-[\kappa_i]) > (d+1)/2$, so that the estimate in Theorem 3.3 can be applied. Replacing $\mathbf{x}$ by $\{\mathbf{x}\}^2 := (x_1^2,\ldots,x_d^2)$ and $\mathbf{y}$ by $\{\mathbf{y}\}^2 := (y_1^2,\ldots,y_d^2)$ in the estimate of Theorem 3.3, we get

$$|\mathbf{K}_n^\delta(W_{\kappa,\mu}^T;\{\mathbf{x}\}^2,\{\mathbf{y}\}^2)| \le c\Big[\frac{\prod_{j=1}^{d+1}(x_jy_j+n^{-1}|\bar{\mathbf{x}}'-\bar{\mathbf{y}}'|+n^{-2})^{-\kappa_j}}{n^{\delta-(d-1)/2}(|\bar{\mathbf{x}}'-\bar{\mathbf{y}}'|+n^{-1})^{\delta+(d+1)/2}} + \frac{\prod_{j=1}^{d+1}(x_jy_j+|\bar{\mathbf{x}}'-\bar{\mathbf{y}}'|^2+n^{-2})^{-\kappa_j}}{n(|\bar{\mathbf{x}}'-\bar{\mathbf{y}}'|+n^{-1})^{d+1}}\Big],$$

where $\mathbf{x}' = (x_1,\ldots,x_d,x_{d+1})$ and $\mathbf{y}' = (y_1,\ldots,y_d,y_{d+1})$ both are points in S^d. The right hand side is the same as in the estimate of Theorem 2.4 with d replaced by $d+1$. Therefore, using the formula

$$\int_{S^d} f(u_1^2,\ldots,u_{d+1}^2)d\omega(\mathbf{u}) = 2\int_{T^d} f(y_1,\ldots,y_d,1-|\mathbf{y}|_1)\frac{d\mathbf{y}}{\sqrt{y_1\ldots y_d(1-|\mathbf{y}|_1)}},$$

which can be easily verified (see, for example, [17]), we conclude that

$$\int_{T^d}|\mathbf{K}_n^\delta(W_{\kappa,\mu}^T;\{\mathbf{x}\}^2,\mathbf{y})|W_{\kappa,\mu}^T(\mathbf{y})d\mathbf{y} = \frac{1}{2}\int_{S^d}|\mathbf{K}_n^\delta(W_{\kappa,\mu}^T;\{\mathbf{x}\}^2,\{\mathbf{y}\}^2)|h_{\kappa,\mu}^2(\mathbf{y}')d\omega(\mathbf{y}') \tag{3.5}$$

Consequently, the proof of the sufficient part follows from that of Theorem 2.1.

We now prove the necessary part. First consider the case $\kappa_j = \min_{1\le i\le d+1}\kappa_i$ for some j with $1\le j\le d$. We may assume $j=1$. Using [12, (4.3.4) and (4.1.5)] it is easy to verify that

$$\frac{2n+\lambda}{\lambda}C_{2n}^{(\lambda)}(t) = p_n^{(\lambda-1/2,-1/2)}(1)p_n^{(\lambda-1/2,-1/2)}(2t^2-1). \tag{3.6}$$

Taking (C,δ) means of $\mathbf{P}_n(W_{\kappa,\mu}^T;\mathbf{x},\mathbf{y})$ in (1.15) and using (3.6), it follows that

$$\mathbf{K}_n^\delta(W_{\kappa,\mu}^T;\mathbf{x},\mathbf{y}) = c_\kappa\int_{[-1,1]^{d+1}} K_n^\delta(w^{(|\kappa|_1+\frac{d-2}{2},-\frac{1}{2})};1,2z^2-1)\prod_{i=1}^{d+1}(1-t_i^2)^{\kappa_i-1}d\mathbf{t}, \tag{3.7}$$

where $z = \sqrt{x_1y_1}\,t_1+\ldots+\sqrt{x_dy_d}t_d+\sqrt{1-|\mathbf{x}|_1}\sqrt{1-|\mathbf{y}|_1}\,t_{d+1}$, and we have used $|\kappa|_1 = \sum_{i=1}^{d+1}\kappa_i$ (recall $\mu=\kappa_{d+1}$). In particular, taking $\mathbf{x}=\mathbf{e}_1=(1,0,\ldots,0)$, we get

$$\mathbf{K}_n^\delta(W_{\kappa,\mu}^T;\mathbf{e}_1,\mathbf{y}) = c_{\kappa_1}\int_{-1}^1 K_n^\delta(w^{(|\kappa|_1+\frac{d-2}{2},-\frac12)};1,2y_1t_1^2-1)(1-t_1^2)^{\kappa_1-1}dt_1.$$

By definition $K_n^\delta(w^{(\lambda-1/2,-1/2)};1,s)$ is the Cesàro (C,δ) mean of the polynomials $p_n^{(\lambda-1/2,-1/2)}(1)p_n^{(\lambda-1/2,-1/2)}(s)$. Using (3.6) and (1.14), it follows as in (3.4) that

$$\begin{aligned}
& c_\tau p_k^{(\lambda-1/2,-1/2)}(1)\int_{-1}^1 p_k^{(\lambda-1/2,-1/2)}(2yt^2-1)(1-t^2)^{\tau-1}dt \\
&= c_\tau\frac{2k+\lambda}{\lambda}\int_{-1}^1 C_{2k}^{(\lambda)}(\sqrt{y}t)(1-t^2)^{\tau-1}dt \\
&= \tilde C_{2k}^{(\lambda-\tau,\tau)}(1)\tilde C_{2k}^{(\lambda-\tau,\tau)}(\sqrt{y}) = p_k^{(\lambda-\tau-1/2,\tau-1/2)}(1)p_k^{(\lambda-\tau-1/2,\tau-1/2)}(2y-1),
\end{aligned}$$

where the last equality follows from (1.13) and checking the normalization constants. Consequently, this shows

$$\mathbf{K}_n^\delta(W_{\kappa,\mu}^T;\mathbf{e}_1,\mathbf{y}) = K_n^\delta(w^{(\lambda-\kappa_1-\frac12,\kappa_1-\frac12)};1,2y_1-1)\quad\text{with}\quad \lambda=|\kappa|_1+(d-1)/2.$$

Hence, changing variable $y_i=(1-y_1)u_i$ for $2\le i\le d$, we get

$$\begin{aligned}
&\int_{T^d}|\mathbf{K}_n^\delta(W_{\kappa,\mu}^T;\mathbf{e}_1,\mathbf{y})|W_{\kappa,\mu}^T(\mathbf{y})d\mathbf{y} \\
&\qquad = c\int_{-1}^1 |K_n^\delta(w^{(\lambda-\kappa_1-\frac12,\kappa_1-\frac12)};1,t)|w^{(\lambda-\kappa_1-\frac12,\kappa_1-\frac12)}(t)dt.
\end{aligned}$$

Therefore, it follows from the summability of the Jacobi expansion that the orthogonal expansion with respect to $W_{\kappa,\mu}^T$ converges at $\mathbf{e}_1$ if and only if $\delta>\lambda-\kappa_1$ ([12, p. 246, Theorem 9.1.4]).

We are left with the case $\mu=\kappa_{d+1}=\min_{1\le i\le d+1}\kappa_i$. In this case, taking $\mathbf{x}=0$ in (3.7) and following the same argument as above, we obtain

$$\begin{aligned}
\mathbf{K}_n^\delta(W_{\kappa,\mu}^T;0,\mathbf{y}) &= c_{\kappa_{d+1}}\int_{-1}^1 K_n^\delta(w^{(|\kappa|_1+\frac{d-2}{2},-1/2)};1,2(1-|\mathbf{y}|_1)t^2-1)(1-t^2)^{\kappa_{d+1}-1}dt \\
&= K_n^\delta(w^{(\lambda-\kappa_{d+1}-\frac12,\kappa_{d+1}-\frac12)};1,1-2|\mathbf{y}|_1).
\end{aligned}$$

Hence, changing coordinates $\mathbf{y}=s\mathbf{y}'$ with $|\mathbf{y}'|_1=1$ (the ℓ^1 polar coordinates), we get

$$\begin{aligned}
&\int_{T^d}|\mathbf{K}_n^\delta(W_{\kappa,\mu}^T;0,\mathbf{y})|W_{\kappa,\mu}^T(\mathbf{y})d\mathbf{y} \\
&= c\int_0^1 s^{|\kappa|_1-\kappa_{d+1}+\frac{d-2}{2}}K_n^\delta(w^{(\lambda-\kappa_{d+1}-\frac12,\kappa_{d+1}-\frac12)};1,1-2s)(1-s)^{\kappa_{d+1}-\frac12}ds \\
&= c\int_{-1}^1 |K_n^\delta(w^{(\lambda-\kappa_{d+1}-\frac12,\kappa_{d+1}-\frac12)};1,t)|w^{(\lambda-\kappa_{d+1}-\frac12,\kappa_{d+1}-\frac12)}(t)dt,
\end{aligned}$$

where the constant can be determined by setting $n=0$. Hence, the necessity of $\delta>\lambda-\kappa_{d+1}$ follows as in the previous case. □

For the proof of Theorem 2.11, we need the pointwise estimate of the kernel. The statement, however, is simpler in this case. As in Theorem 3.3, we again use the notations ξ and ζ associated with $\mathbf{x}$ and $\mathbf{y}\in T^d$, respectively.

Theorem 3.4. *Let* $\mathbf{x}$ *be a point in the interior of* T^d. *Assume* (2.7) *with* $\kappa_{d+1} = \mu$ *and* $(d-1)/2 < \delta \le (d+1)/2$. *For* $\mathbf{y} \in T^d$,
(i) if $y_j \le x_j/2$ *for some* j, *then*

$$|\mathbf{K}_n^\delta(W^T_{\kappa,\mu}; \mathbf{x}, \mathbf{y})| \le c(\mathbf{x}) n^{-\delta+(d-1)/2} \prod_{i=1}^{d+1} (\sqrt{y_i} + n^{-1})^{-\kappa_i};$$

(ii) if for all $1 \le j \le d$, $y_j > x_j/2$, *then*

$$|\mathbf{K}_n^\delta(W^T_{\kappa,\mu}; \mathbf{x}, \mathbf{y})| \le c(\mathbf{x}) n^{-\delta+(d-1)/2} (|\xi - \zeta| + n^{-1})^{-\delta-(d+1)/2}.$$

The proof of this estimate is again being delayed to Section 5. Here we use it to prove Theorem 2.11.

Proof of Theorem 2.11. As in the proof of Theorem 2.9, we replace $\mathbf{x}$ by $\{\mathbf{x}\}^2 := (x_1^2, \ldots, x_d^2)$ and $\mathbf{y}$ by $\{\mathbf{y}\}^2 := (y_1^2, \ldots, y_d^2)$ in the estimate of Theorem 3.4, and use the notation $\mathbf{x}' = (x_1, \ldots, x_d, x_{d+1})$ and $\mathbf{y}' = (y_1, \ldots, y_d, y_{d+1})$ which are in S^d. This gives an estimate that takes the form:

(i) if $y_j^2 \le x_j^2/2$ for some j, then

$$|\mathbf{K}_n^\delta(W^T_{\kappa,\mu}; \{\mathbf{x}\}^2, \{\mathbf{y}\}^2)| \le c(\mathbf{x}) n^{-\delta+(d-1)/2} \prod_{i=1}^{d+1} (y_i + n^{-1})^{-\kappa_i};$$

(ii) if for all $1 \le j \le d$, $y_j^2 > x_j^2/2$, then

$$|\mathbf{K}_n^\delta(W^T_{\kappa,\mu}; \{\mathbf{x}\}^2, \{\mathbf{y}\}^2)| \le c(\mathbf{x}) n^{-\delta+(d-1)/2} (|\mathbf{x}' - \mathbf{y}'| + n^{-1})^{-\delta-(d+1)/2},$$

where we can assume that $x_i \ge 0$ and $y_i \ge 0$ for $1 \le i \le d+1$. Note that the estimate in (i) is the same, with $d+1$ replaced by d, as that of (i) in the proof of Theorem 2.4 and (ii) is the same as the first term in the estimate of (ii) in the proof of Theorem 2.4 (the assumption $x_j y_j > 0$ holds automatically here). Therefore, rewriting (3.5) as

$$\int_{T^d} |\mathbf{K}_n^\delta(W^T_{\kappa,\mu}; \{\mathbf{x}\}^2, \mathbf{y})| W^T_{\kappa,\mu}(\mathbf{y}) d\mathbf{y} = 2^{d+1} \int_{S^d_+} |\mathbf{K}_n^\delta(W^T_{\kappa,\mu}; \{\mathbf{x}\}^2, \{\mathbf{y}\}^2| h^2_{\kappa,\mu}(\mathbf{y}') d\omega(\mathbf{y}'),$$

since the integrand in the right hand side is even for each of its variables, we see that the proof of this theorem follows as in the proof of Theorem 2.5 and it is easier since we only need the case (i) and (ii) here due to the fact that x_i and y_i are all nonnegative. $\square$

4. Generalized Gegenbauer expansion

In this section we prove the results on the generalized Gegenbauer expansion. Essentially we give a proof of Proposition 2.8, since the proof of Theorem 2.7 follows from it. Indeed, as indicated in Section 2.3, the sufficient part of Theorem 2.7 follows from the case $d = 1$ in Theorem 2.4, and the necessary part follows from the lower bound of $T_n^\mu(0)$ and $T_n^\lambda(1)$ in Proposition 2.8.

To prove Proposition 2.8 we first recall a formula about (C, δ) means of the Jacobi expansion ([12, p. 261, (9.41.13)]). Recall that the Jacobi weight function is denoted by $w^{(\alpha,\beta)}$.

Lemma 4.1. *For the (C,δ) means of the Jacobi polynomials,*

$$K_n^\delta(w^{(\alpha,\beta)},1,u) = c(\alpha,\beta,\delta,n)P_n^{(\alpha+\delta+1,\beta)}(u) + \sum_{j=1}^{\infty} c_j(\alpha,\beta,\delta,n)K_n^{\delta+j}(w^{(\alpha,\beta)},1,u),$$

where the coefficients are given by

$$c(\alpha,\beta,\delta,n) = \frac{\Gamma(\delta+1)\Gamma(n+\alpha+\beta+\delta+2)\Gamma(2n+\alpha+\beta+\delta+3)\Gamma(n+1)}{2^{\alpha+\beta+1}\Gamma(\alpha+\delta+2)\Gamma(n+\delta+1)\Gamma(n+\beta+1)\Gamma(2n+\alpha+\beta+2\delta+3)},$$

$$c_j(\alpha,\beta,\delta,n) = (-1)^{j+1}\binom{\delta}{j}\frac{\Gamma(n+j+\delta+1)\Gamma(2n+\alpha+\beta+\delta+3)}{\Gamma(n+\delta+1)\Gamma(2n+j+\alpha+\beta+\delta+3)}.$$

Using the fact that $\Gamma(n+a+1)/\Gamma(n+1) = n^a(1+\mathcal{O}(n^{-1}))$, it is easy to see that $|c(\alpha,\beta,\delta,n)| \sim n^{\alpha+1-\delta}$; moreover, as shown in ([1] and [3]), $|c_j(\alpha,\beta,\delta,n)| \le cj^{-\alpha-\beta-\delta-4}$ and it is bounded as a function of n.

Proof of Proposition 2.8. First it follows from (1.14) that

$$\frac{1}{2}\Big[\tilde{C}_n^{(\lambda,\mu)}(1)\tilde{C}_n^{(\lambda,\mu)}(x) + \tilde{C}_n^{(\lambda,\mu)}(1)\tilde{C}_n^{(\lambda,\mu)}(-x)\Big] = \tilde{C}_n^{(\mu,\lambda)}(0)\tilde{C}_n^{(\mu,\lambda)}(\sqrt{1-x^2}).$$

Hence we have

$$\begin{aligned}T_n^\delta(w_{\lambda,\mu};1) &\ge \frac{1}{2}\int_{-1}^{1}|K_n^\delta(w_{\lambda,\mu};1,y) + K_n^\delta(w_{\lambda,\mu};1,-y)|w_{\lambda,\mu}(y)dy\\ &= \int_{-1}^{1}|K_n^\delta(w_{\mu,\lambda};0,\sqrt{1-y^2})|w_{\lambda,\mu}(y)dy\\ &= \int_{-1}^{1}|K_n^\delta(w_{\mu,\lambda};0,y)|w_{\mu,\lambda}(y)dy = T_n^\delta(w_{\mu,\lambda};0).\end{aligned}$$

Thus to finish the proof, we will only need to prove that $T_n^\mu(w_{\lambda,\mu};0) \ge c\log n$ when $\mu \ge \lambda$. In the following we assume that $\delta = \mu \ge \lambda$. In this case, using (2.5) and Lemma 4.1,

$$\begin{aligned}K_n^\delta(w_{\lambda,\mu};0,y) &= c_\lambda\int_{-1}^{1}K_n^\delta(w_{\lambda+\mu};1,s\sqrt{1-y^2})(1-s^2)^{\lambda-1}ds\\ &= cn^{\lambda+\mu+1/2-\delta}\int_{-1}^{1}P_n^{(\lambda+\mu+\delta+1/2,\lambda+\mu-1/2)}(s\sqrt{1-y^2})(1-s^2)^{\lambda-1}ds\\ &\quad+\sum_{j=1}^{\infty}c_j(\lambda+\mu-1/2,\lambda+\mu-1/2,\delta,n)K_n^{\delta+j}(w_{\lambda,\mu};0,y).\end{aligned}$$

For $j \ge 1$, $\delta+j = \mu+j > \mu$, it follows from the sufficient part of Theorem 2.7 that $\int_{-1}^1|K_n^{\delta+j}(w_{\lambda,\mu},0,y)|w_{\lambda,\mu}(y)dy$ is uniformly bounded and

$$\sum_{j=1}^{\infty}|c_j(\lambda+\mu-1/2,\lambda+\mu-1/2,\delta,n)| \le c\sum_{j=1}^{\infty}j^{-2\lambda-2\mu-\delta-3} < \infty.$$

Hence, since $\delta = \mu$, it follows from the definition (2.6) of T_n^μ that

$$\begin{aligned}T_n^\delta(w_{\mu,\lambda};0) = cn^{\lambda+\frac{1}{2}}\int_0^1\Big|\int_{-1}^{1}P_n^{(\lambda+2\mu+\frac{1}{2},\lambda+\mu-\frac{1}{2})}(s\sqrt{1-y^2})(1-s^2)^{\lambda-1}ds\Big|\\ \times|y|^{2\mu}(1-y^2)^{\lambda-\frac{1}{2}}dy + \mathcal{O}(1),\end{aligned}$$

where the outer integral is taken over $[0,1]$ instead of $[-1,1]$ since the function is even in y. Changing variable $t=\sqrt{1-y^2}$ in the outer integral, we obtain

$$T_n^\delta(w_{\mu,\lambda};0) = cn^{\lambda+\frac12}\int_0^1\Big|\int_{-1}^1 P_n^{(\lambda+2\mu+\frac12,\lambda+\mu-\frac12)}(st)(1-s^2)^{\lambda-1}ds\Big|t^{2\lambda}(1-t^2)^{\mu-\frac12}dt+\mathcal{O}(1).$$

Hence, we need to derive a lower bound on the double integral of the Jacobi polynomial from below, which will be given in the next proposition. □

We are left to derive the lower bound of the double integral of a Jacobi polynomial. The difficulty lies in the fact that we cannot take the absolute value inside the inner integral. The task requires rather delicate estimate.

Proposition 4.2. *Let $a=\lambda+2\mu$ and $b=\lambda+\mu-1$. Then*

$$\int_0^1\Big|\int_{-1}^1 P_n^{(a+\frac12,b+\frac12)}(ty)(1-t^2)^{\lambda-1/2}dt\Big||y|^{2\lambda}(1-y^2)^{\mu-1/2}dy\ge c\,n^{-\lambda-1/2}\log n.$$

Proof. First we assume that $0<\lambda<1$. Let us denote the left hand side by I_n. We start with an obvious inequality followed by a change of variables:

$$\begin{aligned} I_n &\ge c\int_0^{1-dn^{-2}}\Big|\int_{-1}^1 P_n^{(a+\frac12,b+\frac12)}(ty)(1-t^2)^{\lambda-1}dt\Big|y^{2\lambda}(1-y^2)^{\mu-1/2}dy\\ &\ge c\int_0^{1-dn^{-2}}\Big|\int_{-y}^y P_n^{(a+\frac12,b+\frac12)}(u)y(y^2-u^2)^{\lambda-1}du\Big|(1-y^2)^{\mu-1/2}dy, \end{aligned}$$

where d is any fixed positive constant. We need the asymptotics of the Jacobi polynomials as given in [12, p. 198],

$$P_n^{(\alpha,\beta)}(\cos\theta)=n^{-\frac12}k(\theta)\{\cos(N\theta+\tau)+\mathcal{O}(1)(n\sin\theta)^{-1}\}$$

for $dn^{-1}\le\theta\le\pi-dn^{-1}$, where

$$k(\theta)=\pi^{-\frac12}\Big(\sin\frac{\theta}{2}\Big)^{-\alpha-\frac12}\Big(\cos\frac{\theta}{2}\Big)^{-\beta-\frac12},\quad N=n+\frac{\alpha+\beta+1}{2},\quad \tau=-\frac{\pi}{2}\Big(\alpha+\frac12\Big),$$

and d is a fixed positive constant. Using the asymptotic formula for $P_n^{(a+\frac12,b+\frac12)}(y)$, we see that the error term is

$$\begin{aligned} E_n(u) &:= \mathcal{O}(1)n^{-\frac32}(\sin\theta)^{-1}\Big(\sin\frac{\theta}{2}\Big)^{-a-1}\Big(\cos\frac{\theta}{2}\Big)^{-b-1}\\ &=\mathcal{O}(1)n^{-\frac32}(1-u)^{-\frac{a+2}{2}}(1+u)^{-\frac{b+2}{2}} \end{aligned}$$

upon using $u=\cos\theta$. Hence, changing the order of integrals, we see that

$$\begin{aligned} &\int_0^{1-dn^{-2}}\Big|\int_{-y}^y E_n(u)y(y^2-u^2)^{\lambda-1}du\Big|(1-y^2)^{\mu-1/2}dy\\ &\le cn^{-\frac32}\int_{-1+dn^{-2}}^{1-dn^{-2}}\int_{|u|}^1(y^2-u^2)^{\lambda-1}y(1-y^2)^{\mu-1/2}dy\frac{du}{(1-u)^{\frac{a+2}{2}}(1+u)^{\frac{b+2}{2}}}\\ &\le cn^{-\frac32}\int_{-1+dn^{-2}}^{1-dn^{-2}}\frac{(1-u^2)^{\lambda+\mu-1/2}}{(1-u)^{\frac{a+2}{2}}(1+u)^{\frac{b+2}{2}}}du\\ &\le cn^{-\frac32}\big[1+n^{-\lambda-\mu}+n^{-\lambda+1}\big]=cn^{-\lambda-\frac12}, \end{aligned}$$

where in the last inequality we have broken the integral into two parts, one over $[-1+dn^{-2}, 0]$ and the other over $[0, 1-dn^{-2}]$, to get the proper estimate. Hence, using the asymptotic formula for $P_n^{(a+\frac{1}{2}, b+\frac{1}{2})}(u)$ we conclude that

$$I_n \ge c\Big[n^{-1/2}\int_{\sqrt{3}/2}^{1-dn^{-2}} |M_n(y)|(1-y)^{\mu-1/2}dy - n^{-\lambda-1/2}\Big], \tag{4.1}$$

where we define M as the integral of the main part of the asymptotic formula, namely

$$M_n(y) := \int_{-y}^{y} \frac{\cos(N\theta+\tau)}{(\sin\frac{\theta}{2})^{a+1}(\cos\frac{\theta}{2})^{b+1}}(y^2-u^2)^{\lambda-1}du$$

with $u=\cos\theta$. Note that in (4.1) we have taken the outside integral over smaller range $[\sqrt{3}/2, 1-dn^{-2}]$, which implies that if $y=\cos\phi$, then $cn^{-1} \le \phi \le \pi/6$. Introducing the function

$$f_\phi(\theta) = \frac{(\cos^2\phi - \cos^2\theta)^{\lambda-1}}{(\sin\frac{\theta}{2})^{a}(\cos\frac{\theta}{2})^{b}},$$

and changing variable $u=\cos\theta$, we can write $M_n(\cos\phi)$ as

$$M_n(\cos\phi) = \int_{\phi}^{\pi-\phi} f_\phi(\theta)\cos(N\theta+\tau)d\theta.$$

In order to give a sharp lower bound of M_n, we denote the zeros of $\cos(N\theta+\tau)$ by θ_k; that is,

$$\theta_k := \theta_{k,n} = ((k+1/2)\pi - \tau)/N.$$

We assume that ϕ lies between θ_{k-1} and θ_k, that is,

$$\theta_{k-1} \le \phi \le \theta_k, \qquad k \text{ fixed}.$$

We divided the estimate of M_n into several steps, starting with

Claim 4.1. As a function of θ, $f'_\phi(\theta) \le 0$ on $\phi \le \theta \le \pi/2$.

In fact, this follows from the explicit formula of $f'_\phi(\theta)$ given by

$$\begin{aligned} f'_\phi(\theta) =& \frac{(\cos^2\phi-\cos^2\theta)^{\lambda-2}}{(\sin\frac{\theta}{2})^{a+1}(\cos\frac{\theta}{2})^{b+1}}\Big\{(\lambda-1)\cos\theta\sin^2\theta \\ & - \frac{1}{4}(\cos^2\phi-\cos^2\theta)\big[\big((a-b)+(a+b)\cos\theta\big)\big]\Big\}, \end{aligned}$$

and the fact that $a>b$, $\lambda<1$ and $\cos\theta \ge 0$ in the given range of θ.

Claim 4.2. For $0 \le \phi \le \pi/6$,

$$\Big|\int_{\pi/2}^{\pi-\phi} f_\phi(\theta)\cos(N\theta+\tau)d\theta\Big| \le cn^{-1}(\sin(\theta_k-\phi)/2)^{\lambda-1}(\sin\phi/2)^{\lambda-b-1}.$$

To prove the claim, we further divided the integral into two pieces. First we have,

$$\begin{aligned}\Big|\int_{\pi-\theta_k}^{\pi-\phi} f_\phi(\theta)\cos(N\theta+\tau)d\theta\Big| &\le \frac{c}{(\sin\phi/2)^b}\int_{\pi-\theta_k}^{\pi-\phi}(\cos^2\phi-\cos^2\theta)^{\lambda-1}d\theta\\ &\le \frac{c}{(\sin\phi/2)^b}\int_{\phi}^{\theta_k}\Big(\sin\frac{\theta-\phi}{2}\sin\frac{\theta+\phi}{2}\Big)^{\lambda-1}d\theta\\ &\le c(\sin(\theta_k-\phi)/2)^{\lambda}(\sin\phi/2)^{\lambda-b-1}\\ &\le cn^{-1}(\sin(\theta_k-\phi)/2)^{\lambda-1}(\sin\phi/2)^{\lambda-b-1}.\end{aligned}$$

The second piece is integral over $[\pi-\theta_k,\pi/2]$. Integrating by parts gives

$$\begin{aligned}&\int_{\pi/2}^{\pi-\theta_k} f_\phi(\theta)\cos(N\theta+\tau)d\theta\\ &\qquad=\frac{1}{N}f_\phi(\theta)\sin(N\theta+\tau)\Big|_{\pi/2}^{\pi-\theta_k}-\frac{1}{N}\int_{\pi/2}^{\pi-\theta_k} f'_\phi(\theta)\sin(N\theta+\tau)d\theta.\end{aligned}$$

Using the fact that $0\le\phi\le\pi/6$, $\sin((\pi-\theta_k)/2)=\cos(\theta_k/2)\sim 1$, $\cos((\pi-\theta_k)/2)=\sin(\theta_k/2)\sim\sin(\phi/2)$ and

$$\cos^2\phi-\cos^2(\pi-\theta_k)\sim\cos\phi-\cos\theta_k\sim\sin\frac{\theta_k-\phi}{2}\sin\frac{\theta_k+\phi}{2}\ge c\sin\frac{\theta_k-\phi}{2}\sin\frac{\phi}{2},$$

it follows that the absolute value of the first term is bounded by

$$\Big|\frac{1}{N}f_\phi(\theta)\sin(N\theta+\tau)\Big]_{\pi/2}^{\pi-\theta_k}\Big|\le\frac{c}{n}\Big[\Big(\sin\frac{\theta_k-\phi}{2}\Big)^{\lambda-1}\Big(\sin\frac{\phi}{2}\Big)^{\lambda-1-b}+1\Big].$$

For the second term, we use the formula for the derivative of $f'_\phi(\theta)$ to show that its absolute value is bounded by

$$\frac{c}{n}\Big[\int_{\pi/2}^{\pi-\theta_k}\frac{(\cos^2\phi-\cos^2\theta)^{\lambda-2}}{(\sin\frac{\theta}{2})^{a+1}(\cos\frac{\theta}{2})^{b+1}}\cos\theta\sin^2\theta d\theta+\int_{\pi/2}^{\pi-\theta_k}\frac{(\cos^2\phi-\cos^2\theta)^{\lambda-1}}{(\sin\frac{\theta}{2})^{a+1}(\cos\frac{\theta}{2})^{b+1}}d\theta\Big],$$

in which the first term is bounded by, using $\sin\theta=2\sin(\theta/2)\cos(\theta/2)$,

$$\begin{aligned}\frac{c}{n}\int_{\pi/2}^{\pi-\theta_k}\frac{(\cos^2\phi-\cos^2\theta)^{\lambda-2}}{(\cos\frac{\theta}{2})^b}\sin\theta d\theta &\le\frac{c}{n}\frac{1}{(\sin\frac{\theta_k}{2})^b}\int_{\pi/2}^{\pi-\theta_k}(\cos\phi+\cos\theta)^{\lambda-2}\sin\theta d\theta\\ &\le cn^{-1}\Big(\sin\frac{\theta_k-\phi}{2}\Big)^{\lambda-1}\Big(\sin\frac{\phi}{2}\Big)^{\lambda-b-1};\end{aligned}$$

and the second term is bounded by, since $\lambda<1$,

$$\begin{aligned}&\frac{c}{n}(\cos^2\phi-\cos^2\theta_k)^{\lambda-1}\int_{\pi/2}^{\pi-\theta_k}\frac{d\theta}{(\cos\frac{\theta}{2})^{b+1}}\\ &\qquad\le\frac{c}{n}cn^{-1}\Big(\sin\frac{\theta_k-\phi}{2}\sin\frac{\phi}{2}\Big)^{\lambda-1}\Big(\sin\frac{\theta_k}{2}\Big)^{-b}\\ &\qquad\le cn^{-1}\Big(\sin\frac{\theta_k-\phi}{2}\Big)^{\lambda-1}\Big(\sin\frac{\phi}{2}\Big)^{\lambda-b-1}.\end{aligned}$$

Putting these estimates together, we establish the Claim 4.2.

Claim 4.3. There exists an $\varepsilon>0$, such that for $\phi\in I_k:=[\theta_k-\varepsilon/n,\theta_k]$,

$$\Big|\int_{\phi}^{\pi/2} f_\phi(\theta)\cos(N\theta+\tau)d\theta\Big|\ge c\Big[n^{-\lambda}(\sin\phi/2)^{\lambda-a-1}-n^{-1}\Big].$$

Let m denote the largest integer such that $\theta_k \le \pi/2$ (m depends on n). Evidently, $\theta_m - \pi/2 \sim 1/n$. To prove the claim, we split the integral over $[\phi, \pi/2]$ into three pieces over $[\phi, \theta_k]$, $[\theta_k, \theta_m]$ and $[\theta_m, \pi/2]$, respectively. First of all, we have

$$J_{n,1}(\phi) := \Big| \int_{\theta_m}^{\pi/2} f_\phi(\theta) \cos(N\theta + \tau) d\theta \Big| \le c \int_{\theta_m}^{\pi/2} \frac{(\cos^2 \phi - \cos^2 \theta)^{\mu-1}}{(\sin \frac{\theta}{2})^a (\cos \frac{\theta}{2})^b} d\theta \le cn^{-1}$$

since $0 \le \phi \le \pi/6$. We note the above estimate will not change if the integral is over $[\theta_{m-1}, \pi/2]$. Hence, we can assume that $m - k$ is even. Then our second, and main part, is given by

$$\begin{aligned}
J_{n,2}(\phi) &:= \Big| \int_{\theta_k}^{\theta_m} f_\phi(\theta) \cos(N\theta + \tau) d\theta \Big| = \Big| \sum_{j=k+1}^{m} \int_{\theta_{j-1}}^{\theta_j} f_\phi(\theta) \cos(N\theta + \tau) d\theta \Big| \\
&= \frac{1}{N} \Big| \sum_{j=k+1}^{m} (-1)^j \int_0^\pi f_\phi\Big(\frac{\xi + (j - \frac{1}{2})\pi - \tau}{N} \Big) \sin \xi d\xi \Big| \\
&= \frac{1}{N} \Big| \sum_{j=1}^{(m-k)/2} \int_0^\pi \Big[f_\phi\Big(\frac{\xi + (k + 2j - \frac{1}{2})\pi - \tau}{N} \Big) \\
&\qquad\qquad - f_\phi\Big(\frac{\xi + (k + 2j + \frac{1}{2})\pi - \tau}{N} \Big) \Big] \sin \xi d\xi \Big| \\
&= \frac{2}{N^2} \sum_{j=1}^{(m-k)/2} |f'_\phi(\xi_{2j})|,
\end{aligned}$$

where $\xi_{2j} = \frac{\eta_j + (k+2j-\frac{1}{2})\pi - \tau}{N}$ and $\eta_j \in (0, 1)$ as follows from the mean value theorem and Claim 4.1. Since the formula of $f'_\phi(\theta)$ shows that it is a sum of two terms of the same sign, we can drop one term and get

$$J_{n,2}(\phi) \ge \frac{2}{N^2} |f'_\phi(\xi_2)| \ge \frac{c}{N^2} \frac{(\cos^2 \phi - \cos^2 \xi_2)^{\lambda-2}}{(\sin \frac{\xi_2}{2})^{a+1} (\cos \frac{\xi_2}{2})^{b+1}} \cos \xi_2 \sin^2 \xi_2.$$

Since $cn^{-1} \le \phi \le \pi/6$ and $\theta_{k-1} \le \phi \le \theta_k$, we see that $\xi_2 = \theta_k + \mathcal{O}(1/n) = \phi + \mathcal{O}(1/n)$, so that $\cos \xi_2 \sim 1$, $\cos \phi + \cos \xi_2 \sim 1$, and $\sin \frac{\xi_2 + \phi}{2} \sim \sin \frac{\phi}{2}$. Then we have

$$J_{n,2}(\phi) \ge \frac{c}{N^2} \frac{(\sin \frac{\xi_2 - \phi}{2} \sin \frac{\xi_2 + \phi}{2})^{\lambda-2}}{(\sin \frac{\xi_2}{2})^{a-1}} \ge \frac{c}{N^2} \Big(\sin \frac{\phi}{2} \Big)^{\lambda-1-a} \Big(\sin \frac{\xi_2 - \phi}{2} \Big)^{\lambda-2}.$$

Hence, using the fact that $\xi_2 - \phi \le \xi_2 - \theta_{k-1} \le c/N$, we conclude that

$$J_{n,2}(\phi) \ge cn^{-\lambda} (\sin(\phi/2))^{\lambda-1-a}.$$

We note that this estimate holds for all $\phi \in [cn^{-1}, \pi/6]$. For the remaining case, we assume that $\phi \in [\theta_k - \varepsilon/n, \theta_k]$, where ε is a positive number whose value is to be determined. Then

$$\begin{aligned} J_{n,3}(\phi) :=& \Big| \int_\phi^{\theta_k} f_\phi(\theta)\cos(N\theta+\tau)d\theta \Big| \\ \le& \frac{c}{\sin(\phi/2)^a} \int_\phi^{\theta_k} \Big(\sin\big(\frac{\theta-\phi}{2}\big) \sin\big(\frac{\theta+\phi}{2}\big)\Big)^{\lambda-1} d\theta \\ \le& c(\sin(\phi/2))^{\lambda-a-1}(\theta_k-\phi)^\lambda \le c\varepsilon^\lambda n^{-\lambda}(\sin(\phi/2))^{\lambda-a-1}. \end{aligned}$$

Notice that the upper bound of $J_{n,3}$ is the same as the lower bound of $J_{n,2}$ saving the ε^λ term. Choosing ε small but fixed so that the upper bound of $J_{n,3}$ is less than half of the lower bound of $J_{n,2}$, we have proved Claim 4.3 upon combining the estimate for $J_{n,1}$, $J_{n,2}$ and $J_{n,3}$.

Claim 4.4 There exists an $\varepsilon > 0$, such that for $\phi \in I_k = [\theta_k - \varepsilon/n, \theta_k]$,

$$M_n(\cos\phi) \ge c\Big[n^{-\lambda}\Big(\sin\frac{\phi}{2})\Big)^{\lambda-a-1} - n^{-1}\Big(\sin\frac{\theta_k-\phi}{2}\Big)^{\lambda-1}\Big(\sin\frac{\phi}{2}\Big)^{\lambda-b-1}\Big].$$

This follows as the consequence of Claim 4.2, Claim 4.3 and $b = \lambda + \mu - 1$.

We are now in position to prove the proposition in the case of $0 < \lambda < 1$. Let p be the largest positive integer such that $\sqrt{3}/2 \le \cos\theta_p$ and q be the smallest positive integer such that $\cos\theta_q \le 1 - dn^{-2}$. Evidently, $q < p$, $p = p(n) \sim n$ and $q = q(n) \sim n$. It follows from (4.1) and the Claim 4.4 that

$$I_n \ge c(A_n - B_n - n^{-\lambda-1/2}),$$

where

$$\begin{aligned} A_n \ &= n^{-\lambda-1/2}\sum_{j=q}^p \int_{\theta_j-\frac{\varepsilon}{n}}^{\theta_j} \Big(\sin\frac{\phi}{2}\Big)^{\lambda-a-1}(\sin\phi)^{2\mu}d\phi \\ &\ge cn^{-\lambda-1/2}\sum_{j=q}^p \int_{\theta_j-\frac{\varepsilon}{n}}^{\theta_j}(\sin\phi)^{-1}d\phi \\ &\ge cn^{-\lambda-1/2}\sum_{j=q}^p j^{-1} \ge cn^{-\lambda-1/2}\log n, \end{aligned}$$

in which we have used the fact that $a = \lambda + 2\mu$, and

$$\begin{aligned} B_n \ &= n^{-3/2}\sum_{j=q}^p \int_{\theta_j-\frac{\varepsilon}{n}}^{\theta_j} \Big(\sin\frac{\theta_j-\phi}{2}\Big)^{\lambda-1}\Big(\sin\frac{\phi}{2}\Big)^{-\mu}(\sin\phi)^{2\mu}d\phi \\ &\le cn^{-3/2}\sum_{j=q}^p \theta_j^\mu \int_{\theta_j-\frac{\varepsilon}{n}}^{\theta_j}(\theta_j-\phi)^{\lambda-1}d\phi \\ &\le cn^{-3/2-\lambda-\mu}\sum_{j=q}^p j^\mu \\ &\le cn^{-\lambda-1/2}. \end{aligned}$$

Consequently, we conclude that $I_n \ge cn^{-\lambda-1/2}\log n$, which gives the stated result for $0 < \lambda < 1$.

If $\lambda = 0$, then I_n in limit form reduces to

$$\begin{aligned} I_n &= \int_0^1 \left|P_n^{(2\mu+1/2,\mu-1/2)}(y) + P_n^{(2\mu+1/2,\mu-1/2)}(-y)\right|(1-y^2)^{\mu-1/2}dy \\ &\geq \int_0^1 \left|P_n^{(2\mu+1/2,\mu-1/2)}(y)\right|(1-y^2)^{\mu-1/2}dy \\ &\qquad - \int_0^1 \left|P_n^{(2\mu+1/2,\mu-1/2)}(-y)\right|(1-y^2)^{\mu-1/2}dy. \end{aligned}$$

By the estimate of the Jacobi polynomial (see Lemma 5.3 below), the second integral has the upper bound $cn^{-1/2}$; the first term is asymptotically $n^{-1/2}\log n$ by [12, p. 173, (7.34.1)]. Thus, $I_n \geq cn^{-1/2}\log n$ for $\lambda = 0$, which agrees with the case of $\lambda > 0$.

Next we turn our attention to the case $\lambda \geq 1$. Let $r = [\lambda]$. Using the identity

$$P_n^{(a+\frac{1}{2},b+\frac{1}{2})}(y) = \frac{2}{n+a+b+1}\frac{d}{dy}P_{n+1}^{(a-\frac{1}{2},b-\frac{1}{2})}(y) \tag{4.2}$$

and integrating by parts, we get

$$\begin{aligned} &\int_{-y}^{y} P_n^{(a+\frac{1}{2},b+\frac{1}{2})}(u)(y^2-u^2)^{\lambda-1}du \\ =&\frac{(-2)^r}{\prod_{i=1}^r(n+a+b+2-i)}\int_{-y}^{y} P_{n+r}^{(a+\frac{1}{2}-r,b+\frac{1}{2}-r)}(u)\frac{d^r}{du^r}\left[(y^2-u^2)^{\lambda-1}\right]du. \end{aligned}$$

Using induction if necessary, it is easy to verify that

$$\frac{d^r}{du^r}\left[(y^2-u^2)^{\lambda-1}\right] = Au^r(y^2-u^2)^{\lambda-r-1} + q(u)(y^2-u^2)^{\lambda-r},$$

where $A = (-2)^r\Gamma(\lambda)/\Gamma(\lambda-r)$ and $q(u)$ is a polynomial. We conclude that

$$\begin{aligned} &\Big|\int_{-y}^{y} P_n^{(a+\frac{1}{2},b+\frac{1}{2})}(u)(y^2-u^2)^{\lambda-1}dt\Big| \\ &\qquad \geq cn^{-r}\Big[\Big|\int_{-y}^{y} P_{n+r}^{(a+\frac{1}{2}-[\lambda],b+\frac{1}{2}-[\lambda])}(u)u^{[\lambda]}(y^2-u^2)^{\lambda-[\lambda]-1}du\Big| \\ &\qquad\qquad -\Big|\int_{-y}^{y} |P_{n+r}^{(a+\frac{1}{2}-[\lambda],b+\frac{1}{2}-[\lambda])}(u)q(u)(y^2-u^2)^{\lambda-[\lambda]}du\Big|\Big]. \end{aligned}$$

We then use the asymptotics of the Jacobi polynomials as in the proof of the case $0 < \lambda < 1$. Evidently, with $\lambda^* = \lambda - [\lambda]$, $a^* = a - [\lambda] = \lambda^* + 2\mu$ and $b^* = b - [\lambda] = \lambda^* + \mu - 1$, we can derive the lower bound (with $\log n$) of the first part exactly as in the case $0 < \lambda < 1$. Moreover, for the second part, we need to show essentially the following estimate with $0 < \lambda < 1$:

$$\begin{aligned} L_n := \int_0^{1-dn^{-2}} &\Big|\int_{-y}^{y} P_n^{(a+1/2,b+1/2)}(u)q(u)\times(y^2-u^2)^{\lambda}du\Big| \\ &(1-y^2)^{\mu-1/2}dy \leq cn^{-\lambda-1/2}. \end{aligned} \tag{4.3}$$

To prove (4.3), integrating by parts for the inner integral (using (4.2)) gives

$$L_n \leq \frac{c}{n}\int_0^{1-dn^{-2}} \Big|\int_{-y}^{y} P_{n+1}^{(a-1/2,b-1/2)}(u)\frac{d}{du}\left[q(u)(y^2-u^2)^{\lambda}\right]du\Big|(1-y^2)^{\mu-1/2}dy;$$

then using the estimate of the Jacobi polynomial in Lemma 5.3 we obtain

$$\begin{aligned} L_n &\le cn^{-3/2}\int_0^{1-dn^{-2}}\int_0^y (1-u)^{-a/2}(y-u)^{\lambda-1}du(1-y)^{\mu-1/2}dy \\ &\le cn^{-3/2}\int_0^{1-dn^{-2}}(1-y)^{-a/2+\mu-1/2}dy \\ &\le cn^{-3/2}\le cn^{-\lambda-1/2}, \end{aligned}$$

since $b<a$, $a>0$ and $-a/2+\mu-1/2=-(\lambda+1)/2>-1$.

If λ is a positive integer, we do integration by parts a number of times to get

$$\begin{aligned} \int_{-y}^{y} P_n^{(a+1/2,b+1/2)}(u)(y^2-u^2)^{\lambda-1}du &= \frac{(-1)^{\lambda-1}2^{\lambda}\Gamma(\lambda)}{\prod_{i=1}^{\lambda}(n+a+b+2-i)} \\ \times\Big[y^{\lambda-1}P_{n+\lambda}^{(a+1/2-\lambda,b+1/2-\lambda)}(y) &- (-y)^{\lambda-1}P_{n+\lambda}^{(a+1/2-\lambda,b+1/2-\lambda)}(-y) \\ &- \int_{-y}^{y}P_{n+\lambda}^{(a+1/2-\lambda,b+1/2-\lambda)}(u)s(u)du\Big], \end{aligned}$$

where $s(u)$ is a polynomial of degree 2λ. Thus,

$$\begin{aligned} I_n \ge &\frac{c}{n^{\lambda}}\Big[\int_0^1\Big|P_{n+\lambda}^{(a+1/2-\lambda,b+1/2-\lambda)}(y)\Big|y^{\lambda-1}(1-y^2)^{\mu-1/2}dy \\ &-\int_0^1\Big|P_{n+\lambda}^{(a+1/2-\lambda,b+1/2-\lambda)}(-y)\Big|y^{\lambda-1}(1-y^2)^{\mu-1/2}dy \\ &-\int_0^1\Big|\int_{-y}^{y}P_{n+\lambda}^{(a+1/2-\lambda,b+1/2-\lambda)}(u)s(u)du\Big|(1-y^2)^{\mu-1/2}dy\Big]. \end{aligned}$$

Just as the case $\lambda=0$, the first term above has the lower bound $cn^{-\lambda-1/2}\log n$, and the second term has the upper bound $cn^{-\lambda-1/2}$. For the third term, integrating by parts again by the use of (4.2), the inner integral becomes a sum of the terms

$$\frac{1}{\prod_{i=1}^{j}(n+a+b-\lambda+2-i)}P_{n+\lambda+j}^{(a+1/2-\lambda-j,b+1/2-\lambda-j)}(\pm y)s^{(j-1)}(\pm y),$$

for $j=1,2,\ldots,2\lambda+1$. Hence, applying the estimate of the Jacobi polynomial in Lemma 5.3 again, the part of L_n corresponding to each of these terms has an upper bound $cn^{-\lambda-1/2}$. This shows that $L_n\ge cn^{-\lambda-1/2}\log n$ for λ being a positive integer. The proof of the proposition is completed. □

5. The estimate of the kernels for h-harmonics

In this section we prove the estimates of the kernels of the Cesàro means for h-harmonics given in Theorem 3.1 and Theorem 3.2. The proof of the estimates is rather long. We will break the proof into a number of steps, starting with a series of reductions.

Since the Gegenbauer polynomial and the Jacobi polynomial are related by

$$\frac{n+\lambda}{\lambda}C_n^{(\lambda)}(t)=p_n^{(\lambda-1/2,\lambda-1/2)}(1)p_n^{(\lambda-1/2,\lambda-1/2)}(t), \tag{5.1}$$

it follows from the formula (1.5) of the reproducing kernel that

$$K_n^\delta(h_\kappa^2;\mathbf{x},\mathbf{y}) = c_\kappa \int_{[-1,1]^d} K_n^\delta(w^{(\gamma,\gamma)};1, x_1y_1t_1+\ldots+x_dy_dt_d) \times \prod_{i=1}^d (1+t_i)(1-t_i^2)^{\kappa_i-1}d\mathbf{t}, \tag{5.2}$$

where $\gamma = |\kappa|_1 + (d-3)/2$. The first step is to replace the kernel for the Jacobi expansion by a sharp estimate. For this we use the formula in 4.1 repeatedly as in [9] leads to the following lemma.

Lemma 5.1. *For any $\alpha, \beta > -1$ such that $\alpha+\beta+\delta+3>0$,*

$$K_n^\delta(w^{(\alpha,\beta)},1,u) = \sum_{j=0}^J b_j(\alpha,\beta,\delta,n)P_n^{(\alpha+\delta+j+1,\beta)}(u) + G_n^\delta(u),$$

where J is a fixed integer and

$$G_n^\delta(u) = \sum_{j=J+1}^\infty d_j(\alpha,\beta,\delta,n)K_n^{\delta+j}(w^{(\alpha,\beta)},1,u);$$

moreover, the coefficients satisfy the inequalities,

$$|b_j(\alpha,\beta,\delta,n)| \le cn^{\alpha+1-\delta-j} \quad \text{and} \quad |d_j(\alpha,\beta,\delta,n)| \le cj^{-\alpha-\beta-\delta-4}.$$

Since the kernel function $K_n^{\delta+j}(w^{(\alpha,\beta)},1,u)$ contained in the G_n^δ term has large index $\delta+j>\delta$, it can be handled using the following estimate of the kernel function, which is used in [1] and [3] (see Theorem 3.9 there).

Lemma 5.2. *Let $\alpha,\beta \ge -1/2$. Then if $0 \le \delta \le \alpha+3/2$,*

$$|K_n^\delta(w^{(\alpha,\beta)},1,u)| \le cn^{\alpha+1/2-\delta}\Big[(1-u+n^{-2})^{-(\delta+\alpha+3/2)/2} + (1+u+n^{-2})^{-(\beta+1/2)/2}\Big];$$

if $\alpha+3/2 \le \delta \le \alpha+\beta+2$,

$$|K_n^\delta(w^{(\alpha,\beta)},1,u)| \le cn^{-1}\Big[(1-u+n^{-2})^{-(\alpha+3/2)} + (1+u+n^{-2})^{-(\alpha+\beta+2-\delta)/2}\Big];$$

if $\delta \ge \alpha+\beta+2$,

$$|K_n^\delta(w^{(\alpha,\beta)},1,u)| \le cn^{-1}(1-u+n^{-2})^{-(\alpha+3/2)}.$$

We emphasize that the above estimates are not strong enough to yield our estimate for the kernel (5.2) on the sphere. The singular integrals involved in (5.2) require a far more delicate analysis. Upon using Lemma 5.1, the essential part is dealt with using the following estimate of the Jacobi polynomials ([12, (7.32.5) and (4.1.3)]).

Lemma 5.3. *For an arbitrary real number α and $t\in[0,1]$,*

$$|P_n^{(\alpha,\beta)}(t)| \le cn^{-1/2}(1-t+n^{-2})^{-(\alpha+1/2)/2},$$

and the estimate on $[-1,0]$ follows from the fact that $P_n^{(\alpha,\beta)}(t) = (-1)^nP_n^{(\beta,\alpha)}(-t)$.

In particular, in the proof we will use the estimate in a unified form

$$|P_n^{(\alpha,\beta)}(t)| \le cn^{-1/2}\Big[(1-t+n^{-2})^{-(\alpha+1/2)/2} + (1+t+n^{-2})^{-(\beta+1/2)/2}\Big]. \tag{5.3}$$

Moreover, the product of the two terms on the right hand side of (5.3) is also an upper bound of $P_n^{(\alpha,\beta)}(t)$ on $[-1,1]$, which shows, in particular, that if $\beta+1/2 \le 0$, then $cn^{-1/2}(1-u+n^{-2})^{-(\alpha+1/2)/2}$ is an upper bound of $P_n^{(\alpha,\beta)}(t)$ on $[-1,1]$.

Let $\alpha = \beta = |\kappa|_1 + (d-3)/2$ and let $J = [\alpha+\beta+2] = [2|\kappa|_1 + d - 1]$, where $[x]$ denotes the integer part of x. Combining the formula (5.2) and Lemma 5.1, we have

$$K_n^\delta(h_\kappa^2; \mathbf{x}, \mathbf{y}) = \sum_{j=0}^{J} b_j(\alpha,\beta,\delta,n)\Omega_j(\mathbf{x},\mathbf{y}) + \Omega_*(\mathbf{x},\mathbf{y}), \tag{5.4}$$

where

$$\Omega_j(\mathbf{x},\mathbf{y}) = c_\kappa \int_{[-1,1]^d} P_n^{(\alpha+\delta+j+1,\beta)}(u(\mathbf{x},\mathbf{y},\mathbf{t})) \prod_{i=1}^{d}(1+t_i)(1-t_i^2)^{\kappa_i-1}d\mathbf{t}, \tag{5.5}$$

and

$$\Omega_*(\mathbf{x},\mathbf{y}) = c_\kappa \int_{[-1,1]^d} G_n^\delta(u(\mathbf{x},\mathbf{y},\mathbf{t})) \prod_{i=1}^{d}(1+t_i)(1-t_i^2)^{\kappa_i-1}d\mathbf{t}, \tag{5.6}$$

here and in the following we write

$$u(\mathbf{x},\mathbf{y},\mathbf{t}) = x_1y_1t_1 + \ldots + x_dy_dt_d.$$

Our main task is to estimate Ω_0, since Ω_j for $j \ge 1$ can be estimated similarly and Ω_* is relatively easy to handle. We state the estimate for Ω_0 as the following lemma.

Lemma 5.4. *For* $\mathbf{x},\mathbf{y} \in S^{d-1}$, $\delta \ge (d-2)/2$,

$$|\Omega_0(\mathbf{x},\mathbf{y})| \le c\frac{\prod_{j=1}^d (|x_jy_j| + n^{-1}|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-2})^{-\kappa_j}}{n^{|\kappa|_1+1/2}(|\bar{\mathbf{x}}-\bar{\mathbf{y}}| + n^{-1})^{\delta+d/2}}. \tag{5.7}$$

The proof of Lemma 5.4 is long. We begin with further reductions to reduce the task to its essential part. The idea is to use integration by part as much as we can to separate the part of n to the negative power. Let

$$\kappa_j = p_j + \lambda_j, \qquad p_j \in \mathbb{N}_0, \quad \lambda_j \in [0,1), \quad 1 \le j \le d.$$

Assume first that $\lambda_j \in (0,1)$, $1 \le j \le d$. Using the well-known relation

$$\frac{d}{dt}P_{n+1}^{(\alpha,\beta)}(t) = \frac{1}{2}(n+\alpha+\beta+2)P_n^{(\alpha+1,\beta+1)}(t) \tag{5.8}$$

([12, (4.21.7)]), we integrate Ω_0 by parts p_i times for t_i for each i to conclude that

$$\Omega_0(\mathbf{x},\mathbf{y}) = \frac{c_\kappa(-1)^{|\mathbf{p}|_1}\prod_{i=1}^d (x_iy_i)^{-p_i}}{\prod_{i=1}^{|\mathbf{p}|_1}(n+2|\kappa|_1+\delta+d-i-1)/2} \times \tag{5.9}$$

$$\int_{[-1,1]^d} P_{n+|\mathbf{p}|_1}^{(|\lambda|_1+\delta+(d-1)/2,|\lambda|_1+(d-3)/2)}(u(\mathbf{x},\mathbf{y},\mathbf{t})) \prod_{i=1}^{d}\frac{\partial^{p_i}}{\partial t_i^{p_i}}\Big[(1+t_i)(1-t_i^2)^{\kappa_i-1}\Big]d\mathbf{t}.$$

In order to estimate the integrals, we break each integral over $[-1,1]$ into two integrals over $[-1,1-\varepsilon_{n,i}]$ and $[1-\varepsilon_{n,i},1]$, respectively, where $0 \le \varepsilon_{n,i} \le 1/2$ are to be determined. Then the set $[-1,1]^d$ is the union of the sets

$$E_m = (1-\varepsilon_{n,1},1] \times \cdots \times (1-\varepsilon_{n,m},1] \times [-1,1-\varepsilon_{n,m+1}] \times \cdots \times [-1,1-\varepsilon_{n,d}]$$

and the permutations of E_m; that is,

$$[-1,1]^d = \bigcup_{\sigma\in\mathcal{S}_d}\bigcup_{m=0}^{d} \sigma E_m, \qquad \sigma E_m := \{\mathbf{x} : (x_{\sigma_1},\ldots,x_{\sigma_d}) \in E_m\},$$

where $\mathcal{S}_d$ denotes the symmetric group of d objects. Consequently, we have

$$\Omega_0 = \frac{(-1)^{|\mathbf{p}|_1}\prod_{i=1}^d (x_i y_i)^{-p_i}}{\prod_{i=1}^{|\mathbf{p}|_1}(n+2|\kappa|_1+\delta+d-i-1)/2}\Big[\sum_{m=0}^{d}\Omega_{0,m} + \sum_{\sigma}\sum_{m=1}^{d-1}\sigma\Omega_{0,m}\Big] \tag{5.10}$$

where, we will write $\Omega_{0,m}$ in stead of $\Omega_{0,m}(\mathbf{x},\mathbf{y})$ from now on,

$$\Omega_{0,m} = c_\kappa \int_{E_m} P_{n+|\mathbf{p}|_1}^{(|\lambda|_1+\delta+(d-1)/2,|\lambda|_1+(d-3)/2)}(u(\mathbf{x},\mathbf{y},\mathbf{t})) \times \prod_{i=1}^{d}\frac{\partial^{p_i}}{\partial t_i^{p_i}}\Big[(1+t_i)(1-t_i^2)^{\kappa_i-1}\Big]d\mathbf{t}$$

and $\sigma\Omega_{0,m}$ denotes the integral over σE_m. Since the desired results in Theorem 3.1 and Theorem 3.2 are independent of the choice of the order of x_j, we only need to deal with $\Omega_{0,m}$ for $0\le m\le d$.

We make a further reduction by making use of (5.8) again and integrating by parts of $\Omega_{0,m}$. Let us introduce the notations

$$u_j(\mathbf{x},\mathbf{y},\mathbf{t}) = x_1y_1t_1+\ldots+x_jy_jt_j+x_{j+1}y_{j+1}(1-\varepsilon_{n,j+1})+\ldots+x_dy_d(1-\varepsilon_{n,d})$$

for $j=0,1,\ldots,d$. We note that $u_0(\mathbf{x},\mathbf{y},\mathbf{t})$ is independent of $\mathbf{t}$ and $u_d(\mathbf{x},\mathbf{y},\mathbf{t}) = u(\mathbf{x},\mathbf{y},\mathbf{t})$. We consider $\Omega_{0,0}$ first. If we use (5.8) and integrate by parts with respect to t_d in $\Omega_{0,0}$, we have

$$\begin{aligned}\Omega_{0,0} = {} & \frac{2(x_dy_d)^{-1}}{n+|\mathbf{p}|_1+2|\lambda|_1+d+\delta-2}\int_{-1}^{1-\varepsilon_{n,1}}\cdots\int_{-1}^{1-\varepsilon_{n,d-1}} \\ & \Big\{\frac{\partial^{p_d}}{\partial t_d^{p_d}}\Big[(1+t_d)(1-t_d^2)^{\kappa_d-1}\Big]\Big|_{t_d=1-\varepsilon_{n,d}} P_{n+|\mathbf{p}|_1+1}^{(|\lambda|_1+\delta+(d-3)/2,|\lambda|_1+(d-5)/2)}(u_{d-1}(\mathbf{x},\mathbf{y},\mathbf{t})) \\ & \quad - \int_{-1}^{1-\varepsilon_{n,d}} P_{n+|\mathbf{p}|_1+1}^{(|\lambda|_1+\delta+(d-3)/2,|\lambda|_1+(d-5)/2)}(u(\mathbf{x},\mathbf{y},\mathbf{t})) \\ & \times \frac{\partial^{p_d+1}}{\partial t_d^{p_d+1}}\Big[(1+t_d)(1-t_d^2)^{\kappa_d-1}\Big]dt_d\Big\}\prod_{i=1}^{d-1}\frac{\partial^{p_i}}{\partial t_i^{p_i}}\Big[(1+t_i)(1-t_i^2)^{\kappa_i-1}\Big]dt_1\ldots dt_{d-1}.\end{aligned}$$

Continuing integration by parts once for each of the variables t_i, $1\le i\le d-1$, we conclude that

$$\Omega_{0,0} = \frac{2^d\prod_{i=1}^d(x_iy_i)^{-1}}{\prod_{i=1}^d(n+|\mathbf{p}|_1+2|\lambda|_1+d+\delta-i-1)}\Big\{\sum_{j=0}^{d}\Lambda_{0,j}+\sum_{\sigma}\sum_{j=1}^{d-1}\sigma\Lambda_{0,j}\Big\}, \tag{5.11}$$

where $\Lambda_{0,j}$ is given by

$$\begin{aligned}\Lambda_{0,j} = {} & (-1)^j\prod_{i=j+1}^{d}\frac{\partial^{p_i}}{\partial t_i^{p_i}}\Big[(1+t_i)(1-t_i^2)^{\kappa_i-1}\Big]\Big|_{t_i=1-\varepsilon_{n,i}}\int_{-1}^{1-\varepsilon_{n,1}}\cdots\int_{-1}^{1-\varepsilon_{n,j}} \\ & P_{n+|\mathbf{p}|_1+d}^{(|\lambda|_1+\delta-(d+1)/2,|\lambda|_1-(d+3)/2)}(u_j(\mathbf{x},\mathbf{y},\mathbf{t}))\prod_{i=1}^{j}\frac{\partial^{p_i+1}}{\partial t_i^{p_i+1}}\Big[(1+t_i)(1-t_i^2)^{\kappa_i-1}\Big]dt_1\ldots dt_j,\end{aligned}$$

and $\sigma\Lambda_{0,j}$ denotes a permutation of the order of variables in $\Lambda_{0,j}$; that is, $\sigma\Lambda_{0,j}$ is the same as $\Lambda_{0,j}$ after a permutation of variables. We will establish an estimate of $\Lambda_{0,j}$ that is independent of the order of variables, from which the estimate of $\sigma\Lambda_{0,j}$, hence that of $\Omega_{0,0}$, follows.

Similarly, for $\Omega_{0,m}$, we can use (5.8) and integrate by parts once for each of the variables t_j, $m+1 \le j \le d$, it then follows that

(5.12)

$$\Omega_{0,m} = \frac{2^{d-m}\prod_{i=m+1}^{d}(x_iy_i)^{-1}}{\prod_{i=1}^{d-m}(n+|\mathbf{p}|_1+2|\lambda|_1+d+\delta-i-1)} \int_{1-\varepsilon_{n,1}}^{1}\cdots\int_{1-\varepsilon_{n,m}}^{1}$$
$$\Big\{\sum_{j=m}^{d}\Lambda_{m,j}+\sum_{\sigma}\sum_{j=m+1}^{d-1}\sigma\Lambda_{m,j}\Big\}\prod_{i=1}^{m}\frac{\partial^{p_i}}{\partial t_i^{p_i}}\Big[(1+t_i)(1-t_i^2)^{\kappa_i-1}\Big]dt_1\dots dt_m,$$

where $\sigma\Lambda_{m,j}$ has the same meaning as before, and $\Lambda_{m,j}$ are given by

$$\Lambda_{m,m} = \prod_{i=m+1}^{d}\frac{\partial^{p_i}}{\partial t_i^{p_i}}\Big[(1+t_i)(1-t_i^2)^{\kappa_i-1}\Big]\Big|_{t_i=1-\varepsilon_{n,i}}$$
$$\times P_{n+|\mathbf{p}|_1+d-m}^{(|\lambda|_1+\delta-(d+1)/2+m,|\lambda|_1-(d+3)/2+m)}(u_m(\mathbf{x},\mathbf{y},\mathbf{t}))$$

and for $j = m+1,\dots,d$,

$$\Lambda_{m,j} = (-1)^{j-m}\prod_{i=j+1}^{d}\frac{\partial^{p_i}}{\partial t_i^{p_i}}\Big[(1+t_i)(1-t_i^2)^{\kappa_i-1}\Big]\Big|_{t_i=1-\varepsilon_{n,i}}\times$$
$$\int_{-1}^{1-\varepsilon_{n,m+1}}\cdots\int_{-1}^{1-\varepsilon_{n,j}} P_{n+|\mathbf{p}|_1+d-m}^{(|\lambda|_1+\delta-(d+1)/2+m,|\lambda|_1-(d+3)/2+m)}(u_j(\mathbf{x},\mathbf{y},\mathbf{t}))$$
$$\times\prod_{i=m+1}^{j}\frac{\partial^{p_i+1}}{\partial t_i^{p_i+1}}\Big[(1+t_i)(1-t_i^2)^{\kappa_i-1}\Big]dt_{m+1}\dots dt_j.$$

We note that for $m=d$, there is only one term $\Lambda_{d,d}$ in the formula of $\Omega_{0,d}$, and we take the convention that $\prod_{i=a}^{b}A_i = 1$ whenever $a>b$ in the formula.

In order to estimate $\Omega_{0,m}$, it is then sufficient to give a sharp estimate of $\Lambda_{m,j}$ that is independent of the order of its variables. Considering $t\in[0,1]$ and $t\in[-1,0]$ separately, if necessary, we see that

$$\Big|\frac{\partial^p}{\partial t^p}\Big[(1+t)(1-t^2)^{\kappa-1}\Big]\Big| \le c(1+t)^{\kappa-p}(1-t)^{\kappa-p-1}. \tag{5.13}$$

Using the inequality (5.13) and the estimate (5.3) of the Jacobi polynomial in $\Lambda_{m,j}$ and recalling that $\kappa_j = p_j+\lambda_j$, we conclude that

$$|\Lambda_{m,j}| \le cn^{-1/2}\prod_{i=j+1}^{d}\varepsilon_{n,i}^{\lambda_i-1}(U_{m,j}+V_{m,j}), \tag{5.14}$$

where $V_{m,j}$ does not appear in the case of $m=0$ and $|\lambda|_1 \le (d+2)/2$,

$$U_{m,m} := (1-u_m(\mathbf{x},\mathbf{y},\mathbf{t})+n^{-2})^{-(|\lambda|_1+\delta+m-d/2)/2},$$

$$V_{m,m} := (1+u_m(\mathbf{x},\mathbf{y},\mathbf{t})+n^{-2})^{-(|\lambda|_1+m-(d+2)/2)/2},$$

and for $j = m+1, \ldots, d$,

$$U_{m,j} := \int_{-1}^{1-\varepsilon_{n,m+1}} \cdots \int_{-1}^{1-\varepsilon_{n,j}} \frac{\prod_{i=m+1}^{j} [(1+t_i)^{\lambda_i - 1}(1-t_i)^{\lambda_i - 2}]}{(1-u_j(\mathbf{x},\mathbf{y},\mathbf{t})+n^{-2})^{(|\lambda|_1+\delta+m-d/2)/2}} dt_{m+1}\ldots dt_j,$$

$$V_{m,j} := \int_{-1}^{1-\varepsilon_{n,m+1}} \cdots \int_{-1}^{1-\varepsilon_{n,j}} \frac{\prod_{i=m+1}^{j} [(1+t_i)^{\lambda_i - 1}(1-t_i)^{\lambda_i - 2}]}{(1+u_j(\mathbf{x},\mathbf{y},\mathbf{t})+n^{-2})^{(|\lambda|_1+m-(d+2)/2)/2}} dt_{m+1}\ldots dt_j.$$

Remark 5.1. In the case of $|\lambda|_1 - (d+2)/2 + m \le 0$, the term $(1 - u_j(\mathbf{x},\mathbf{y},\mathbf{t}) + n^{-2})^{-(|\lambda|_1+m-(d+2)/2)/2}$ in $V_{m,j}$ does not appear. Indeed, this comes from the remark that follows right after Lemma 5.3, since (5.14) comes from the estimate of the Jacobi polynomial in Lemma 5.3, and the second parameter of the corresponding Jacobi polynomial is $|\lambda|_1 - (d+3)/2 + m \le -1/2$ if $|\lambda|_1 - (d+2)/2 + m \le 0$. In particular, if $m = 0$ and $|\lambda|_1 \le (d+2)/2$, then $V_{m,j}$ does not appear in (5.14).

After this series of reduction, we are ready to derive the desired estimate for Ω_0. We need to consider two separate cases. First we deal with

Case 1. $|\lambda|_1 + \delta - d/2 \ge 0$.

In this case, we choose $\varepsilon_{n,j}$ as

$$\varepsilon_{n,j} := \frac{n^{-1}(|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})}{2d(|x_j y_j| + n^{-1}|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-2})}. \tag{5.15}$$

Evidently, we have $0 \le \varepsilon_{n,j} \le 1/2d$. By the definition of $u_j(\mathbf{x},\mathbf{y},\mathbf{t})$ we have

$$\begin{aligned} 1 \pm u_j(\mathbf{x},\mathbf{y},\mathbf{t}) + n^{-2} &\ge 1 - |u_j(\mathbf{x},\mathbf{y},\mathbf{t})| + n^{-2} \\ &\ge 1 - \sum_{i=1}^{d} |x_i y_i| + n^{-2} = |\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2/2 + n^{-2} \sim (|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^2. \end{aligned} \tag{5.16}$$

Hence, it follows from the definition of $U_{m,j}$ and the fact that $|\lambda|_1 + \delta - d/2 \ge 0$,

$$|U_{m,j}| \le c \frac{1}{(|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^{|\lambda|_1+\delta+m-d/2}} \prod_{i=m+1}^{j} g_{\lambda_i}(1 - \varepsilon_{n,i}),$$

where the function $g_\lambda(t)$, $0 < \lambda < 1$, is defined by

$$g_\lambda(t) = \int_{-1}^{t} (1+s)^{\lambda-1}(1-s)^{\lambda-2} ds \sim (1+t)^\lambda (1-t)^{\lambda-1} \tag{5.17}$$

for $t \in [-1,1]$. It follows from the definition of $\varepsilon_{n,i}$ that $g_{\lambda_i}(1-\varepsilon_{n,i}) \sim \varepsilon_{n,i}^{\lambda_i - 1}$. Consequently, we conclude that

$$|U_{m,j}| \le c \prod_{i=m+1}^{j} \varepsilon_{n,i}^{\lambda_i - 1} \frac{1}{(|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^{|\lambda|_1+\delta+m-d/2}}. \tag{5.18}$$

Similarly, we have the estimate for $V_{m,j}$

$$|V_{m,j}| \le c \prod_{i=m+1}^{j} \varepsilon_{n,i}^{\lambda_i - 1} \Big[1 + \frac{1}{(|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^{|\lambda|_1+m-(d+2)/2}}\Big], \tag{5.19}$$

while we have two terms since if $|\lambda|_1 + m - (d+2)/2 < 0$, then the term $(|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^{-(|\lambda|_1+m-(d+2)/2)}$ does not appear in the above estimate of $V_{m,j}$ by Remark 5.1. Both of these estimates also hold in the case of $j = m$. However, we have

$|\lambda|_1 + m - (d+2)/2 < |\lambda|_1 + \delta + m - d/2$, which is equivalent to $\delta > -1$; hence, using the fact that $|\lambda|_1 + \delta + m - d/2 \geq 0$, we conclude that

$$|\Lambda_{m,j}| \leq cn^{-1/2} \prod_{i=m+1}^{d} \varepsilon_{n,i}^{\lambda_i - 1} \frac{1}{(|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^{|\lambda|_1 + \delta + m - d/2}}$$

for $m \leq j \leq d$. Therefore, it follows from (5.12), (5.13) and (5.14) that for $0 \leq m \leq d$,

$$\begin{aligned} |\Omega_{0,m}| &\leq cn^{-1/2} \frac{\prod_{i=m+1}^{d} \varepsilon_{n,i}^{\lambda_i - 1}}{(|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^{|\lambda|_1 + \delta + m - d/2}} \frac{\prod_{i=m+1}^{d} |x_i y_i|^{-1}}{n^{d-m}} \\ &\quad \times \int_{1-\varepsilon_{n,1}}^{1} \cdots \int_{1-\varepsilon_{n,m}}^{1} \prod_{i=1}^{m} [(1+t_i)^{\lambda_i} (1-t_i)^{\lambda_i - 1}] dt_1 \dots dt_m \\ &\leq cn^{m-d-1/2} \frac{\prod_{i=1}^{m} \varepsilon_{n,i}^{\lambda_i} \prod_{i=m+1}^{d} |x_i y_i|^{-1} \varepsilon_{n,i}^{\lambda_i - 1}}{(|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^{|\lambda|_1 + \delta + m - d/2}}, \end{aligned}$$

where we have used the fact that $\int_{1-\varepsilon_{n,j}}^{1} (1+t_j)^{\lambda_j} (1-t_j)^{\lambda_j - 1} dt_j \leq c\varepsilon_{n,j}^{\lambda_j}$. Consequently, using the formula of $\varepsilon_{n,j}$ we end up with

$$\begin{aligned} |\Omega_{0,m}| &\leq c \frac{\prod_{j=1}^{d} (|x_j y_j| + n^{-1}|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-2})^{-\lambda_j}}{n^{|\lambda|_1 + 1/2} (|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^{\delta + d/2}} \\ &\quad \times \prod_{j=m+1}^{d} \frac{|x_j y_j| + n^{-1}|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-2}}{|x_j y_j|}. \end{aligned} \tag{5.20}$$

Remark 5.2. In fact, this estimate (5.20) of $\Omega_{0,m}$ works whenever $m > 0$ without assuming the condition $|\lambda|_1 + \delta - d/2 \geq 0$. Indeed, the only place that we need the assumption $|\lambda|_1 + \delta - d/2 \geq 0$ is at the estimate of $\Lambda_{m,j}$, where we used the fact that $|\lambda|_1 + \delta - d/2 + m \geq 0$; however, since $\delta > (d-2)/2$, this inequality holds without any condition when $m > 0$.

Substituting the above estimate of $\Omega_{0,m}$ into (5.10), we obtain that for $|\lambda|_1 + \delta - d/2 \geq 0$,

$$\begin{aligned} \Omega_0 &\leq c \frac{\prod_{j=1}^{d} |x_j y_j|^{-p_j} (|x_j y_j| + n^{-1}|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-2})^{-\lambda_j}}{n^{|\kappa|_1 + 1/2} (|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^{\delta + d/2}} \\ &\quad \times \prod_{j=1}^{d} \frac{|x_j y_j| + n^{-1}|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-2}}{|x_j y_j|}. \end{aligned} \tag{5.21}$$

The estimate (5.7) in Lemma 5.4 in the case of $|\lambda|_1 + \delta - d/2 \geq 0$ follows from the above estimate if

$$|x_j y_j| \geq n^{-1}|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-2}, \qquad 1 \leq j \leq d. \tag{5.22}$$

It turns out that $\mathbf{x}, \mathbf{y}$ satisfying (5.22) is in fact the most delicate case that we have to deal with. To consider the other cases, we need one more notation. Let D be a subset of $\{1, 2, \dots, d\}$ and let D^c be its complement; that is, $D^c = \{1, 2, \dots, d\} \backslash D$. We denote by $|\lambda|_D$ the index $|\lambda|_D = \sum_{i \in D} \lambda_i$ and define $|\lambda|_{D^c}$ similarly. In the following we choose D to be the set

$$D := \{j : |x_j y_j| \geq n^{-1}|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-2}\},$$

and we assume that (5.22) does not hold; that is, D is a proper subset of $\{1,2,\ldots,d\}$ or D^c is not empty. We then follow the process that leads to the estimate of Ω_0 in (5.21) for $j \in D$.

Using (5.8) and integrating Ω_0 in (5.5) by parts p_j times for each t_j, $j \in D$, it follows that

$$\begin{aligned}\Omega_0 = & \frac{c_\kappa (-1)^{|\mathbf{p}|_D} \prod_{i\in D}(x_i y_i)^{-p_i}}{\prod_{i=1}^{|\mathbf{p}|_D}(n+2|\kappa|_1+\delta+d-i-1)/2} \\ & \times \int_{[-1,1]^d} P_{n+|\mathbf{p}|_D}^{(|\kappa|_{D^c}+|\lambda|_D+\delta+(d-1)/2,|\kappa|_{D^c}+|\lambda|_D+(d-3)/2)}(u(\mathbf{x},\mathbf{y},\mathbf{t})) \\ & \times \prod_{i\in D}\frac{\partial^{p_i}}{\partial t_i^{p_i}}\Big[(1+t_i)(1-t_i^2)^{\kappa_i-1}\Big]\prod_{i\in D^c}(1+t_i)(1-t_i^2)^{\kappa_i-1}d\mathbf{t}.\end{aligned}$$

For each $j \in D$, we break each integral over $t_j \in [-1,1]$ into two integrals over $[-1, 1-\varepsilon_{n,j}]$ and $[1-\varepsilon_{n,j}, 1]$, respectively, where $\varepsilon_{n,j}$ is as before. Let $|D|$ denote the number of elements in D and let $D = \{i_1, \ldots, i_{|D|}\}$. We define E_m^D by

$$E_m^D = (1-\varepsilon_{n,i_1},1]\times\cdots\times(1-\varepsilon_{n,i_m},1]\times[-1,1-\varepsilon_{n,i_{m+1}}]\times\ldots\times[-1,1-\varepsilon_{n,i_{|D|}}],$$

and define σE_m^D as a permutation of the order of intervals in E_m. Then we can write

$$\begin{aligned}\Omega_0 = & \frac{c_\kappa (-1)^{|\mathbf{p}|_D} \prod_{i\in D}(x_i y_i)^{-p_i}}{\prod_{i=1}^{|\mathbf{p}|_D}(n+2|\kappa|_1+\delta+d-i-1)/2} \\ & \times \int_{[-1,1]^{|D^c|}}\Big[\sum_{m=0}^{|D|}\Omega_{0,m}^D+\sum_\sigma\sum_{m=1}^{|D|-1}\sigma\Omega_{0,m}^D\Big]\prod_{i\in D^c}(1+t_i)(1-t_i^2)^{\kappa_i-1}dt_i,\end{aligned}$$

where $\Omega_{0,m}^D$ is like in (5.10) but for indices in D,

$$\begin{aligned}\Omega_{0,m}^D = c_\kappa \int_{E_m^D} & P_{n+|\mathbf{p}|_D}^{(|\kappa|_{D^c}+|\lambda|_D+\delta+(d-1)/2,|\kappa|_{D^c}+|\lambda|_D+(d-3)/2)}(u(\mathbf{x},\mathbf{y},\mathbf{t})) \\ & \times\prod_{i\in D}\frac{\partial^{p_i}}{\partial t_i^{p_i}}\Big[(1+t_i)(1-t_i^2)^{\kappa_i-1}\Big]dt_i\end{aligned}$$

and $\sigma\Omega_{0,m}^D$ denotes the integral over σE_m^D. We can now follow the procedure that leads to (5.21) to estimate $\Omega_{0,m}^D$; that is, we will use (5.8) and integrate $\Omega_{0,m}^D$ by parts once for each variable t_j, $j = i_{m+1},\ldots,i_{|D|}$, and write the $\Omega_{0,m}^D$ as a sum of $\Lambda_{m,j}^D$ like (5.12), where the Jacobi polynomial in $\Lambda_{m,j}^D$ is

$$P_{n+|\mathbf{p}|_D+|D|-m}^{(|\kappa|_{D^c}+|\lambda|_D+\delta+(d-1)/2+m-|D|,|\kappa|_{D^c}+|\lambda|_D+(d-3)/2+m-|D|)}(u(\mathbf{x},\mathbf{y},\mathbf{t})),$$

whose first parameter satisfies

$$|\kappa|_{D^c}+|\lambda|_D+\delta+(d-1)/2+m-|D| \ge |\lambda|_1+\delta+(d-1)/2-|D| \ge -1/2$$

for all $m \ge 0$ if $|D^c| \ne 0$, $|\lambda|_1+\delta-d/2 \ge 0$ or $\delta \ge (d-2)/2$. Consequently, we can use (5.13) and Lemma 5.3 to obtain an analogy of (5.14); then following the procedure that leads to (5.21) almost verbatim, we obtain the following estimate

$$\begin{aligned} |\Omega_0| &\le c \frac{\prod_{j\in D} |x_j y_j|^{-p_j} (|x_j y_j| + n^{-1}|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-2})^{-\lambda_j}}{n^{|\kappa|_D + 1/2} (|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^{|\kappa|_{D^c} + \delta + d/2}} \\ &\qquad\qquad \times \prod_{j\in D} \frac{|x_j y_j| + n^{-1}|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-2}}{|x_j y_j|} \\ &\le c \frac{\prod_{j\in D} (|x_j y_j| + n^{-1}|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-2})^{-\kappa_j}}{n^{|\kappa|_D + 1/2} (|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^{|\kappa|_{D^c} + \delta + d/2}}, \end{aligned}$$

where the last step follows from the definition of D. Since for $j \in D^c$,

$$|x_j y_j| + n^{-1}|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-2} \sim n^{-1}|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-2},$$

it follows from the last estimate that the estimate (5.7) holds when $|D^c| \ne 0$, or when (5.22) fails to hold. Consequently, we have established Lemma 5.4 under the condition that $|\lambda|_1 + \delta - d/2 \ge 0$.

Remark 5.3. We emphasize that, under the condition $\delta \ge (d-2)/2$, the proof of (5.7) when $|D^c| \ne 0$ holds without the condition $|\lambda|_1 + \delta - d/2 \ge 0$. It is from this perspective that we consider the case $\mathbf{x}, \mathbf{y}$ satisfying (5.22) the most delicate case.

Remark 5.4. We would like to point out that the above proof of Case 1 works without the assumption $\delta \ge (d-2)/2$; that is, the estimate (5.7) works under the condition $|\lambda|_1 + \delta - d/2 \ge 0$ with $\delta > 0$.

Case 2. $|\lambda|_1 + \delta - d/2 < 0$.

This case turns out to be rather delicate. The proof is far more involved than the previous case. First we notice that it may be assumed that $|\lambda|_1 < 1$, since $|\lambda|_1 \ge 1$ and $\delta \ge (d-2)/2$ implies that $|\lambda|_1 + \delta - d/2 \ge 0$. We then have $(d-2)/2 \le \delta < d/2 - |\lambda|_1$. Second, by Remark 5.3, we can assume that (5.22) holds.

We reduce the essential part of the estimate to that of $\Omega_{0,m}$ as before. We should emphasize that the partition of Ω_0 into $\Omega_{0,m}$ depends on the choice of $\varepsilon_{n,j}$. As we pointed out in Remark 5.2 that the estimate of $\Omega_{0,m}$ in (5.20) holds for all $m > 0$ independent of the sign of $|\lambda|_1 + \delta - d/2$. Unfortunately, it turns out that the same choice of $\varepsilon_{n,j}$ does not work with $\Omega_{0,0}$. This forces us to choose another $\varepsilon_{n,j}$, which we denote by $\varepsilon^*_{n,j}$,

$$\varepsilon^*_{n,j} := \frac{n^{-1}(|\mathbf{x} - \mathbf{y}| + n^{-1})}{2d(|x_j y_j| + n^{-1}|\mathbf{x} - \mathbf{y}| + n^{-2})}.$$

We again have $0 \le \varepsilon^*_{n,j} \le 1/2d$. Let us denote the corresponding partition of $[-1,1]^d$ as a union of E^*_m and Ω_0 is partitioned as a sum over $\Omega^*_{0,m}$ like (5.10). We then need to derive estimates not only for $\Omega^*_{0,0}$ but also for $\Omega^*_{0,m}$. Our goal is to establish, for $|\lambda|_1 < 1$ and $(d-2)/2 \le \delta < d/2 - |\lambda|_1$,

$$|\Omega^*_{0,m}| \le c \frac{\prod_{i=1}^d (|x_i y_i| + n^{-1}|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-2})^{-\lambda_i}}{n^{|\lambda|_1 + 1/2} (|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^{\delta + d/2}}. \tag{5.23}$$

Substituting (5.23) into (5.10) yields (5.7). Hence, the proof of (5.23) will ensure the proof of (5.7) in Case 2.

Remark 5.5. Since $|\bar{\mathbf{x}} - \bar{\mathbf{y}}| \le |\mathbf{x} - \mathbf{y}|$, we have $\varepsilon_{n,j} \le \varepsilon^*_{n,j}$, which implies that $E^*_0 \subset E_0$. Hence, the estimate of $\Omega^*_{0,0}$ is over a region smaller than that of $\Omega_{0,0}$. Consequently, we cannot combine the estimate of $\Omega^*_{0,0}$ in (5.23) and that of $\Omega_{0,m}$ to finish the proof of Lemma 5.4. We need to establish the estimate (5.23) of $\Omega^*_{0,m}$ for $m \ge 0$ in a different way. Let us also mention that the choice $\varepsilon^*_{n,j}$ does not work for the Case 1.

Case 2.1, the estimate of $\Omega^*_{0,0}$ *for* $|\lambda|_1 < 1$ *and* $(d-2)/2 \le \delta < d/2 - |\lambda|_1$ We notice that the formulae (5.10) to (5.14) hold with $\varepsilon_{n,j}$ replaced by $\varepsilon^*_{n,j}$. From (5.12), we need to give an estimate for $\Lambda_{0,j}$. Since $|\lambda|_1 \le (d+2)/2$, this is the case that $V_{0,j}$ does not appear in (5.14) (Remark 5.1). Hence, from (5.14) and the formulae for $U_{0,j}$, it follows that

$$\Lambda_{0,0} \le cn^{-1/2} \prod_{i=1}^{d} {\varepsilon^*_{n,i}}^{\lambda_i - 1} (1 - u_0(\mathbf{x}, \mathbf{y}, \mathbf{t}) + n^{-2})^{-(|\lambda|_1 + \delta - d/2)/2} := J_0,$$

and for $j = 1, 2, \ldots, d$,

$$\Lambda_{0,j} \le cn^{-1/2} \prod_{i=j+1}^{d} {\varepsilon^*_{n,i}}^{\lambda_i - 1} \int_{-1}^{1-\varepsilon^*_{n,1}} \cdots \int_{-1}^{1-\varepsilon^*_{n,j}} \frac{\prod_{i=1}^{j} (1+t_i)^{\lambda_i - 1} (1-t_i)^{\lambda_i - 2}}{(1 - u_j(\mathbf{x}, \mathbf{y}, \mathbf{t}) + n^{-2})^{(|\lambda|_1 + \delta - d/2)/2}} dt_1 \ldots dt_j := J_j.$$

In order to estimate these terms, we prove that $J_j \le cJ_{j-1}$ first, so that we only need to estimate J_0. We use the fact that

$$1 - u_j(\mathbf{x}, \mathbf{y}, \mathbf{t}) + n^{-2} = 1 - u_{j-1}(\mathbf{x}, \mathbf{y}, \mathbf{t}) + n^{-2} + x_j y_j (1 - \varepsilon^*_{n,j} - t_j). \tag{5.24}$$

Thus, if $x_j y_j \le 0$, then $1 - u_j(\mathbf{x}, \mathbf{y}, \mathbf{t}) + n^{-2} \le 1 - u_{j-1}(\mathbf{x}, \mathbf{y}, \mathbf{t}) + n^{-2}$, from which it follows that

$$\begin{aligned} J_j &\le cn^{-1/2} \prod_{i=j+1}^{d} {\varepsilon^*_{n,i}}^{\lambda_i - 1} \int_{-1}^{1-\varepsilon^*_{n,1}} \cdots \int_{-1}^{1-\varepsilon^*_{n,j-1}} \\ &\qquad \frac{\prod_{i=1}^{j-1} (1+t_i)^{\lambda_i - 1} (1-t_i^2)^{\lambda_i - 2}}{(1 - u_{j-1}(\mathbf{x}, \mathbf{y}, \mathbf{t}) + n^{-2})^{(|\lambda|_1 + \delta - d/2)/2}} dt_1 \ldots dt_j g_{\lambda_j}(1 - \varepsilon^*_{n,j}) \\ &\le cJ_{j-1}, \end{aligned}$$

where in the last step we have used the fact that $g_{\lambda_j}(1 - \varepsilon^*_{n,j}) \sim {\varepsilon^*_{n,j}}^{\lambda_j - 1}$ which follows from (5.17). In the following, we consider the case that $x_j y_j > 0$.

It turns out that we need to consider the case of $j = 1$ and the case of $j \ge 2$ separately. For $j = 1$, we integrate J_1 by parts and using (5.17) to conclude that

$$\begin{aligned} J_1 &= cn^{-1/2} \prod_{i=2}^{d} {\varepsilon^*_{n,i}}^{\lambda_i - 1} \Big[\frac{g_{\lambda_1}(1 - \varepsilon^*_{n,1})}{(1 - u_0(\mathbf{x}, \mathbf{y}, \mathbf{t}) + n^{-2})^{(|\lambda|_1 + \delta - d/2)/2}} \\ &\quad - \frac{1}{2} x_1 y_1 (|\lambda|_1 + \delta - d/2) \int_{-1}^{1-\varepsilon^*_{n,1}} \frac{g_{\lambda_1}(t_1) dt_1}{(1 - u_1(\mathbf{x}, \mathbf{y}, \mathbf{t}) + n^{-2})^{(|\lambda|_1 + \delta - d/2)/2 + 1}} \Big] \\ &\le c \Big[J_0 + n^{-1/2} x_1 y_1 \prod_{i=2}^{d} {\varepsilon^*_{n,i}}^{\lambda_i - 1} \int_{-1}^{1-\varepsilon^*_{n,1}} \frac{(1 - t_1)^{\lambda_1 - 1} dt_1}{(1 - u_1(\mathbf{x}, \mathbf{y}, \mathbf{t}) + n^{-2})^{(|\lambda|_1 + \delta - d/2 + 2)/2}} \Big]. \end{aligned}$$

Using (5.24), it follows that

$$J_1 \le cJ_0 + cn^{-1/2} x_1 y_1 \prod_{i=2}^{d} \varepsilon_{n,i}^{*\,\lambda_i - 1} \times \int_{-1}^{1-\varepsilon_{n,1}^*} \frac{(1-\varepsilon_{n,1}^* - t_1)^{\lambda_1 - 1} dt_1}{[1 - u_0(\mathbf{x},\mathbf{y},\mathbf{t}) + n^{-2} + x_1 y_1 (1 - \varepsilon_{n,1}^* - t_1)]^{(|\lambda|_1 + \delta - d/2 + 2)/2}}.$$

In the last integral we make the change of variable

$$s = \frac{x_1 y_1 (1 - \varepsilon_{n,1}^* - t_1)}{1 - u_0(\mathbf{x},\mathbf{y},\mathbf{t}) + n^{-2}}$$

to conclude that it is equal to

$$\frac{(x_1 y_1)^{-\lambda_1}}{(1 - u_0(\mathbf{x},\mathbf{y},\mathbf{t}) + n^{-2})^{(|\lambda|_1 + \delta - d/2 + 2)/2 - \lambda_1}} \times \int_0^{\frac{x_1 y_1 (2 - \varepsilon_{n,1}^*)}{1 - u_0(\mathbf{x},\mathbf{y},\mathbf{t}) + n^{-2}}} \frac{s^{\lambda_1 - 1}}{(1+s)^{(|\lambda|_1 + \delta - d/2 + 2)/2}} ds;$$

hence, upon using the fact that $(|\lambda|_1 + \delta - d/2 + 2)/2 - \lambda_1 > 0$, which follows from $\delta \ge (d-2)/2$ and $\lambda_1 < 1$, and the elementary equation

$$\int_0^t \frac{s^{a-1}}{(1+s)^b} ds \sim \frac{t^a}{(1+t)^a}, \qquad b > a > 0,$$

it then follows that

$$J_1 \le cJ_0 + cn^{-1/2} x_1 y_1 \prod_{i=2}^{d} \varepsilon_{n,i}^{*\,\lambda_i - 1} \frac{(1 - u_0(\mathbf{x},\mathbf{y},\mathbf{t}) + n^{-2} + x_1 y_1 (2 - \varepsilon_{n,1}^*))^{-\lambda_1}}{(1 - u_0(\mathbf{x},\mathbf{y},\mathbf{t}) + n^{-2})^{(|\lambda|_1 + \delta - d/2 + 2)/2 - \lambda_1}}.$$

From the definition of $\varepsilon_{n,j}^*$, it follows that $|\sum_{i=1}^d x_i y_i \varepsilon_{n,i}^*| \le n^{-1}(|\mathbf{x}-\mathbf{y}| + n^{-1})/2$, hence, we have that

$$\begin{aligned} 1 - u_0(\mathbf{x},\mathbf{y},\mathbf{t}) + n^{-2} &= \frac{1}{2}|\mathbf{x}-\mathbf{y}|^2 + \sum_{i=1}^d x_i y_i \varepsilon_{n,i}^* + n^{-2} \\ &\sim (|\mathbf{x}-\mathbf{y}| + n^{-1})^2 \ge n^{-1}(|\mathbf{x}-\mathbf{y}| + n^{-1}), \end{aligned} \tag{5.25}$$

so that by $x_1 y_1 \le (1 - u_0(\mathbf{x},\mathbf{y},\mathbf{t}) + n^{-2} + x_1 y_1 (2 - \varepsilon_{n,i}^*))$ and $0 \le \varepsilon_{n,i}^* \le 1/2d$,

$$\begin{aligned} &\frac{x_1 y_1 (1 - u_0(\mathbf{x},\mathbf{y},\mathbf{t}) + n^{-2} + x_1 y_1 (2 - \varepsilon_{n,i}^*))^{-\lambda_1}}{(1 - u_0(\mathbf{x},\mathbf{y},\mathbf{t}) + n^{-2})^{1-\lambda_1}} \\ &\qquad \le c\Big(\frac{|x_1 y_1| + n^{-1}|\mathbf{x}-\mathbf{y}| + n^{-2}}{n^{-1}(|\mathbf{x}-\mathbf{y}| + n^{-1})}\Big)^{1-\lambda_1} = c\,\varepsilon_{n,1}^{*\,\lambda_1 - 1}; \end{aligned}$$

consequently, we have that $J_1 \le J_0$.

Next we consider the case $j \ge 2$. Since $|\lambda|_1 < 1$ and $0 < \lambda_i < 1$, there is at most one $k \in \{1, 2, \ldots, d\}$ such that $\lambda_k \in [1/2, 1)$, and all other $\lambda_i \in (0, 1/2)$. Thus, there exists at least one $\lambda_i \in (0, 1/2)$ for $1 \le i \le j$ since $j \ge 2$. Let us assume that $i = j$, that is, $\lambda_j \in (0, 1/2)$. We then integrate by parts with respect to t_j. From

(5.17) and (5.24) we conclude that

$$J_j \le cn^{-1/2} \prod_{i=j+1}^{d} \varepsilon_{n,i}^{*\,\lambda_i-1} \int_{-1}^{1-\varepsilon_{n,1}^*} \cdots \int_{-1}^{1-\varepsilon_{n,j-1}^*}$$

$$\Big[\frac{g_{\lambda_j}(1-\varepsilon_{n,j}^*)}{(1-u_{j-1}(\mathbf{x},\mathbf{y},\mathbf{t})+n^{-2})^{(|\lambda|_1+\delta-d/2)/2}} - \frac{1}{2}x_jy_j(|\lambda|_1+\delta-d/2)$$

$$\times \int_{-1}^{1-\varepsilon_{n,j}^*} \frac{g_{\lambda_j}(t_j)dt_j}{(1-u_j(\mathbf{x},\mathbf{y},\mathbf{t})+n^{-2})^{(|\lambda|_1+\delta-d/2)/2+1}}\Big]$$

$$\times \prod_{i=1}^{j-1}(1+t_i)^{\lambda_i-1}(1-t_i)^{\lambda_i-2}dt_1\dots dt_{j-1}$$

$$\le c\Big[J_{j-1} + n^{-1/2}x_jy_j \prod_{i=j+1}^{d} \varepsilon_{n,i}^{*\,\lambda_i-1} \int_{-1}^{1-\varepsilon_{n,1}^*} \cdots \int_{-1}^{1-\varepsilon_{n,j-1}^*}$$

$$\frac{1}{(1-u_{j-1}(\mathbf{x},\mathbf{y},\mathbf{t})+n^{-2})^{(|\lambda|_1+\delta-d/2+1)/2}}$$

$$\times \int_{-1}^{1-\varepsilon_{n,j}^*} \frac{(1-t_j)^{\lambda_j-1}dt_j}{(1-u_j(\mathbf{x},\mathbf{y},\mathbf{t})+n^{-2})^{1/2}} \prod_{i=1}^{j-1}(1+t_i)^{\lambda_i-1}(1-t_i)^{\lambda_i-2}dt_1\dots dt_{j-1}\Big],$$

where we have used the fact that $1-u_j(\mathbf{x},\mathbf{y},\mathbf{t}) \ge 1-u_{j-1}(\mathbf{x},\mathbf{y},\mathbf{t})$ since $x_jy_j>0$, and $|\lambda|_1+\delta-d/2+1>0$. Using (5.24), $1-u_j(\mathbf{x},\mathbf{y},\mathbf{t})+n^{-2} \ge x_jy_j(1-\varepsilon_{n,j}^*-t_j)$, it follows that

$$\int_{-1}^{1-\varepsilon_{n,j}^*} \frac{(1-t_j)^{\lambda_j-1}dt_j}{(1-u_j(\mathbf{x},\mathbf{y},\mathbf{t})+n^{-2})^{1/2}} \le (x_jy_j)^{-1/2}\int_{-1}^{1-\varepsilon_{n,j}^*} \frac{(1-t_j)^{\lambda_j-1}dt_j}{(1-\varepsilon_{n,j}^*-t_j)^{1/2}}$$

$$\le c(x_jy_j)^{-1/2}\Big[\int_{-1}^{1-3\varepsilon_{n,j}^*/2}(1-t_j)^{\lambda_j-3/2}dt_j$$

$$+ \varepsilon_{n,j}^{*\,\lambda_j-1}\int_{1-3\varepsilon_{n,j}^*/2}^{1-\varepsilon_{n,j}^*}(1-\varepsilon_{n,j}^*-t_j)^{-1/2}dt_j\Big]$$

$$\le c(x_jy_j)^{-1/2}\varepsilon_{n,j}^{*\,\lambda_j-1/2},$$

where in the last step we have used the fact that $0<\lambda_j<1/2$. Therefore, using the fact that

$$1-u_{j-1}(\mathbf{x},\mathbf{y},\mathbf{t})+n^{-2} = (1-\sum_{i=1}^{j-1}x_iy_it_i - x_jy_j - \sum_{i=j+1}^{d}x_jy_j(1-\varepsilon_{n,i}^*)+n^{-2})$$

$$+ x_jy_j\varepsilon_{n,j}^* \ge x_jy_j\varepsilon_{n,j}^*,$$

and the definition of J_{j-1}, it follows that

$$J_j \le cJ_{j-1} + cn^{-1/2}x_jy_j\prod_{i=j+1}^{d}\varepsilon_{n,i}^{*\,\lambda_i-1}\frac{(x_jy_j)^{-1/2}\varepsilon_{n,j}^{*\,\lambda_j-1/2}}{(x_jy_j\varepsilon_{n,j}^*)^{1/2}}\int_{-1}^{1-\varepsilon_{n,1}^*}\cdots\int_{-1}^{1-\varepsilon_{n,j-1}^*}$$

$$\frac{1}{(1-u_{j-1}(\mathbf{x},\mathbf{y},\mathbf{t})+n^{-2})^{(|\lambda|_1+\delta-d/2)/2}}\prod_{i=1}^{j-1}(1+t_i)^{\lambda_i-1}(1-t_i^2)^{\lambda_i-2}dt_1\dots dt_{j-1}$$

$$\le cJ_{j-1}$$

for $j \ge 2$. Consequently, we conclude that $J_j \le cJ_0$ for all $j \ge 1$. Hence, from (5.11), the definition of J_0 and (5.25), it follows that

$$|\Omega^*_{0,0}| \le c\frac{\prod_{i=1}^d |x_iy_i|^{-1}(|x_iy_i| + n^{-1}|\mathbf{x}-\mathbf{y}| + n^{-2})^{1-\lambda_i}}{n^{|\lambda|_1+1/2}(|\mathbf{x}-\mathbf{y}| + n^{-1})^{\delta+d/2}}. \tag{5.26}$$

Recall that we assume that (5.22) holds. Now, if for all $j \in \{1,\dots,d\}$,

$$n^{-1}|\bar{\mathbf{x}}-\bar{\mathbf{y}}| + n^{-2} \le |x_jy_j| \le n^{-1}|\mathbf{x}-\mathbf{y}| + n^{-2},$$

which implies, in particular, that $|x_jy_j| \le cn^{-1}$ for $1 \le j \le d$, then for n sufficiently large, we have

$$|\mathbf{x}-\mathbf{y}| \ge |\bar{\mathbf{x}}-\bar{\mathbf{y}}| = 2 - 2\sum_i |x_iy_i| \ge 2 - c/n \ge c|\mathbf{x}-\mathbf{y}|^2.$$

Thus, in this case, we may replace $|\mathbf{x}-\mathbf{y}|$ by $|\bar{\mathbf{x}}-\bar{\mathbf{y}}|$ in (5.26) to obtain that

$$\begin{aligned}|\Omega^*_{0,0}| &\le c\frac{\prod_{i=1}^d |x_jy_j|^{-1}(|x_iy_i| + n^{-1}|\bar{\mathbf{x}}-\bar{\mathbf{y}}| + n^{-2})^{1-\lambda_i}}{n^{|\lambda|_1+1/2}(|\bar{\mathbf{x}}-\bar{\mathbf{y}}| + n^{-1})^{\delta+d/2}}\\ &\le c\frac{\prod_{i=1}^d (|x_iy_i| + n^{-1}|\bar{\mathbf{x}}-\bar{\mathbf{y}}| + n^{-2})^{-\lambda_i}}{n^{|\lambda|_1+1/2}(|\bar{\mathbf{x}}-\bar{\mathbf{y}}| + n^{-1})^{\delta+d/2}}.\end{aligned}$$

On the other hand, if there exist $k \in \{1,2,\dots,d\}$ such that

$$|x_ky_k| \ge n^{-1}|\mathbf{x}-\mathbf{y}| + n^{-2},$$

let us assume that one such index is $k = d$, then it follows from (5.26) that

$$|\Omega^*_{0,0}| \le c\frac{\prod_{i=1}^d (|x_iy_i| + n^{-1}|\mathbf{x}-\mathbf{y}| + n^{-2})^{-\lambda_i}}{n^{|\lambda|_1+1/2}(|\mathbf{x}-\mathbf{y}| + n^{-1})^{\delta+d/2}} \prod_{i=1}^{d-1} \frac{|x_iy_i| + n^{-1}|\mathbf{x}-\mathbf{y}| + n^{-2}}{|x_iy_i|}.$$

Using (5.22) we have

$$\begin{aligned}\prod_{i=1}^{d-1} \frac{|x_iy_i| + n^{-1}|\mathbf{x}-\mathbf{y}| + n^{-2}}{|x_iy_i|} &\le 2^{d-1}\prod_{i=1}^{d-1} \frac{|x_iy_i| + n^{-1}|\mathbf{x}-\mathbf{y}| + n^{-2}}{|x_iy_i| + n^{-1}|\bar{\mathbf{x}}-\bar{\mathbf{y}}| + n^{-2}}\\ &\le 2^{d-1}\prod_{i=1}^{d-1} \frac{|\mathbf{x}-\mathbf{y}| + n^{-1}}{|\bar{\mathbf{x}}-\bar{\mathbf{y}}| + n^{-1}} = 2^{d-1}\Big(\frac{|\mathbf{x}-\mathbf{y}| + n^{-1}}{|\bar{\mathbf{x}}-\bar{\mathbf{y}}| + n^{-1}}\Big)^{d-1},\end{aligned} \tag{5.27}$$

where the second inequality follows from the elementary inequality $(t+a)/(t+b) \le a/b$ which holds for $t > 0$ and $a > b > 0$. Consequently, we conclude that

$$|\Omega^*_{0,0}| \le c\frac{\prod_{i=1}^d (|x_iy_i| + n^{-1}|\mathbf{x}-\mathbf{y}| + n^{-2})^{-\lambda_i}}{n^{|\lambda|_1+1/2}(|\mathbf{x}-\mathbf{y}| + n^{-1})^{\delta-(d-2)/2}} \frac{1}{(|\bar{\mathbf{x}}-\bar{\mathbf{y}}| + n^{-1})^{d-1}}.$$

Since $\delta \ge (d-2)/2$, $\lambda_i > 0$ and $|\mathbf{x}-\mathbf{y}| \ge |\bar{\mathbf{x}}-\bar{\mathbf{y}}|$, it follows that

$$|\Omega^*_{0,0}| \le c\frac{\prod_{i=1}^d (|x_iy_i| + n^{-1}|\bar{\mathbf{x}}-\bar{\mathbf{y}}| + n^{-2})^{-\lambda_i}}{n^{|\lambda|_1+1/2}(|\bar{\mathbf{x}}-\bar{\mathbf{y}}| + n^{-1})^{\delta+d/2}},$$

which holds for all $\mathbf{x}$ and $\mathbf{y}$ satisfy (5.22). This is the estimate (5.23) for $\Omega^*_{0,0}$.

Case 2.2, the estimate of $\Omega^*_{0,m}$, $1 \le m \le d$, for $|\lambda|_1 < 1$ and $(d-2)/2 \le \delta < d/2 - |\lambda|_1$ Since $0 \le |\lambda|_1 < 1$ and $\delta \ge (d-2)/2$, we have

$$|\lambda|_1 + \delta + m - d/2 \ge 0, \qquad m = 1,2,\dots,d.$$

Consequently, we see that the estimate (5.18) of $U_{m,j}$ holds with $\varepsilon_{n,j}$ replaced by $\varepsilon^*_{n,j}$; that is, we have

$$U_{m,j} \le c \prod_{i=m+1}^{j} \varepsilon^{*\,\lambda_i-1}_{n,i} (|\bar{\mathbf{x}}-\bar{\mathbf{y}}|+n^{-1})^{-(|\lambda|_1+\delta+m-d/2)}.$$

Similarly, the estimate (5.19) of $V_{m,j}$ holds with $\varepsilon_{n,i}$ replaced by $\varepsilon^*_{n,i}$. Therefore, following the steps that lead to (5.20) and using the definition of $\varepsilon^*_{n,j}$, we conclude that

$$\begin{aligned} |\Omega^*_{0,m}| &\le c n^{m-d-1/2} \frac{\prod_{j=1}^m \varepsilon^{*\,\lambda_j}_{n,j} \prod_{j=m+1}^d |x_j y_j|^{-1} \varepsilon^{*\,\lambda_j-1}_{n,j}}{(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|+n^{-1})^{|\lambda|_1+\delta+m-d/2}} \\ &\le c \frac{\prod_{j=1}^m (|x_j y_j| + n^{-1}|\mathbf{x}-\mathbf{y}|+n^{-2})^{-\lambda_j}}{n^{|\lambda|_1+1/2}(|\mathbf{x}-\mathbf{y}|+n^{-1})^{d-m-|\lambda|_1}} \\ &\quad \times \frac{\prod_{j=m+1}^d |x_j y_j|^{-1}(|x_j y_j| + n^{-1}|\mathbf{x}-\mathbf{y}|+n^{-2})^{1-\lambda_j}}{(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|+n^{-1})^{|\lambda|_1+\delta+m-d/2}}. \end{aligned} \tag{5.28}$$

This estimate by itself is not strong enough for proving (5.23). We shall use it only in the case that there is a k, $m+1 \le k \le d$, and $x_k y_k$ satisfies

$$x_k y_k < 0 \quad \text{and} \quad |x_k y_k| \ge (|\mathbf{x}-\mathbf{y}|^2 + n^{-2})/4d, \tag{5.29}$$

which does not happen for $m = d$. Let us assume that $1 \le m \le d-1$ and $k = d$. Then, under the condition (5.29) which implies that $|x_d y_d| \ge c(n^{-1}|\mathbf{x}-\mathbf{y}|+n^{-2})$, we have

$$\begin{aligned} |\Omega^*_{0,m}| \le c &\frac{\prod_{j=1}^d (|x_j y_j| + n^{-1}|\mathbf{x}-\mathbf{y}|+n^{-2})^{-\lambda_j}}{n^{|\lambda|_1+1/2}(|\mathbf{x}-\mathbf{y}|+n^{-1})^{d-m-|\lambda|_1}(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|+n^{-1})^{|\lambda|_1+\delta+m-d/2}} \\ &\times \prod_{j=m+1}^{d-1} \frac{(|x_j y_j| + n^{-1}|\mathbf{x}-\mathbf{y}|+n^{-2})}{|x_j y_j|}. \end{aligned}$$

Using (5.22) it follows as in (5.27) that

$$\prod_{i=m+1}^{d-1} \frac{|x_i y_i| + n^{-1}|\mathbf{x}-\mathbf{y}|+n^{-2}}{|x_i y_i|} \le 2^{d-m-1}\Big(\frac{|\mathbf{x}-\mathbf{y}|+n^{-1}}{|\bar{\mathbf{x}}-\bar{\mathbf{y}}|+n^{-1}}\Big)^{d-m-1},$$

from which we conclude that

$$|\Omega^*_{0,m}| \le c \frac{\prod_{i=1}^d (|x_i y_i| + n^{-1}|\bar{\mathbf{x}}-\bar{\mathbf{y}}|+n^{-2})^{-\lambda_i}}{n^{|\lambda|_1+1/2}(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|+n^{-1})^{\delta+d/2}} \Big(\frac{|\bar{\mathbf{x}}-\bar{\mathbf{y}}|+n^{-1}}{|\mathbf{x}-\mathbf{y}|+n^{-1}}\Big)^{1-|\lambda|_1}.$$

The use of (5.29) helps us to reduce the exponent of the last term by 1 so that it has a positive power $1-|\lambda|_1 \ge 0$. Hence, using the fact that $|\mathbf{x}-\mathbf{y}| \ge |\bar{\mathbf{x}}-\bar{\mathbf{y}}|$, we have proved (5.23) under the condition (5.29).

On the other hand, if (5.29) does not hold, then for all $m+1 \le j \le d$ we have $x_j y_j \ge 0$ or $x_j y_j < 0$ but $|x_j y_j| \le (|\mathbf{x}-\mathbf{y}|^2 + n^{-2})/4d$. In this case, recalling the formulae (5.12) and (5.14), for $\Omega^*_{0,m}$ we have $t_i \in (1-\varepsilon^*_{n,i}, 1)$, $1 \le i \le m$; hence, by the definition of $u_j(\mathbf{x},\mathbf{y},\mathbf{t})$ and the definition of $\varepsilon^*_{n,j}$, we have that for $j \ge m$,

$$
\begin{aligned}
1- u_j(\mathbf{x},\mathbf{y},\mathbf{t}) + n^{-2} &= |\mathbf{x}-\mathbf{y}|^2/2 + \sum_{i=1}^{j} x_i y_i (1-t_i) + \sum_{i=j+1}^{d} x_i y_i \varepsilon^*_{n,i} + n^{-2} \\
&\ge |\mathbf{x}-\mathbf{y}|^2/2 - \sum_{i=1}^{m} |x_i y_i| \varepsilon^*_{n,i} - 2\sum_{i=m+1}^{j} |x_i y_i| - \sum_{i=j+1}^{d} |x_i y_i| \varepsilon^*_{n,i} + n^{-2} \\
&\ge |\mathbf{x}-\mathbf{y}|^2/2 - \frac{d-j+m}{2d} n^{-1}(|\mathbf{x}-\mathbf{y}| + n^{-1}) - \frac{j-m}{2d}(|\mathbf{x}-\mathbf{y}|^2 + n^{-2}) + n^{-2} \\
&\sim (|\mathbf{x}-\mathbf{y}| + n^{-1})^2.
\end{aligned}
$$

Using this inequality in the definition of $U_{m,j}$, since $|\lambda|_1 + \delta + m - d/2 \ge 0$, and using (5.17) we conclude as in (5.18) that

$$
U_{m,j} \le c \prod_{i=m+1}^{j} \varepsilon^{*\,\lambda_i - 1}_{n,i} (|\mathbf{x}-\mathbf{y}| + n^{-1})^{-(|\lambda|_1 + \delta + m - d/2)}.
$$

For $V_{m,j}$, we still have the estimate (5.19) with $\varepsilon_{n,i}$ replaced by $\varepsilon^*_{n,i}$, and from (5.12), (5.13) and (5.14) we conclude as in (5.20) that

$$
\begin{aligned}
|\Omega^*_{0,m}| &\le c n^{m-d-1/2} \prod_{j=1}^{m} \varepsilon^{*\,\lambda_j}_{n,j} \prod_{j=m+1}^{d} |x_j y_j|^{-1} \varepsilon^{*\,\lambda_j - 1}_{n,j} \Big[(|\mathbf{x}-\mathbf{y}| + n^{-1})^{-(|\lambda|_1 + \delta + m - d/2)} \\
&\qquad\qquad + (|\bar{\mathbf{x}}-\bar{\mathbf{y}}| + n^{-1})^{-(|\lambda|_1 + m - (d+2)/2)} \Big] \\
&= c \Big[\frac{\prod_{j=1}^{d} |x_j y_j|^{-1} (|x_j y_j| + n^{-1}|\mathbf{x}-\mathbf{y}| + n^{-2})^{1-\lambda_j}}{n^{|\lambda|_1 + 1/2} (|\mathbf{x}-\mathbf{y}| + n^{-1})^{\delta + d/2}} \prod_{j=1}^{m} \frac{|x_j y_j|}{|x_j y_j| + n^{-1}|\mathbf{x}-\mathbf{y}| + n^{-2}} \\
&\quad + \frac{\prod_{j=1}^{d} (|x_j y_j| + n^{-1}|\mathbf{x}-\mathbf{y}| + n^{-2})^{-\lambda_j}}{n^{|\lambda|_1 + 1/2} (|\mathbf{x}-\mathbf{y}| + n^{-1})^{d-m-|\lambda|_1} (|\bar{\mathbf{x}}-\bar{\mathbf{y}}| + n^{-1})^{|\lambda|_1 + m - (d+2)/2}} \\
&\qquad\qquad \times \prod_{j=m+1}^{d} \frac{|x_j y_j| + n^{-1}|\mathbf{x}-\mathbf{y}| + n^{-2}}{|x_j y_j|} \Big].
\end{aligned}
$$

The first term is bounded by

$$
c \frac{\prod_{j=1}^{d} |x_j y_j|^{-1} (|x_j y_j| + n^{-1}|\mathbf{x}-\mathbf{y}| + n^{-2})^{1-\lambda_j}}{n^{|\lambda|_1 + 1/2} (|\mathbf{x}-\mathbf{y}| + n^{-1})^{\delta + d/2}},
$$

which is the same as the right hand side of (5.26), and has been proved to be bounded by the right hand side of (5.23). Similarly to (5.27), from (5.22) we have

$$
\prod_{j=m+1}^{d} \frac{|x_j y_j| + n^{-1}|\mathbf{x}-\mathbf{y}| + n^{-2}}{|x_j y_i|} \le 2^{d-m} \Big(\frac{|\mathbf{x}-\mathbf{y}| + n^{-1}}{|\bar{\mathbf{x}}-\bar{\mathbf{y}}| + n^{-1}} \Big)^{d-m},
$$

so that the second term is bounded by

$$
\begin{aligned}
& c \, \frac{\prod_{j=1}^{d} (|x_j y_j| + n^{-1}|\mathbf{x}-\mathbf{y}| + n^{-2})^{-\lambda_j} (|\mathbf{x}-\mathbf{y}| + n^{-1})^{|\lambda|_1}}{n^{|\lambda|_1 + 1/2} (|\bar{\mathbf{x}}-\bar{\mathbf{y}}| + n^{-1})^{|\lambda|_1 + d/2 - 1}} \\
&\quad \le c \frac{\prod_{j=1}^{d} (|x_j y_j| + n^{-1}|\mathbf{x}-\mathbf{y}| + n^{-2})^{-\lambda_j}}{n^{|\lambda|_1 + 1/2} (|\bar{\mathbf{x}}-\bar{\mathbf{y}}| + n^{-1})^{\delta + d/2}},
\end{aligned}
$$

since $|\lambda|_1 + d/2 - 1 \le \delta + d/2$. Thus, we have proved that $|\Omega^*_{0,m}|$, $0 \le m \le d$, has the estimate of (5.23) for the case $|\lambda|_1 < 1$ and $(d-2)/2 \le \delta < d/2 - |\lambda|_1$. Substituting (5.23) into (5.10) and using (5.22), we complete the proof of (5.7) in Case 2. Combining the estimates in Case 1 and Case 2, the proof of Lemma 5.4 is completed when $0 < \lambda_i < 1$ for $1 \le i \le d$.

Recall that $\kappa_i = p_i + \lambda_i$. If one of the $\kappa_j = 0$, then the formula (5.2) holds under the limit (1.6), so that the integral against t_j will not appear. In this case, it is not hard to see that the estimate in Lemma 5.4 also holds under the above limit. In case one of the κ_j is an integer, that is, $\lambda_j = 0$, then when we integrate by parts p_j times, as in (5.9), one additional term will appear. For example, let $\kappa_1 = p_1$; then integrating by parts p_1 times in Ω_0, we have

$$\begin{aligned}\Omega_0 = {} & \frac{c_\kappa (-1)^{\kappa_1}(x_1y_1)^{-\kappa_1}}{\prod_{i=1}^{\kappa_1}(n+2|\kappa|_1+\delta+d-j-1)/2}\int_{[-1,1]^{d-1}}\Big\{P_{n+\kappa_1}^{(\alpha_1,\beta_1)}(u(\mathbf{x},\mathbf{y},\mathbf{t}))\Big|_{t_1=1} \\ & \times \frac{\partial^{\kappa_1-1}}{\partial t_1^{\kappa_1-1}}\Big[(1+t_1)(1-t_1^2)^{\kappa_1-1}\Big]\Big|_{t_1=1} - \int_{-1}^{1} P_{n+\kappa_1}^{(\alpha_1,\beta_1)}(u(\mathbf{x},\mathbf{y},\mathbf{t})) \\ & \times \frac{\partial^{\kappa_1}}{\partial t_1^{\kappa_1}}\Big[(1+t_1)(1-t_1^2)^{\kappa_1-1}\Big]dt_1\Big\}\prod_{i=2}^{d}(1+t_i)(1-t_i^2)^{\kappa_i-1}dt_i \\ = {} & \frac{c_\kappa (-1)^{\kappa_1}(x_1y_1)^{-\kappa_1}}{\prod_{i=1}^{\kappa_1}(n+2|\kappa|_1+\delta+d-i-1)/2}\Big\{(\kappa_1-1)!2^{\kappa_1} \\ & \times \int_{[-1,1]^{d-1}} P_{n+\kappa_1}^{(\alpha_1,\beta_1)}(x_1y_1+x_2y_2t_2+\ldots+x_dy_dt_d)\prod_{i=2}^{d}(1+t_i)(1-t_i^2)^{\kappa_i-1}dt_i \\ & - \int_{[-1,1]^d} P_{n+\kappa_1}^{(\alpha_1,\beta_1)}(u(\mathbf{x},\mathbf{y},\mathbf{t}))\frac{\partial^{\kappa_1}}{\partial t_1^{\kappa_1}}\Big[(1+t_1)(1-t_1^2)^{\kappa_1-1}\Big]dt_1 \\ & \qquad\qquad \times \prod_{i=2}^{d}(1+t_i)(1-t_i^2)^{\kappa_i-1}dt_i\Big\},\end{aligned}$$

where we write $\alpha_1 = |\kappa|_1 - \kappa_1 + \delta + (d-1)/2$ and $\beta_1 = |\kappa|_1 - \kappa_1 + (d-3)/2$. If all other κ_i are not integers, then continuing integration by parts with respect to other variables, we see that the second term in the right hand side is the same as the right hand side of (5.9), and the first term is similar but with one integral less. If more than one κ_i are integers, then we will have more terms. The extremal case is a term containing no integrals, which appears when all κ_i are integers; the term takes the form

$$\frac{c_\kappa(-1)^{|\kappa|_1}\prod_{i=1}^{d}(x_iy_i)^{-\kappa_i}\prod_{i=1}^{d}(\kappa_i-1)!2^{\kappa_i}}{\prod_{i=1}^{|\kappa|_1}(n+2|\kappa|_1+\delta+d-i-1)/2}P_{n+|\kappa|_1}^{(\delta+(d-1)/2,(d-3)/2)}(x_1y_1+\ldots+x_dy_d),$$

which is bounded, upon using Lemma 5.3 and the fact that $1 \pm (x_1y_1+\ldots+x_dy_d)+n^{-2} = |\mathbf{x}\pm\mathbf{y}|^2/2+n^{-2} \sim (|\mathbf{x}\pm\mathbf{y}|+n^{-1})^2 \ge (|\bar{\mathbf{x}}-\bar{\mathbf{y}}|+n^{-1})^2$, by

$$\frac{c}{n^{|\kappa|_1+1/2}}\prod_{i=1}^{d}|x_iy_i|^{-\kappa_i}\Big[(|\mathbf{x}-\mathbf{y}|+n^{-1})^{-\delta-d/2}+(|\mathbf{x}+\mathbf{y}|+n^{-1})^{-(d-2)/2}\Big].$$

Hence, in case that (5.22) holds, we see that this estimate is bounded by the right hand side of (5.7). In case that (5.22) does not hold, then we use the set D and follow the method in the second half of Case 1. All other terms are between the above term and the right hand side of (5.9), and can be handled similarly. This completes the proof of Lemma 5.4. $\square$

With the hard estimate of Ω_0 taking care of, we are now ready to prove the main estimates in Theorem 3.1 and Theorem 3.2.

Proof of Theorem 3.1. We use the formula (5.4) of $K_n^\delta(h_\kappa^2; \mathbf{x}, \mathbf{y})$. The estimate of Ω_0 is given in Lemma 5.4. Replacing δ by $\delta + j$, we obtain the estimate of Ω_j for $j \ge 0$. Hence, by Lemma 5.1 and (5.4) it follows that, with $\alpha = \beta = |\kappa|_1 + (d-3)/2$ and $J = [\alpha + \beta + 2]$,

$$(5.30) \qquad |\sum_{j=0}^{J} b_j(\alpha,\beta,\delta,n)| \cdot |\Omega_j(\mathbf{x},\mathbf{y})| \le c \frac{\prod_{j=1}^d (|x_j y_j| + n^{-1}|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-2})^{-\kappa_j}}{n^{\delta-(d-2)/2}(|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^{\delta+(d/2)}},$$

which is the first term in the desired estimate in Theorem 3.1. Next we estimate the second term Ω_* of (5.4). By Lemma 5.1 and Lemma 5.2, it follows from (5.6) that

$$(5.31) \quad |\Omega_*| \le cn^{-1} \int_{[-1,1]^d} \frac{1}{(1 - u(\mathbf{x},\mathbf{y},\mathbf{t}) + n^{-2})^{|\kappa|_1 + d/2}} \prod_{i=1}^d (1+t_i)(1-t_i^2)^{\kappa_i - 1} d\mathbf{t}.$$

By the definition of $u(\mathbf{x},\mathbf{y},\mathbf{t})$, we have

$$1 - u(\mathbf{x},\mathbf{y},\mathbf{t}) + n^{-2} \ge 1 - \sum_{i=1}^d |x_i y_i t_i| + n^{-2} = \frac{1}{2}|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2 + \sum_{i=1}^d |x_i y_i|(1 - |t_i|) + n^{-2},$$

and it follows that

$$|\Omega_*| \le cn^{-1} \int_{[0,1]^d} \frac{\prod_{i=1}^d (1-t_i)^{\kappa_i - 1}}{(|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2/2 + \sum_{i=1}^d |x_i y_i|(1-t_i) + n^{-2})^{|\kappa|_1 + d/2}} d\mathbf{t}.$$

Changing variables $t_j \to s_j$ by

$$s_j = \frac{|x_j y_j|}{|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2/2 + n^{-2}}(1 - t_j), \qquad 1 \le j \le d$$

in th above integrals, we obtain

$$\begin{aligned} |\Omega_*| \ &\le c \prod_{i=1}^d \Big(\frac{|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2 + n^{-2}}{|x_i y_i|} \Big)^{\kappa_i} \frac{1}{n(|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2 + n^{-2})^{|\kappa|_1 + d/2}} \\ &\quad \times \int_0^{\frac{|x_1 y_1|}{|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2/2 + n^{-2}}} \cdots \int_0^{\frac{|x_d y_d|}{|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2/2 + n^{-2}}} \frac{1}{(1 + s_1 + \ldots + s_d)^{|\kappa|_1 + d/2}} \prod_{i=1}^d s_i^{\kappa_i - 1} d\mathbf{s}. \end{aligned}$$

Using the elementary inequality

$$\int_0^{r_1} \cdots \int_0^{r_d} \frac{\prod_{i=1}^d s_i^{\kappa_i - 1} d\mathbf{s}}{(1 + s_1 + \ldots + s_d)^{|\kappa|_1 + d/2}} \le \prod_{i=1}^d \int_0^{r_i} \frac{s_i^{\kappa_i - 1} ds_i}{(1 + s_i)^{\kappa_i + 1/2}} \le c \prod_{i=1}^d \Big(\frac{r_i}{1 + r_i} \Big)^{\kappa_i},$$

it then follows that

$$|\Omega_*| \le c \frac{\prod_{i=1}^d (|x_i y_i| + |\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2 + n^{-2})^{-\kappa_i}}{n(|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^d}.$$

The above estimate still holds even if one of the $\kappa_i = 0$. Indeed, in that case, the formula (5.2) holds under the limit (1.6), so that the integral in (5.31) against t_j will not appear. Repeating the above process leads to the same estimate of Ω_*

as above. Together with (5.30), the desired estimate of $K_n^\delta(h_\kappa^2;\mathbf{x},\mathbf{y})$ follows from (5.4). This proves Theorem 3.1. □

Proof of Theorem 3.2. Recall that $\kappa_i = p_i + \lambda_i$. We give the proof under the assumption that $0 < \lambda_i < 1$ for $1 \le i \le d$. The cases that some or all $\lambda_i = 0$ are proved similarly, as in the consideration at the end of the proof of Lemma 5.4.

If $|y_j| \le |x_j|/2$ for some j, then $|\bar{\mathbf{x}} - \bar{\mathbf{y}}| \ge \big||x_j| - |y_j|\big| \ge |x_j|/2$. Hence, it follows that for $1 \le i \le d$

$$|x_i y_i| + n^{-1}|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-2} \ge |x_i y_i| + n^{-1}|x_j|/2.$$

Using this inequality in the estimate of $K_n^\delta(h_\kappa^2;\mathbf{x},\mathbf{y})$ in Theorem 3.1, we get

$$\begin{aligned}|K_n^\delta(h_\kappa^2;\mathbf{x},\mathbf{y})| &\le c\Big[\frac{\prod_{i=1}^d (|x_i y_i| + n^{-1}|x_j|/2)^{-\kappa_i}}{n^{\delta-(d-2)/2}(|x_j|/2 + n^{-1})^{\delta+d/2}} + \frac{\prod_{i=1}^d (|x_i y_i| + n^{-1}|x_j|/2)^{-\kappa_i}}{n(|x_j|/2 + n^{-1})^d}\Big] \\ &\le c(\mathbf{x}) n^{(d-2)/2-\delta}\prod_{i=1}^d (|y_i| + n^{-1})^{-\kappa_i},\end{aligned}$$

where in the second inequality we use the fact that $0 \le \delta - (d-2)/2 \le 1$. This proves the estimate in (i).

If for all $j = 1, 2, \ldots, d$, $|y_j| > |x_j|/2$ and $x_j y_j \ge 0$, then $|\mathbf{x} - \mathbf{y}| = |\bar{\mathbf{x}} - \bar{\mathbf{y}}|$ and

$$|x_j y_j| + n^{-1}|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-2} \ge x_j^2/2 + n^{-2}.$$

Hence, from the estimate of $K_n^\delta(h_\kappa^2;\mathbf{x},\mathbf{y})$ in Theorem 3.1, it follows that

$$\begin{aligned}|K_n^\delta(h_\kappa^2;\mathbf{x},\mathbf{y})| &\le c\prod_{j=1}^d (|x_j|^2/2 + n^{-2})^{-\kappa_j} \\ &\quad\times\Big[n^{(d-2)/2-\delta}(|\mathbf{x}-\mathbf{y}| + n^{-1})^{-\delta-d/2} + n^{-1}(|\mathbf{x}-\mathbf{y}| + n^{-1})^{-d}\Big] \\ &\le c(\mathbf{x}) n^{-\delta+(d-2)/2}(|\mathbf{x}-\mathbf{y}| + n^{-1})^{-\delta-d/2},\end{aligned}$$

since $(d-2)/2 < \delta \le d/2$, which is the desired estimate in (ii).

Finally, we consider the case (iii) that for all $j = 1, 2, \ldots, d$, $|y_j| \ge |x_j|/2$, and $x_k y_k < 0$ for some k. In this case, we have

$$|x_j y_j| \ge x_j^2/2 \qquad \text{and} \qquad |\mathbf{x}-\mathbf{y}|^2 \ge (x_k - y_k)^2 \ge x_k^2. \tag{5.32}$$

To derive the desired estimate, we need to go back to the proof of Lemma 5.4. We use the decomposition (5.4) and again follow the argument from (5.10) to (5.14), so that the essential part of the estimate is reduced to the $U_{m,j}$ and $V_{m,j}$ terms. First we consider the case $|\lambda|_1 + \delta - d/2 > 0$. In the present case, we choose $\varepsilon_{n,j}$ as

$$\varepsilon_n := \varepsilon_{n,j} := \frac{1}{2dn}(|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})$$

instead of (5.15). We note that this choice is independent of the index j. From the definition of $U_{m,j}$, the variables $t_{m+1}, \ldots, t_j$ that appear in $U_{m,j}$ are each within the range $(-1, 1-\varepsilon_n)$. For $i \ge j+1$, let us write $t_i = 1 - \varepsilon_n$ in the definition of

$u_j(\mathbf{x},\mathbf{y},\mathbf{t})$. Then, if $x_k y_k < 0$ and $k \le m$ or $k \ge j+1$, we have

$$1 - u_j(\mathbf{x},\mathbf{y},\mathbf{t}) + n^{-2} = 1 - \sum_{i=1}^{d} x_i y_i t_i + n^{-2}$$

$$= -x_k y_k \ + 1 - \sum_{i \ne k} x_i y_i t_i + x_k y_k (1 - t_k) + n^{-2} \ge -x_k y_k = |x_k y_k| \ge |x_k|^2/2.$$

Using this inequality in the definition of $U_{m,j}$ and using (5.17), we conclude that

$$U_{m,j} \le c(\mathbf{x}) \prod_{i=m+1}^{j} \varepsilon_n^{\lambda_i - 1},$$

where we used the fact that $|\lambda|_1 + \delta + m - d/2 \ge 0$. This estimate of $U_{m,j}$ also holds for $U_{m,m}$, for which we take the convention that $\prod_{m+1}^{m} \varepsilon_n^{\lambda_i-1} = 1$. If $x_k y_k < 0$ and $m+1 \le k \le j$, then we split the integral over t_k in the definition of $U_{m,j}$ into two integrals, one over $[-1,0]$ and another over $[0, 1-\varepsilon_n]$. The above lower bound of $1 - u_j(\mathbf{x},\mathbf{y},\mathbf{t}) + n^{-2}$ still holds for the part that includes the integral of t_k over $[0, 1-\varepsilon_n]$, and we use $1 - u_j(\mathbf{x},\mathbf{y},\mathbf{t}) + n^{-2} \ge c(|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^2$, see (5.16), on the part that includes the integral over $[-1,0]$; this way, we derive using the definition of ε_n that

$$\begin{aligned} U_{m,j} &\le c \prod_{i=m+1}^{j} \varepsilon_n^{\lambda_i-1} \frac{\varepsilon_n^{1-\lambda_k}}{(|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^{|\lambda|_1 + \delta + m - d/2}} + c(\mathbf{x}) \prod_{i=m+1}^{j} \varepsilon_n^{\lambda_i-1} \\ &= c(\mathbf{x}) \prod_{i=m+1}^{j} \varepsilon_n^{\lambda_i-1} \Big[\frac{1}{n^{1-\lambda_k}(|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^{|\lambda|_1 + \delta + m - d/2 - (1-\lambda_k)}} + 1 \Big] \\ &\le c(\mathbf{x}) \prod_{i=m+1}^{j} \varepsilon_n^{\lambda_i-1} \Big[\frac{1}{(|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^{|\lambda|_1 + \delta + m - d/2 - (1-\lambda^*)}} + 1 \Big], \end{aligned}$$

where $\lambda^* = \max_{1 \le i \le d} \lambda_i$. For $V_{m,j}$, we note that the estimate (5.19) is valid and recall the remark right after the estimate (5.19), we conclude that

$$V_{m,j} \le c \prod_{i=m+1}^{j} \varepsilon_n^{\lambda_i-1} \Big[1 + \frac{1}{(|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^{|\lambda|_1 + m - (d+2)/2}} \Big].$$

Using the inequality $|\lambda|_1 + m - (d+2)/2 \le |\lambda|_1 + \delta + m - d/2 - (1 - \lambda^*)$, which is equivalent to $\delta \ge -\lambda^*$, we see that $V_{m,j}$ is bounded by the bound of $U_{m,j}$. Consequently, from (5.11) - (5.14) and using (5.32) and the definition of ε_n we conclude that for $0 \le m \le d$,

$$\begin{aligned} |\Omega_{0,m}| &\le \frac{c(\mathbf{x})}{n^{d-m+1/2}} \prod_{i=1}^{m} \varepsilon_n^{\lambda_i} \prod_{i=m+1}^{d} \varepsilon_n^{\lambda_i-1} \Big[\frac{1}{(|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^{|\lambda|_1 + \delta + m - d/2 - (1-\lambda^*)}} + 1 \Big] \\ &\le \frac{c(\mathbf{x})}{n^{|\lambda|_1 + 1/2}} \Big[\frac{1}{|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^{\delta + d/2 - (1-\lambda^*)}} + \frac{1}{(|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^{d-m-|\lambda|_1}} \Big]. \end{aligned}$$

Since $d - m - |\lambda|_1 = (\delta + d/2) - (|\lambda|_1 + \delta + m - d/2)$ and $|\lambda|_1 + \delta + m - d/2 \ge |\lambda|_1 + \delta - d/2 > 0$, it follows that

$$|\Omega_{0,m}| \le c(\mathbf{x}) n^{-|\lambda|_1 - 1/2} (|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^{-(\delta + d/2 - \eta)},$$

where $\eta = \min\{1 - \lambda^*, |\lambda|_1 + \delta - d/2\} > 0$, for $|\lambda|_1 + \delta - d/2 > 0$.

Next we consider the case $|\lambda|_1+\delta-d/2\le 0$. In this case, we follow the Case 2 of the proof of Lemma 5.4. We again have $|\lambda|_1<1$, since $|\lambda|_1\ge 1$ and $\delta>(d-2)/2$ implies that $|\lambda|_1+\delta-d/2>0$. For $\Omega^*_{0,0}$, we use (5.26) and (5.32) to conclude that

$$|\Omega^*_{0,0}|\le c(\mathbf{x})n^{-|\lambda|_1-1/2}\le c(\mathbf{x})n^{-|\lambda|_1-1/2}(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|+n^{-1})^{-(\delta+d/2-\eta)}$$

with $\eta=1-|\lambda|_1\ge 0$, which follows since $\delta+d/2-\eta\ge 0$. For $\Omega^*_{0,m}$ we use (5.28) and (5.32) to conclude that

$$|\Omega^*_{0,m}|\le c(\mathbf{x})\frac{1}{n^{|\lambda|_1+1/2}(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|+n^{-1})^{|\lambda|_1+\delta+m-d/2}}.$$

Since $|\lambda|_1+\delta+m-d/2=\delta+d/2-(d-m-|\lambda|_1)$ and $d-m-|\lambda|_1\ge 1-|\lambda|_1>0$ for $m\le d-1$, we conclude that

$$|\Omega^*_{0,m}|\le c(\mathbf{x})\frac{1}{n^{|\lambda|_1+1/2}(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|+n^{-1})^{\delta+d/2-\eta}}$$

with $\eta=1-|\lambda|_1>0$ for $m=1,2,\ldots,d-1$. Finally, we consider the case $m=d$. The integral of $\Omega^*_{0,d}$ is over E^*_d, we have that $t_j\in[1-\varepsilon^*_{n,j},1]$ for $j=1,2,\ldots,d$, which implies that

$$|\sum_{i=1}^d x_iy_i(1-t_i)|\le\sum_{i=1}^d |x_iy_i|\varepsilon^*_{n,i}\le n^{-1}(|\mathbf{x}-\mathbf{y}|+n^{-1})/2.$$

We have then

$$1-u(\mathbf{x},\mathbf{y},\mathbf{t})+n^{-2}=|\mathbf{x}-\mathbf{y}|^2/2+\sum_{i=1}^d x_jy_j(1-t_j)+n^{-2}\sim(|\mathbf{x}-\mathbf{y}|+n^{-1})^2.$$

Use Lemma 5.3, (5.13) and the above relation, it follows from the definition of $\Omega^*_{0,d}$ (after (5.10)) that

$$|\Omega^*_{0,d}|\le cn^{-1/2}\prod_{j=1}^d {\varepsilon^*_{n,j}}^{\lambda_j}\Big[(|\mathbf{x}-\mathbf{y}|+n^{-1})^{-(|\lambda|_1+\delta+d/2)}+(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|+n^{-1})^{-(|\lambda|_1+(d-2)/2)}\Big];$$

consequently, using (5.32) and the definition of $\varepsilon^*_{n,j}$, we conclude that

$$\begin{aligned}|\Omega^*_{0,d}|&\le c(\mathbf{x})n^{-|\lambda|_1-1/2}\Big[1+(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|+n^{-1})^{-(|\lambda|_1+(d-2)/2)}\Big]\\&\le c(\mathbf{x})n^{-|\lambda|_1-1/2}(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|+n^{-1})^{-(\delta+d/2-\eta)}\end{aligned}$$

with $\eta=1-|\lambda|_1$, since $\delta+d/2-\eta\ge|\lambda|_1+(d-2)/2\ge 0$. Substituting these estimates into (5.10) and using (5.32), we conclude that

$$|\Omega_0|\le c(\mathbf{x})n^{-|\kappa|_1-1/2}(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|+n^{-1})^{-\delta-d/2+\eta}$$

with some $\eta>0$ for both $|\lambda|_1+\delta-d/2>0$ and $|\lambda|_1+\delta-d/2\le 0$. Similarly, upon replacing δ by $\delta+j$, we get the estimate for Ω_j, which is given by

$$|\Omega_j|\le c(\mathbf{x})n^{-|\kappa|_1-1/2}(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|+n^{-1})^{-\delta-d/2-j+\eta}$$

with some $\eta>0$. Putting all these estimates together, we conclude from Lemma 5.1 and (5.4) that, with $\alpha=\beta=|\kappa|_1+(d-3)/2$,

$$|\sum_{j=0}^m b_j(\alpha,\beta,\delta,n)|\cdot|\Omega_j|\le c(\mathbf{x})n^{-\delta+(d-2)/2}(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|+n^{-1})^{-\delta-d/2+\eta}$$

with some $\eta > 0$. Finally we estimate Ω_* in (5.6). We use (5.31). Let us assume that $k = d$ in (5.32). We then have

$$\begin{aligned} 1 - u(\mathbf{x},\mathbf{y},\mathbf{t}) + n^{-2} &\ge 1 - \sum_{i=1}^{d-1} |x_i y_i t_i| - |x_d y_d| - x_d y_d (1+t_d) + n^{-2} \\ &= |\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2/2 + \sum_{i=1}^{d-1} |x_j y_j|(1-|t_i|) + |x_d y_d|(1+t_d) + n^{-2}, \end{aligned}$$

which implies, in particular, that for $t_d \in [0,1]$, $1 - u(\mathbf{x},\mathbf{y},\mathbf{t}) + n^{-2} \ge |x_d y_d| \ge x_d^2/2$ by (5.32); and for $t_d \in (-1,0)$,

$$1 - u(\mathbf{x},\mathbf{y},\mathbf{t}) + n^{-2} \ge |\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2/2 + \sum_{i=1}^{d} |x_j y_j|(1-|t_i|) + n^{-2}.$$

Therefore, using these inequalities in (5.31) we obtain

$$\begin{aligned} |\Omega_*| &\le \frac{c(\mathbf{x})}{n} \int_{[-1,1]^{d-1}} \Big[\int_{-1}^{0} \frac{(1+t_d)^{\kappa_d} dt_d}{(|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2/2 + \sum_{i=1}^{d} |x_j y_j|(1-|t_i|) + n^{-2})^{|\kappa|_1 + d/2}} + 1 \Big] \\ &\qquad\qquad \times \prod_{i=1}^{d-1} (1-t_i^2)^{\kappa_i - 1} d\mathbf{t}' \\ &\le \frac{c(\mathbf{x})}{n} \Big[\int_{[0,1]^d} \frac{(1-t_d)^{\kappa_d} \prod_{i=1}^{d-1} (1-t_i)^{\kappa_i - 1}}{(|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2/2 + \sum_{i=1}^{d} |x_j y_j|(1-t_i) + n^{-2})^{|\kappa|_1 + d/2}} d\mathbf{t} + 1 \Big]. \end{aligned}$$

For each $1 \le i \le d$, changing variables $t_i \to s_i$ with

$$s_i = \frac{|x_i y_i|}{|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2/2 + n^{-2}} (1 - t_i)$$

and using (5.32) we have

$$\begin{aligned} |\Omega_*| &\le \frac{c(\mathbf{x})}{n} \Big[(|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2/2 + n^{-2})^{1-d/2} \\ &\qquad \times \int_0^{\frac{|x_1 y_1|}{|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2/2 + n^{-2}}} \cdots \int_0^{\frac{|x_d y_d|}{|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2/2 + n^{-2}}} \frac{s_d^{\kappa_d} \prod_{i=1}^{d-1} s_i^{\kappa_i - 1}}{(1 + s_1 + \ldots + s_d)^{|\kappa|_1 + d/2}} d\mathbf{s} + 1 \Big] \\ &\le \frac{c(\mathbf{x})}{n} \Big[(|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2/2 + n^{-2})^{1-d/2} \int_0^{\frac{|x_d y_d|}{|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2/2 + n^{-2}}} \frac{s_d^{\kappa_d}}{(1+s_d)^{\kappa_d + 1/2}} ds_d \\ &\qquad \times \prod_{i=1}^{d-1} \int_0^{\frac{|x_i y_i|}{|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2/2 + n^{-2}}} \frac{s_i^{\kappa_i - 1}}{(1+s_i)^{\kappa_i + 1/2}} ds_i + 1 \Big]. \end{aligned}$$

Hence, using the elementary inequalities that for $a \ge 0$ and $r \ge \sigma > 0$,

$$\int_0^r \frac{s^{a-1}}{(1+s)^{a+1/2}} ds \le c \quad \text{and} \quad \int_0^r \frac{s^{a}}{(1+s)^{a+1/2}} ds \le c r^{1/2},$$

we conclude that

$$\begin{aligned} |\Omega_*| &\le \frac{c(\mathbf{x})}{n} \Big[(|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2/2 + n^{-2})^{1-d/2} \Big(\frac{|x_d y_d|}{|\bar{\mathbf{x}} - \bar{\mathbf{y}}|^2/2 + n^{-2}} \Big)^{1/2} + 1 \Big] \\ &\le \frac{c(\mathbf{x})}{n} \Big[(|\bar{\mathbf{x}} - \bar{\mathbf{y}}| + n^{-1})^{1-d} + 1 \Big]. \end{aligned}$$

Since $(d-2)/2 < \delta \le d/2$, we have

$$n^{-1} = n^{-\delta+(d-2)/2}(n^{-1})^{d/2-\delta} \le n^{-\delta+(d-2)/2}(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|+n^{-1})^{d/2-\delta},$$

so that

$$|\Omega_*| \le c(\mathbf{x}) n^{-\delta-(d-2)/2}(|\bar{\mathbf{x}}-\bar{\mathbf{y}}|+n^{-1})^{-\delta-d/2+1}.$$

Consequently, the estimate in (iii) follows from (5.4). □

6. The estimate of the kernels for $W^T_{\kappa,\mu}$

In this section we prove the estimate of the Cesàro kernels for $W^T_{\kappa,\mu}$ in Theorem 3.3 and Theorem 3.4. The essential part of the proof is similar to that of 3.1 and Theorem 3.2. We shall be brief.

Proof of Theorem 3.3. Throughout the proof we write $\mu = \kappa_{d+1}$ and write $|\kappa|_1 = \sum_{i=1}^{d+1} \kappa_i$. We start with the formula (3.7) for the kernel $K_n^\delta(W^T_{\kappa,\mu};\mathbf{x},\mathbf{y})$ and break it into a sum

$$\mathbf{K}_n^\delta(W^T_{\kappa,\mu};\mathbf{x},\mathbf{y}) = \sum_{j=0}^{J} b_j(\alpha,-1/2,\delta,n)\Omega_j(\mathbf{x},\mathbf{y}) + \Omega_*(\mathbf{x},\mathbf{y}),$$

where $\alpha = |\kappa|_1 + (d-2)/2$ and $J = [\alpha+\beta+2]$,

$$\Omega_j(\mathbf{x},\mathbf{y}) = c_\kappa \int_{[-1,1]^{d+1}} P_n^{(\alpha+\delta+j+1,-\frac{1}{2})}(2z^2-1)\prod_{i=1}^{d+1}(1-t_i^2)^{\kappa_i-1}d\mathbf{t}$$

and

$$\Omega_*(\mathbf{x},\mathbf{y}) = c_\kappa \int_{[-1,1]^{d+1}} G_n^\delta(2z^2-1)\prod_{i=1}^{d+1}(1-t_i^2)^{\kappa_i-1}d\mathbf{t}, \tag{6.1}$$

here and in the following, we use the notation

$$z = z(\mathbf{x},\mathbf{y},\mathbf{t}) = \sqrt{x_1y_1}t_1 + \ldots + \sqrt{x_dy_d}t_d + \sqrt{1-|\mathbf{x}|_1}\sqrt{1-|\mathbf{y}|_1}t_{d+1}.$$

Using Lemma 5.1 and Lemma 5.2, Ω_* is bounded by

$$\Omega_*(\mathbf{x},\mathbf{y}) = \mathcal{O}(n^{-1})c_\kappa \int_{[-1,1]^{d+1}} (1-z+n^{-2})^{-\alpha-\frac{3}{2}}\prod_{i=1}^{d+1}(1-t_i^2)^{\kappa_i-1}d\mathbf{t};$$

Using the quadratic transform $P_n^{(\lambda,-\frac{1}{2})}(2t^2-1) = a_nP_{2n}^{(\lambda,\lambda)}(t)$, in which $a_n = \mathcal{O}(1)$ since $P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}$, it follows that

$$\Omega_j(\mathbf{x},\mathbf{y}) = c_\kappa a_n \int_{[-1,1]^{d+1}} P_{2n}^{(\alpha+\delta+j+1,\alpha+\delta+j+1)}(z)\prod_{i=1}^{d+1}(1-t_i^2)^{\kappa_i-1}d\mathbf{t}.$$

Following the proof of Theorem 3.1, we consider the estimate of Ω_0. Let $\kappa_j = p_j+\lambda_j$, p_j being nonnegative integers and $\lambda_j \in [0,1)$. Recall the notation $\xi = (\sqrt{x_1},\ldots,\sqrt{x_{d+1}})$ with $x_{d+1} = 1-|\mathbf{x}|_1$ and $\zeta = (\sqrt{y_1},\ldots,\sqrt{y_{d+1}})$ with $y_{d+1} = 1-|\mathbf{y}|_1$. The objective is to prove the following lemma.

Lemma 6.1. *For $\mathbf{x},\mathbf{y} \in T^d$, $\delta + |\lambda|_1 \ge (d+1)/2$,*

$$|\Omega_0| \le c\frac{\prod_{j=1}^{d+1}(\sqrt{x_jy_j}+n^{-1}|\xi-\zeta|+n^{-2})^{-\kappa_j}}{n^{|\kappa|_1+1/2}(|\xi-\zeta|+n^{-1})^{\delta+(d+1)/2}}.$$

For $\lambda_j \in (0,1)$, $j = 1, \ldots, d+1$, using (5.8) and integrating Ω_0 by parts p_j times for each t_j, we obtain

$$\Omega_0 = \frac{c_\kappa a_n (-1)^{|p|_1} \prod_{j=1}^{d+1} (\sqrt{x_j y_j})^{-p_j}}{\prod_{j=1}^{|p|_1} \frac{1}{2}(2n + 2(\alpha+\delta+1) - j + 1)} \int_{[-1,1]^{d+1}} P_{2n+|p|_1}^{(|\lambda|_1+\delta+\frac{d}{2}, |\lambda|_1+\delta+\frac{d}{2})}(z) \times \prod_{j=1}^{d+1} \frac{\partial^{p_j}}{\partial t_j^{p_j}} (1-t_j^2)^{\kappa_j - 1} d\mathbf{t}.$$

Note that ξ is in S_+^d, the first quarter of S^d. If, instead, we consider ξ as a vector in its own right, $\xi = (\xi_1, \ldots, \xi_{d+1})$, then we can write z as

$$z = z(\xi, \zeta, \mathbf{t}) = \xi_1 \zeta_1 t_1 + \ldots + \xi_d \zeta_d t_d + \xi_{d+1} \zeta_{d+1} t_{d+1}, \quad \text{with} \quad \zeta = (\zeta_1, \ldots, \zeta_{d+1}).$$

If we let ξ range over the entire S^d, then in order to evaluate Ω_0, it suffices to consider the integral for $(t_1, \ldots, t_{d+1}) \in [0,1]^{d+1}$. Each of the other $2^{d+1} - 1$ parts can be reduced to this part by making the substitutions of variables $t_j = -t_j$ and $\xi_j = -\xi_j$ for some j's. Let

$$Q_0 = n^{-|p|_1} \prod_{j=1}^{d+1} (\xi_j \zeta_j)^{-p_j} \int_{[0,1]^{d+1}} P_{2n+|p|_1}^{(|\lambda|_1+\delta+\frac{d}{2}, |\lambda|_1+\delta+\frac{d}{2})}(z) \prod_{j=1}^{d+1} \frac{\partial^{p_j}}{\partial t_j^{p_j}} (1-t_j^2)^{\kappa_j - 1} d\mathbf{t}.$$

Instead of proving Lemma 6.1, we shall prove the following lemma

Lemma 6.2. *For $\xi, \zeta \in S^d$, $\delta + |\lambda|_1 \geq (d+1)/2$,*

$$|Q_0| \leq c \frac{\prod_{j=1}^{d+1} (|\xi_j \zeta_j| + n^{-1}|\bar{\xi} - \bar{\zeta}| + n^{-2})^{-\kappa_j}}{n^{|\kappa|_1 + 1/2} (|\bar{\xi} - \bar{\zeta}| + n^{-1})^{\delta + (d+1)/2}}.$$

Evidently, since $\bar{\xi} \in S_+^d$, Lemma 6.1 follows from Lemma 6.2.

In order to prove the estimate in Lemma 6.2, we break $[0,1]^{d+1}$ into the union of the sets

$$E_m = (1 - \varepsilon_{n,1}, 1] \times \ldots \times (1 - \varepsilon_{n,m}, 1] \times [0, 1 - \varepsilon_{n,m+1}] \times \ldots \times [0, 1 - \varepsilon_{n,d+1}].$$

and the permutations of E_m just as in the proof of Lemma 5.4, where $\varepsilon_{n,j}$ is chosen as in (5.15) with $\bar{\xi}$ and $\bar{\zeta}$ in place of $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$, respectively. Consequently, we can write,

$$Q_0 = n^{-|p|_1} \prod_{j=1}^{d+1} (\xi_j \zeta_j)^{-p_j} \Big[\sum_{m=0}^{d+1} Q_{0,m} + \sum_{\sigma} \sum_{m=1}^{d} \sigma Q_{0,m} \Big],$$

where, the notation is self-evident when comparing with (5.10),

$$Q_{0,m} = \int_{E_m} P_{2n+|p|_1}^{(|\lambda|_1+\delta+\frac{d}{2}, |\lambda|_1+\delta+\frac{d}{2})}(z) \prod_{j=1}^{d+1} \frac{\partial^{p_j}}{\partial t_j^{p_j}} (1-t_j^2)^{\kappa_j - 1} d\mathbf{t}.$$

and $\sigma Q_{0,m}$ denotes the term with integral over σE_m. Again, since the stated estimate in Lemma 6.2 is independent of the choice of the order of x_j, we only need to deal with $\Omega_{0,m}$. Using (5.8) and integrating $Q_{0,m}$ by parts once for each

t_j, $j = m+1, \ldots, d+1$, we conclude that

$$Q_{0,m} = \mathcal{O}(n^{m-d-1}) \prod_{j=m+1}^{d+1} (\xi_j \zeta_j)^{-1} \int_{1-\varepsilon_{n,1}}^{1} \cdots \int_{1-\varepsilon_{n,m}}^{1} \Big\{ \sum_{j=m}^{d+1} \Lambda_{m,j} + \sum_{\sigma} \sum_{j=m+1}^{d} \sigma \Lambda_{m,j} \Big\} \times \prod_{i=1}^{m} \frac{\partial^{p_i}}{\partial t_i^{p_i}} (1-t_i^2)^{\kappa_i - 1} dt_1 \ldots, dt_m,$$

where the meaning of $\sigma\Lambda_{m,j}$ is as before and $\Lambda_{m,j}$ are given by

$$\Lambda_{m,j} = (-1)^{j-m} \int_0^{1-\varepsilon_{n,m+1}} \cdots \int_0^{1-\varepsilon_{n,j}} \Big[\frac{\partial^{p_{d+1}}}{\partial t_{d+1}^{p_{d+1}}} (1-t_{d+1}^2)^{\kappa_{d+1}-1} \cdots \Big[\frac{\partial^{p_{j+1}}}{\partial t_{j+1}^{p_{j+1}}} (1-t_{j+1}^2)^{\kappa_{j+1}-1} P_{2n+|p|_1+d+1-m}^{(|\lambda|_1+\delta-\frac{d}{2}-1+m, |\lambda|_1+\delta-\frac{d}{2}-1+m)} (z(\xi,\zeta,\mathbf{t})) \Big] \Big|_{t_{j+1}=0}^{1-\varepsilon_{n,j+1}} \cdots \Big] \Big|_{t_{d+1}=0}^{1-\varepsilon_{n,d+1}} \prod_{i=m+1}^{j} \frac{\partial^{p_i+1}}{\partial t_i^{p_i+1}} (1-t_i^2)^{\kappa_i-1} dt_{m+1} \ldots dt_j.$$

Note that unlike the proof of Lemma 5.4, the lower limit $t_i = 0$ plays a more serious role here. Indeed, substituting the lower limit $t_i = 0$ and upper limit $t_i = 1 - \varepsilon_{n,i}$, it follows from (5.3) and (5.13) that

$$|\Lambda_{m,j}| \le c n^{-\frac{1}{2}} \prod_{p=j+1}^{d+1} \varepsilon_{n,p}^{\lambda_p - 1} \Big[\sum_{i=j}^{d+1} U_{m,j,i} + \sum_{\sigma} \sum_{i=j+1}^{d} \sigma U_{m,j,i} \Big]$$

where, the meaning of $\sigma U_{m,j,i}$ is as before and $U_{m,j,i}$ are given by

$$U_{m,j,i} = \int_0^{1-\varepsilon_{n,m+1}} \cdots \int_0^{1-\varepsilon_{n,j}} (1 - z_{j,i}^2 + n^{-2})^{-\frac{1}{2}(|\lambda|_1+\delta+m-\frac{d+1}{2})} \times \prod_{i=m+1}^{j} (1-t_i)^{\lambda_i - 2} dt_{m+1} \ldots dt_j$$

and $z_{j,i} = \xi_1\zeta_1 t_1 + \ldots + \xi_j \zeta_j t_j + \xi_{j+1}\zeta_{j+1}(1-\varepsilon_{n,j+1}) + \ldots + \xi_i \zeta_i (1-\varepsilon_{n,i})$.

From our assumption that $\delta + |\lambda|_1 \ge (d+1)/2$, it follows that $|\lambda|_1 + \delta + m - (d+1)/2 \ge 0$ for all $m \ge 0$. Hence the power of $1 - z_{j,i}^2 + n^{-2}$ is negative and we can use the inequality

$$1 - z_{j,i}^2 + n^{-2} \ge 1 - |z_{j,\mathbf{k}}| + n^{-2} \ge 1 - \sum_{i=1}^{d+1} |\xi_i \zeta_i| + n^{-2} \ge c(|\bar{\xi} - \bar{\zeta}| + n^{-1})^2 \tag{6.2}$$

to enlarge the term under the integral signs (comparing with (5.16)). Using (5.17) for $0 < \lambda < 1$ and $t \in [-1,1]$, this gives

$$U_{m,j,i} \le c \prod_{p=m+1}^{j} \varepsilon_{n,p}^{\lambda_p - 1} \frac{1}{(|\bar{\xi} - \bar{\zeta}| + n^{-1})^{|\lambda|_1+\delta+m-(d+1)/2}}.$$

The right hand side is the same estimate in (5.18); hence, we can follow the proof of Case 1 in Lemma 5.4 to finish the proof of Lemma 6.2.

This way, we have proved Lemma 6.1. The estimate of Ω_* term can be carried out in exactly the same way as in the proof of Theorem 3.1 so is the rest of the proof of the theorem, which gives the proof of Theorem 3.3. $\square$

Note that the restriction $|\lambda|_1 + \delta \ge (d+1)/2$ is used to justify the use of (6.2) in the estimate of $U_{m,j,i}$, which allows us to work with $|\bar{\xi} - \bar{\zeta}|$ and not to deal with $z_{j,i}$. The case $|\lambda|_1 + \delta < (d+1)/2$ poses new difficulties since we need to work with various cases in $z_{j,i}$, and the induction argument showing $J_i \le cJ_0$ in the Case 2 of the proof of Lemma 5.4 no longer seems to work.

Proof of Theorem 3.4. The proof follows essentially from the proof in part (i) and part (ii) of Theorem 3.2, since it comes down to the proof of (i) and (ii) in the proof of Theorem 2.11. We choose to give a brief independent proof below.

Notice that the condition $|\lambda|_1 = \sum_{i=1}^{d+1}(\kappa_i - [\kappa_i]) \ge 1$ implies $(d-1)/2 \ge (d+1)/2 - |\lambda|_1$, so that $\delta > (d-1)/2$ implies that $\delta + |\lambda|_1 \ge (d+1)/2$; that is, the assumption of Theorem 3.3 holds. If $y_j \le x_j/2$ for some j, then

$$|\xi - \zeta| \ge |\sqrt{x_j} - \sqrt{y_j}| = \frac{|x_j - y_j|}{|\sqrt{x_j} + \sqrt{y_j}|} \ge x_j/2.$$

Hence, it follows that for $i = 1, 2, \ldots, d+1$

$$\sqrt{x_i y_i} + n^{-1}|\xi - \zeta| + n^{-2} \ge \sqrt{x_i y_i} + n^{-1} x_j/2.$$

Using this inequality in the estimate of Theorem 3.3 and the fact that $0 \le \delta - (d-1)/2 \le 1$, (i) is proved. If for all $j = 1, 2, \ldots, d$, $y_j > x_j/2$, then $\sqrt{x_j y_j} + n^{-1}|\xi - \zeta| + n^{-2} \ge x_j/\sqrt{2} + n^{-2}$. Using this inequality in the estimate in Theorem 3.3 gives the case (ii). $\square$

Note that the proof of Theorem 3.4 follows from the statement of Theorem 3.3; so the restriction (2.7) can be removed if the condition $|\lambda|_1 + \delta \ge (d+1)/2$ in Theorem 3.3 can be removed.

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