AN INTEGRAL FORMULA FOR GENERALIZED GEGENBAUER POLYNOMIALS AND JACOBI POLYNOMIALS

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Abstract. The generalized Gegenbauer polynomials are orthogonal polynomials with respect to the weight function $|x|^{2\mu}(1-x^2)^{\lambda-1/2}$. An integral formula for these polynomials is proved, which serves as a transformation between h-harmonic polynomials associated with $\mathbb{Z}^2$ invariant weight functions on the plane. The formula also gives a new integral transform for the Jacobi polynomials, which contains several well-known formulae as special cases. The new formulae can be used to prove the positivity of certain sums of the generalized Gegenbauer and Jacobi polynomials.

1. Introduction

For $\lambda > -1$ and $\mu \ge 0$, let $C_n^{(\lambda,\mu)}$ denote the generalized Gegenbauer polynomials of degree n, defined by the generating function

$$\sum_{n=0}^{\infty} C_n^{(\lambda,\mu)}(x) r^n = c_\mu \int_{-1}^{1} \frac{1}{(1-2rtx+r^2)^{\lambda+\mu}}(1+t)(1-t^2)^{\mu-1}dt,$$

where c_μ is a constant defined by

$$c_\mu^{-1} = \int_{-1}^{1}(1+t)(1-t^2)^{\mu-1}dt = \frac{\sqrt{\pi}\,\Gamma(\mu)}{\Gamma(\mu+\frac{1}{2})}.$$

The polynomials $C_n^{(\lambda,\mu)}$ are orthogonal polynomials with respect to the weight function

$$w_{\lambda,\mu}(x) = |x|^{2\mu}(1-x^2)^{\lambda-1/2}, \qquad \lambda,\mu > -1/2.$$

One of the main result of the present paper is the following identity:

Theorem 1.1. *For $\tau_1 > \kappa_1 > -1/2$, $\tau_2 > \kappa_2 > -1/2$ and $0 \le \theta \le \pi$,*

$$\begin{aligned}(1.1)\quad \frac{C_n^{(\tau_2,\tau_1)}(\cos\theta)}{\Gamma(\tau_1+\frac{n+1}{2})C_n^{(\tau_2,\tau_1)}(1)} &= \frac{\Gamma(\tau_2+\frac{1}{2})}{\Gamma(\tau_1-\kappa_1)\Gamma(\tau_2-\kappa_2)\Gamma(\kappa_2+\frac{1}{2})}\\ &\times \int_{-1}^{1}\int_{-1}^{1}(t_1^2\cos^2\theta+t_2^2\sin^2\theta)^{n/2}\frac{C_n^{(\kappa_2,\kappa_1)}(t_1\cos\theta(t_1^2\cos^2\theta+t_2^2\sin^2\theta)^{-1/2})}{\Gamma(\kappa_1+\frac{n+1}{2})C_n^{(\kappa_2,\kappa_1)}(1)}\\ &\times |t_1|^{2\kappa_1}|t_2|^{2\kappa_2}(1-t_1^2)^{\tau_1-\kappa_1-1}(1-t_2^2)^{\tau_2-\kappa_2-1}dt_1dt_2.\end{aligned}$$

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To put this result into perspective, let us relate it to several well-known transformations for Jacobi polynomials and Gegenbauer polynomials. Using the limit

$$\lim_{\mu\to 0} c_\mu \int_{-1}^{1} f(t)(1-t^2)^{\mu-1}dt = \frac{f(1)+f(-1)}{2}, \tag{1.2}$$

the generating function of the generalized Gegenbauer polynomials becomes the generating function of the Gegenbauer polynomials, usually denoted by $C_n^\lambda(x)$, when $\mu = 0$; that is, $C_n^{(\lambda,0)} = C_n^\lambda$. Hence, setting $\tau_1 = \kappa_1 = 0$ gives the following identity of Feldheim and Vilenkin (cf. [2, p. 24]),

$$\begin{aligned}\frac{C_n^\nu(\cos\theta)}{C_n^\nu(1)} =& \frac{2\Gamma(\nu+\frac{1}{2})}{\Gamma(\lambda+\frac{1}{2})\Gamma(\nu-\lambda)} \int_0^\pi \sin^{2\lambda}\varphi \cos^{2\nu-2\lambda-1}\varphi[1-\sin^2\theta\cos^2\varphi]^{n/2} \\ &\times \frac{C_n^\lambda(\cos\theta(1-\sin^2\theta\cos^2\varphi)^{-1/2})}{C_n^\lambda(1)} d\varphi\end{aligned}$$

for $\nu > \lambda > -1/2$, $0 \le \theta \le \pi$. Furthermore, the generalized Gegenbauer polynomials can be given in terms of the Jacobi polynomials $P_n^{(\alpha,\beta)}$, which are orthogonal polynomials associated with the weight function $(1-x)^\alpha(1+x)^\beta$ on the interval $[-1,1]$, normalized by

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha+1}{n} = \frac{(\alpha+1)_n}{n!}$$

(see, for example, [6]). Recall the Pochhammer symbol $(a)_m = a(a+1)\dots(a+m-1)$. The relation is ([5, p. 27]),

$$\begin{aligned} C_{2n}^{(\lambda,\mu)}(x) &= \frac{(\lambda+\mu)_n}{\left(\mu+\frac{1}{2}\right)_n} P_n^{(\lambda-1/2,\mu-1/2)}(2x^2-1), \\ C_{2n+1}^{(\lambda,\mu)}(x) &= \frac{(\lambda+\mu)_{n+1}}{\left(\mu+\frac{1}{2}\right)_{n+1}} x P_n^{(\lambda-1/2,\mu+1/2)}(2x^2-1). \end{aligned} \tag{1.3}$$

If we replace n in (1.1) by $2n$ and use the first relation of (1.3), we get the following integral transform for the Jacobi polynomials:

Theorem 1.2. *For $\gamma > \alpha > -1$, $\delta > \beta > -1$ and $0 \le \theta \le \pi/2$,*

$$\begin{aligned}\frac{P_n^{(\gamma,\delta)}(\cos 2\theta)}{P_n^{(\gamma,\delta)}(1)P_n^{(\delta,\gamma)}(1)} &= c_{\gamma,\alpha}c_{\delta,\beta}\int_0^1\int_0^1 (s^2\cos^2\theta + t^2\sin^2\theta)^n \\ &\times \frac{P_n^{(\alpha,\beta)}(u(s,t,\theta))}{P_n^{(\alpha,\beta)}(1)P_n^{(\beta,\alpha)}(1)} s^{2\beta+1}t^{2\alpha+1}(1-s^2)^{\gamma-\alpha-1}(1-t^2)^{\delta-\beta-1}dsdt,\end{aligned} \tag{1.4}$$

where $u(s,t,\theta) = (s^2\cos\theta^2 - t^2\sin\theta^2)/(s^2\cos\theta^2 + t^2\sin\theta^2)$ and $c_{\gamma,\alpha}$ is a constant given by

$$c_{\gamma,\alpha}^{-1} = \int_{-1}^1 |t|^{2\alpha+1}(1-t^2)^{\gamma-\alpha-1}dt = \frac{\Gamma(\alpha+1)\Gamma(\gamma-\alpha)}{\Gamma(\gamma+1)}.$$

This formula contains several well-known transforms. Setting $\beta = \delta$ or $\gamma = \alpha$ (use the limit (1.2)) in the formula (1.4) and making a proper change of variable, we get the following two identities proved by Askey and Fitch in [3] (see also the discussion in [2, p. 20]):

$$\frac{(1-x)^{\alpha+\mu}}{(1+x)^{n+\alpha+1}} \frac{P_n^{(\alpha+\mu,\beta)}(x)}{P_n^{(\alpha+\mu,\beta)}(1)} = \frac{2^\mu \Gamma(\alpha+\mu+1)}{\Gamma(\alpha+1)\Gamma(\mu)} \times \int_x^1 \frac{(1-y)^\alpha}{(1+y)^{n+\alpha+\mu+1}} \frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)} (y-x)^{\mu-1} dy \tag{1.5}$$

for $\alpha > -1$, $\mu > 0$ and $-1 < x < 1$, and

$$\frac{(1+x)^{\beta+\mu}}{(1-x)^{n+\beta+1}} \frac{P_n^{(\alpha,\beta+\mu)}(x)}{P_n^{(\beta+\mu,\alpha)}(1)} = \frac{2^\mu \Gamma(\beta+\mu+1)}{\Gamma(\beta+1)\Gamma(\mu)} \times \int_{-1}^x \frac{(1+y)^\beta}{(1-y)^{n+\beta+\mu+1}} \frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\beta,\alpha)}(1)} (x-y)^{\mu-1} dy \tag{1.6}$$

for $\beta > -1$, $\mu > 0$ and $-1 < x < 1$. These transformations among the Jacobi and Gegenbauer polynomials have applications in a number of problems, we refer to the discussions in [2].

As almost always the case, a formula for the special functions is discovered when one has the proper interpretation, often while working on something else. The formula (1.1) is discovered through the study of transformation between h-harmonics of different parameters on the plane. The h-harmonics are homogeneous polynomials orthogonal with respect to weight functions that are invariant under reflection groups. They are generalizations of the classical harmonic polynomials. There is a second order differential-difference operator, playing the role of Laplacian, which is in the commutative algebra generated by a family of commuting first order differential-difference operators (Dunkl's operators). For the general theory of h-harmonics, we refer to [4, 5] and the references therein. For the question in hand, we work with h-harmonics orthogonal with respect to the weight function $|x_1|^{2\kappa_1}|x_2|^{2\kappa_2}$, invariant under the abelian group $\mathbb{Z}_2 \times \mathbb{Z}_2$. The background and connection to h-harmonics are given in Section 2, in which Theorem 1.1 will be proved and another transform of the generalized Gegenbauer polynomials and Jacobi polynomials will also be given.

For the Jacobi polynomials, the transform (1.4) in Theorem 1.2 is a natural extension of the well-known formulae (1.5) and (1.6). The formula (1.4) appears to be new. Although such a formula is conceivable from combining the two formulae (1.5) and (1.6), the exact formulation is not obvious. Now the formula is stated, it might be possible to derive (1.4) this way, but we have not been able to find the two-dimensional change of variables that would yield such a deduction. In Section 3, an independent proof of Theorem 1.2 will be given using the hypergeometric functions.

Finally, some applications and consequences of Theorem 1.1 and Theorem 1.2 will be given in Section 4.

2. h-harmonics and integral transforms

We restrict the discussion of h-harmonics to the special case considered in the present paper, which is the simplest case of the general theory as $d = 2$ and the group is abelian, see [4, 5, 7]. Let $\mathcal{P}_n^2$ denote the space of homogeneous polynomials of degree n in two variables and let Π_n^2 denote the space of polynomials of degree n

in two variables. With respect to the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ and two nonnegative parameters $\kappa_1$and κ_2, the Dunkl operators $\mathcal{D}_i$ are defined by

$$\mathcal{D}_1 f(x) = \frac{\partial f}{\partial x_1} + \kappa_1 \frac{f(x) - f(-x_1, x_2)}{x_1}, \quad \mathcal{D}_2 f(x) = \frac{\partial f}{\partial x_2} + \kappa_2 \frac{f(x) - f(x_1, -x_2)}{x_2},$$

where $x = (x_1, x_2)$. These operators commute, that is, $\mathcal{D}_1\mathcal{D}_2 = \mathcal{D}_2\mathcal{D}_1$, and they satisfy $\mathcal{D}_i\mathcal{P}_n^2 \subset \mathcal{P}_{n-1}^2$. The h-Laplacian is defined by $\Delta_h = \mathcal{D}_1^2 + \mathcal{D}_2^2$, which plays the role of the ordinary Laplacian. Indeed, let $h_\kappa(x_1, x_2) = |x_1|^{\kappa_1}|x_2|^{\kappa_2}$; then $P \in \mathcal{P}_n^2$ and

$$\int_{S^1} PQh_\kappa^2(x)d\omega = 0, \qquad \forall Q \in \Pi_{n-1}^2,$$

if and only if $\Delta_h P = 0$. The space of h-harmonic polynomials of degree n is defined by

$$\mathcal{H}_n(h_\kappa^2) = \mathcal{P}_n^2 \cup \ker \Delta_h.$$

When $\kappa_1 = \kappa_2 = 0$, the h-harmonics become the ordinary harmonics. We denote by $\mathcal{H}_n$ the space of ordinary harmonics of degree n. Just as the ordinary harmonic polynomials, $\dim \mathcal{H}_n(h_\kappa^2) = 2$ for $n > 0$ and $\dim \mathcal{H}_0(h_\kappa^2) = 1$. In the polar coordinates

$$x_1 = r\cos\theta, \quad x_2 = r\sin\theta, \qquad r \ge 0, \quad 0 \le \theta \le 2\pi,$$

an orthonormal basis for the space $\mathcal{H}_n(h_\kappa^2)$ can be given in terms of the generalized Gegenbauer polynomials,

$$\begin{aligned} Y_{n,1}^\kappa(x) &= r^n \tilde{C}_n^{(\kappa_2,\kappa_1)}(\cos\theta), \\ Y_{n,2}^\kappa(x) &= r^n \sqrt{\frac{\kappa_1+\kappa_2+1}{\kappa_2+\frac{1}{2}}} \sin\theta\, \tilde{C}_{n-1}^{(\kappa_2+1,\kappa_1)}(\cos\theta), \end{aligned} \tag{2.1}$$

where $\tilde{C}_n^{(\lambda,\mu)}$ denote the orthonormal generalized Gegenbauer polynomials with respect to the normalized weight function and we set $Y_{0,2}^\kappa(x) = 0$. Furthermore, there is an intertwining operator V_κ between the commutative algebras generated by the partial derivatives and by the Dunkl operators, which is the unique linear operator defined by

$$V_\kappa\mathcal{P}_n^2 \subset \mathcal{P}_n^2, \qquad V_\kappa 1 = 1, \qquad \mathcal{D}_i V_\kappa = V_\kappa \partial_i, \quad i = 1, 2,$$

where ∂_i stands for $\partial/\partial x_i$. Moreover, in the present situation, this operator is given explicitly by an integral transform

$$V_\kappa f(x_1, x_2) = c_{\kappa_1}c_{\kappa_2}\int_{-1}^1\int_{-1}^1 f(t_1x_1, t_2x_2)(1+t_1)(1+t_2)(1-t_1^2)^{\kappa_1-1}(1-t_2^2)^{\kappa_2-1}dt_1dt_2.$$

Note that $V_\kappa = id$ if $\kappa_1 = \kappa_2 = 0$ by the limit relation (1.2).

The intertwining operator V_κ is a one-to-one mapping from $\mathcal{H}$ to $\mathcal{H}_n(h_\kappa^2)$. Consequently, the operator $V_{\tau,\kappa}$ defined by

$$V_{\tau,\kappa} = V_\tau V_\kappa^{-1}$$

is a one-to-one mapping from $\mathcal{H}_n(h_\kappa^2)$ to $\mathcal{H}_n(h_\tau^2)$. It turns out that for $\tau_1 > \kappa_1$ and $\tau_2 > \kappa_2$, the operator $V_{\tau,\kappa}$ can be given as an integral transform as well.

Notice that V_κ is a product of two integrals of one variable, we only need to consider one integral. For $\mu > 0$, define

$$V_\mu f(x) = c_\mu \int_{-1}^1 f(tx)(1+t)(1-t^2)^{\mu-1}dt. \tag{2.2}$$

Then $V_\mu f$ is one-to-one on the space of polynomials. We define $V_{\lambda,\mu} = V_\lambda V_\mu^{-1}$.

Lemma 2.1. *For $\lambda > \mu > 0$,*

$$V_{\lambda,\mu} f(x) = \frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda-\mu)\Gamma(\mu+1/2)} \int_{-1}^{1} f(xt)|t|^{2\mu}(1+t)(1-t^2)^{\lambda-\mu-1} dt.$$

Proof. Using the fact that V is one-to-one and taking $f = V_\mu g$, we only need to show that for $\lambda > \mu > 0$ and $g(x) = x^m$, $m \in \mathbb{N}_0$,

$$V_\lambda g(x) = b_{\lambda,\mu} \int_{-1}^{1} V_\mu g(xs)|s|^{2\mu}(1+s)(1-s^2)^{\lambda-\mu-1} ds \tag{2.3}$$

for a proper constant $b_{\lambda,\mu}$. An easy computation shows that

$$V_\mu g(x) = c_\mu \frac{\Gamma((m+1+\varepsilon_m)/2)\Gamma(\mu)}{\Gamma(\mu+(m+1+\varepsilon_m)/2)}\, g(x), \qquad g(x) = x^m,$$

where $\varepsilon = 1$ if m is odd, and $\varepsilon = 0$ if m is even, which implies that for $g(x) = x^m$ the right hand side of (2.3) is equal to

$$\begin{aligned} b_{\lambda,\mu} c_\mu & \frac{\Gamma((m+1+\varepsilon_m)/2)\Gamma(\mu)}{\Gamma(\mu+(m+1+\varepsilon_m)/2)} \int_{-1}^{1} s^m |s|^{2k}(1+s)(1-s^2)^{\lambda-\mu-1} ds\, g(x) \\ &= b_{\lambda,\mu} c_\mu \frac{\Gamma(\mu)\Gamma(\lambda-\mu)}{\Gamma(\lambda)} \frac{\Gamma((m+1+\varepsilon_m)/2)\Gamma(\lambda)}{\Gamma(\lambda+(m+1+\varepsilon_m)/2)}\, g(x) \\ &= b_{\lambda,\mu} c_\mu c_\lambda^{-1} \frac{\Gamma(\mu)\Gamma(\lambda-\mu)}{\Gamma(\lambda)} V_\lambda g(x). \end{aligned}$$

Choosing the constant $b_{\lambda,\mu}$ so that the constant in front of $V_\lambda g(x)$ is 1 gives

$$b_{\lambda,\mu} = c_\mu^{-1} c_\lambda \frac{\Gamma(\lambda)}{\Gamma(\mu)\Gamma(\lambda-\mu)} = \frac{\Gamma(\lambda+1/2)}{\Gamma(\lambda-\mu)\Gamma(\mu+1/2)}.$$

This proves the desired result. □

We note that the condition $\lambda > \mu$ is necessary. In particular, we cannot take $\lambda = 0$ to get a formula for V_μ^{-1}. For the record, we write down a formula for the inverse of V_μ^{-1} below. For $\mu > 0$, let $[\mu]$ denote the integer part of μ.

Proposition 2.2. *Let $\mu > 0$. If g is an even function, then*

$$V_\mu^{-1} g(x) = a_\mu \frac{d}{dx}\Big(\frac{d}{xdx}\Big)^{[\mu]} \Big\{ x^{2[\mu]+1} \int_0^1 g(xs) s^{2\mu}(1-s^2)^{-(\mu-[\mu])} ds \Big\};$$

if g is an odd function, then

$$V_\mu^{-1} g(x) = a_\mu \Big(\frac{d}{xdx}\Big)^{[\mu]+1} \Big\{ x^{2[\mu]+2} \int_0^1 g(xs) s^{2\mu+1}(1-s^2)^{-(\mu-[\mu])} ds \Big\},$$

where $a_\mu = 2^{-[\mu]}\Gamma(1/2)/(\Gamma(\mu+1/2)\Gamma(-\mu+[\mu]+1))$.

The proof amounts to verify that the given formula satisfies $V_\mu^{-1} g_m(x) = x^m$ for $g_m(x) = V(x^m)$, $m \in \mathbb{N}_0$; note that g_m has been computed in the proof of the lemma. We omit the details.

For the weight function $h_\kappa(x) = |x_1|^{\kappa_1}|x_2|^{\kappa_2}$, an integral formula for $V_{\tau,\kappa}$ follows from Lemma 2.1. The formula can be written in the following form:

Corollary 2.3. *For $\tau_1 > \kappa_1 \ge 0$ and $\tau_2 > \kappa_2 \ge 0$,*

$$V_{\tau,\kappa} f(x) = a_{\tau,\kappa} h_\kappa^{-2}(x) V_{\tau-\kappa}(f h_\kappa^2)(x),$$

where

$$a_{\tau,\kappa} = c_{\tau_1-\kappa_1}^{-1} c_{\tau_1-\kappa_1}^{-1} \frac{\Gamma(\tau_1+\frac{1}{2})}{\Gamma(\kappa_1+\frac{1}{2})\Gamma(\tau_1-\kappa_1)} \frac{\Gamma(\tau_2+\frac{1}{2})}{\Gamma(\kappa_2+\frac{1}{2})\Gamma(\tau_2-\kappa_1)}.$$

The integral transform of the generalized Gegenbauer polynomials is the consequence of the formula for $V_{\tau,\kappa}$.

Proof of Theorem 1.1. First we assume that $\kappa_1 \ge 0$ and $\kappa_2 \ge 0$. The orthonormal basis (2.1) of $\mathcal{H}_n(h_\kappa^2)$ shows that $Y_{n,1}^\kappa(x_1,x_2)$ is even in x_2 and $Y_{n,2}^\kappa(x_1,x_2)$ is odd in x_2. Recall that $V_{\tau,\kappa}$ is a one-to-one mapping from $\mathcal{H}_n(h_\kappa^2)$ to $\mathcal{H}_n(h_\tau^2)$. The formula for $V_{\tau,\kappa}$ in the lemma shows that if $f(x_1,x_2)$ is even [resp. odd] in x_2, then $V_{\tau,\kappa} f(x_1,x_2)$ is also even [resp. odd] in x_2. Consequently, we must have $Y_{n,1}^\tau = B_n V_{\tau,\kappa} Y_{n,1}^\kappa$ for some constant B_n. That is,

$$\begin{aligned} (2.4) \qquad & (x_1^2+x_2^2)^{n/2} C_n^{(\tau_2,\tau_1)}\Big(\frac{x_1}{\sqrt{x_1^2+x_2^2}}\Big) \\ &\qquad = B_n V_{\tau,\kappa}\Big[(x_1^2+x_2^2)^{n/2} C_n^{(\kappa_2,\kappa_1)}\Big(\frac{x_1}{\sqrt{x_1^2+x_2^2}}\Big)\Big]. \end{aligned}$$

The constant B_n can be determined by setting $x_1 = 1$ and $x_2 = 0$. Since $C_n^{(\kappa_2,\kappa_1)}$ is even if n is even and is odd if n is odd, we have $t_1^n C_n^{(\kappa_2,\kappa_1)}(t_1/|t|) = |t_1|^n C_n^{(\kappa_2,\kappa_1)}(1)$, which gives

$$\begin{aligned} C_n^{(\tau_2,\tau_1)}(1) &= B_n C_n^{(\kappa_2,\kappa_1)}(1) \\ &\quad \times \int_{-1}^1 |t_1|^{n+2\kappa_1}(1-t_1^2)^{\tau_1-\kappa_1-1} dt_1 \int_{-1}^1 |t_2|^{2\kappa_2}(1-t_2^2)^{\tau_2-\kappa_2-1} dt_2. \end{aligned}$$

Computing the integrals to determine the formula for B_n and setting $x_1 = \cos\theta$ and $x_2 = \sin\theta$ in (2.4) proves the desired result for the case that $\kappa_1 \ge 0$ and $\kappa_2 \ge 0$. The analytic continuation extends the range to $\kappa_1 > -1/2$ and $\kappa_2 > -1/2$. $\square$

Notice that (1.1) is just the transformation (2.4) between h-harmonics of different parameters. The same interpretation for h-harmonics even in x_2 applies to the integral formula (1.4) for the Jacobi polynomials; in particular, a further restriction on the weight function of the h-harmonics gives such an interpretation to the classical formulae (1.5) and (1.6).

With the definition of V_μ at (2.2), the generating function of the generalized Gengebauer polynomials shows that

$$C_n^{(\lambda,\mu)}(x) = V_\mu C_n^{\lambda+\mu}(x), \qquad \mu \ge 0.$$

It follows that for $\sigma > 0$,

$$V_\sigma V_\mu^{-1} C_n^{(\lambda,\mu)}(x) = V_\sigma C_n^{\lambda+\mu}(x) = V_\sigma C_n^{\lambda+\mu-\sigma+\sigma}(x) = C_n^{(\lambda+\mu-\sigma,\sigma)}(x).$$

Consequently, as another application of the formula of $V_{\lambda,\mu}$ in Lemma 2.1, we obtain the following integral formula of the generalized Gegenbauer polynomials:

Proposition 2.4. *For $\mu \ge 0$ and $\lambda \ge \sigma > 0$,*

$$C_n^{(\lambda-\sigma,\mu+\sigma)}(x) = \frac{\Gamma(\mu+\sigma+1/2)}{\Gamma(\sigma)\Gamma(\mu+1/2)} \int_{-1}^1 C_n^{(\lambda,\mu)}(xt)|t|^{2\mu}(1+t)(1-t^2)^{\sigma-1} dt.$$

In particular, using the above formula for $2n$ and the first formula of (1.3), a proper change of variable gives the first of the following well-known formulae for the Jacobi polynomials ([3, p. 420]),

$$(1-x)^{\alpha+\sigma}\frac{P_n^{(\alpha+\sigma,\beta-\sigma)}(x)}{P_n^{(\alpha+\sigma,\beta-\sigma)}(1)} = \frac{\Gamma(\alpha+\sigma+1)}{\Gamma(\alpha+1)\Gamma(\sigma)}\int_x^1 (1-y)^\alpha \frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)}(y-x)^{\sigma-1}dy$$

for $\alpha > -1$, $\sigma > 0$ and $-1 < x < 1$, and

$$(1+x)^{\beta+\sigma}\frac{P_n^{(\alpha-\sigma,\beta+\sigma)}(x)}{P_n^{(\beta+\sigma,\alpha-\sigma)}(1)} = \frac{\Gamma(\beta+\sigma+1)}{\Gamma(\beta+1)\Gamma(\sigma)}\int_{-1}^x (1+y)^\beta \frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\beta,\alpha)}(1)}(x-y)^{\sigma-1}dy$$

for $\beta > -1$, $\sigma > 0$ and $-1 < x < 1$.

3. Hypergeometric functions and Jacobi polynomials

The formula (1.1) is established as a transformation between h-harmonics of different parameters. In this section, we give a direct proof using the hypergeometric functions. Since the generalized Gegenbauer polynomials can be written in terms of the Jacobi polynomials by (1.3), we give such a proof for the formula (1.4). Recall that the hypergeometric function is defined by

$${}_2F_1(a,b;c;x) = \sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n n!}x^n, \qquad |x|<1,$$

where a, b, c are parameters. The Jacobi polynomials can be written as

$$\frac{P_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)}(1)} = {}_2F_1\Big(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\Big). \tag{3.1}$$

We need two lemmas on the hypergeometric function.

Lemma 3.1. *If $\Re(e) > \Re(c) > 0$, then*

$$(u+(1-u)s)^{-a}{}_2F_1\Big(a, e-c+b; e; \frac{u}{u+(1-u)s}\Big)$$
$$= \frac{\Gamma(e)}{\Gamma(c)\Gamma(e-c)}\int_0^1 {}_2F_1\Big(a,b;c;\frac{ut}{ut+(1-u)s}\Big)(ut+(1-u)s)^{-a}t^{c-1}(1-t)^{e-c-1}dt.$$

Proof. We start with Bateman's integral ([1, Theorem 2.2.4, p. 68])

$${}_2F_1(a,c-b;e;x) = \frac{\Gamma(e)}{\Gamma(c)\Gamma(e-c)}\int_0^1 {}_2F_1(a,c-b;c,xt)t^{c-1}(1-t)^{e-c-1}dt$$

and Pfaff's relation ([1, Theorem 2.2.5, p. 68])

$${}_2F_1(a,b;c;x) = (1-x)^{-a}{}_2F_1\Big(a,c-b;c;\frac{x}{x-1}\Big).$$

Together they give the following integral relation

$${}_2F_1(a,c-b;e;x) = \frac{\Gamma(e)}{\Gamma(c)\Gamma(e-c)}\int_0^1 {}_2F_1\Big(a,b;c;\frac{xt}{xt-1}\Big)(1-tx)^{-a}t^{c-1}(1-t)^{e-c-1}dt.$$

Setting $x = z/(z-1)$ so that $1 - xt = 1 - (1-t)z/(1-z)$ and using Pfaff's relation again, we get

$$
\begin{aligned}
&{}_2F_1(a, e-c+b; e; z) = (1-z)^{-a}{}_2F_1\Big(a, c-b; e; \frac{z}{z-1}\Big) \\
=&\frac{\Gamma(e)}{\Gamma(c)\Gamma(e-c)}\int_0^1 {}_2F_1\Big(a,b;c;\frac{zt}{zt+(1-z)}\Big)(1-(1-t)z)^{-a}t^{c-1}(1-t)^{e-c-1}dt.
\end{aligned}
$$

Setting $z = u/(u+(1-u)s)$ gives the stated formula. $\square$

Lemma 3.2. *If* $-a \in \mathbb{N}_0$, $\Re(a+b-c) > -1$, $\Re(d) > \Re(b) > 0$, *then*

$$
\begin{aligned}
{}_2F_1(a,d;c;u) = &\frac{\Gamma(d-c+1)}{\Gamma(b-c+1)\Gamma(d-b)} \\
&\times \int_0^1 ((1-u)t+u)^{-a}{}_2F_1\Big(a,b;c;\frac{u}{u+(1-u)t}\Big)t^{a+b-c}(1-t)^{d-b-1}dt.
\end{aligned}
$$

Proof. Let $n = -a \in \mathbb{N}_0$. The definition of ${}_2F_1$ gives

$$
((1-u)t+u)^{-a}{}_2F_1\Big(a,b;c;\frac{u}{u+(1-u)t}\Big) = \sum_{k=0}^{n}\frac{(-n)_k(b)_k}{(c)_k k!}u^k(t+(1-t)u)^{n-k}.
$$

Using the binomial formula to expand $(t+(1-t)u)^{n-k}$ so that the Beta integral can be used to carry out the integration, we get

$$
\begin{aligned}
&\int_0^1 ((1-u)t+u)^{-a}{}_2F_1\Big(a,b;c;\frac{u}{u+(1-u)t}\Big)t^{a+b-c}(1-t)^{d-b-1}dt \\
&= \sum_{k=0}^{n}\frac{(-n)_k(b)_k}{(c)_k k!}\sum_{j=0}^{n-k}\binom{n-k}{j}\frac{\Gamma(b-c+j-n+1)\Gamma(n-k-j+d-b)}{\Gamma(d-c-k+1)}u^{n-j}.
\end{aligned}
$$

Changing the order of the summations and using the fact that

$$
\binom{n-k}{j} = (-1)^j\frac{(-n)_j(-j)_k}{j!(-n)_k} \quad \text{and} \quad (A-k)_k = (-1)^k(1-A)_k,
$$

the summation over k can be seen to be a ${}_3F_2$ and the above double sum becomes

$$
\begin{aligned}
&\frac{\Gamma(b-c+1)\Gamma(d-b)}{\Gamma(d-c+1)}\sum_{j=0}^{n}\frac{(-n)_j(d-b)_j}{(c-b)_j j!}{}_3F_2\Big(\begin{matrix}-j, b, c-d\\ c, 1-d+b-j\end{matrix};1\Big)u^j \\
&= \frac{\Gamma(b-c+1)\Gamma(d-b)}{\Gamma(d-c+1)}\sum_{j=0}^{n}\frac{(-n)_j(c-b)_j}{(c)_j j!}\frac{(c-b)_j(d)_j}{(c)_j(d-b)_j}u^j \\
&= \frac{\Gamma(b-c+1)\Gamma(d-b)}{\Gamma(d-c+1)}{}_2F_1(-n,d;c;u),
\end{aligned}
$$

where the Pfaff-Saalshütz formula [1, Theorem 2.2.6, p. 69] is used to sum the ${}_3F_2$. This proves the lemma. $\square$

The transformation that gives the formula (1.4) is the following expression of the hypergeometric function:

Proposition 3.3. *If* $-a \in \mathbb{N}_0$, $\Re(a+b-c) > -1$, $\Re(e) > \Re(c)$ *and* $\Re(d-e+b-c) > 0$, *then*

$$\begin{aligned} {}_2F_1(a,d;e;u) =& \frac{\Gamma(d-e+1)}{\Gamma(b-c+1)\Gamma(d-e+c-b)} \frac{\Gamma(e)}{\Gamma(c)\Gamma(e-c)} \\ & \times \int_0^1 \int_0^1 {}_2F_1\Big(a,b;c;\frac{ut}{ut+(1-u)s}\Big)(ut+(1-u)s)^{-a} \\ & \times t^{c-1}(1-t)^{e-c-1}s^{a+b-c}(1-s)^{d-e+c-b-1}dtds \end{aligned}$$

Proof. In the formula of Lemma 3.2 replacing b by $e-c+b$ and c by e gives

$$\begin{aligned} &{}_2F_1(a,d;e;u) = \frac{\Gamma(d-c+1)}{\Gamma(b-c+1)\Gamma(d-e+c-b)} \\ &\times \int_0^1 (u+(1-u)s)^{-a}{}_2F_1\Big(a,e-c+b;e;\frac{u}{u+(1-u)s}\Big)s^{a+b-c}(1-s)^{d-e+c-b-1}ds. \end{aligned}$$

Notice that the ${}_2F_1$ inside the integral is exactly the same as the left hand side in Lemma 3.1. The desired result follows from substituting the formula in Lemma 3.1 into the above formula. □

Using the ${}_2F_1$ expression of the Jacobi polynomials at (3.1), the formula (1.4) follows from the above proposition upon setting $a=-n$, $b=n+\alpha+\beta+1$, $c=\alpha+1$, $d=n+\gamma+\delta+1$, $e=\gamma+1$ and $u=\sin^2\theta$.

4. Applications

We discuss several applications of the formulae (1.1) and (1.4) in this section. These applications are similar to those that are consequences of (1.5) and (1.6) as discussed in [2] and their proofs are also similar, so we shall be brief.

Theorem 4.1. *Let* $\tau_1 > \kappa_1 > -1/2$ *and* $\tau_2 > \kappa_2 > -1/2$ *and* $-1 \le x \le 1$. *Then there exists a measure* $d\mu_x(y)$ *such that*

$$\frac{C_n^{(\tau_2,\tau_1)}(y)}{\Gamma(\tau_1+\frac{n+1}{2})C_n^{(\tau_2,\tau_1)}(1)} = \int_{-1}^1 \frac{C_n^{(\kappa_2,\kappa_1)}(y)}{\Gamma(\kappa_1+\frac{n+1}{2})C_n^{(\kappa_2,\kappa_1)}(1)} d\mu_x(y), \tag{4.1}$$

where $\int_{-1}^1 d\mu_x(y) = 1$, $d\mu_x(y)$ *is independent of* n *and* $d\mu_x(y) > 0$ *for* $-1 < x < 1$ *and* $-1 < y < 1$.

Proof. The Poisson kernel $P^{(\lambda,\mu)}$ of $C_n^{(\lambda,\mu)}$ satisfies (cf. [7]),

$$\begin{aligned} P^{(\lambda,\mu)}(r;x,y) &= \sum_{j=0}^\infty (h_n^{(\lambda,\mu)})^{-1} C_j^{(\lambda,\mu)}(x) C_j^{(\lambda,\mu)}(y) r^j \\ &= c_\lambda c_\mu \int_{-1}^1 \int_{-1}^1 \frac{1-r^2}{(1-2r(xyt_1 - 2\sqrt{1-x^2}\sqrt{1-y^2}t_2)+r^2)^{\lambda+\mu+1}} \\ &\qquad \times (1+t_1)(1+t_2)(1-t_1^2)^{\lambda-1}(1-t_2^2)^{\mu-1}dt_1dt_2 \end{aligned}$$

for $|r|<1$, $-1 \le x,y \le 1$, where $h_n^{(\lambda,\mu)}$ is the L^2 norm of $C_n^{(\lambda,\mu)}$,

$$h_n^{(\lambda,\mu)} = \int_{-1}^1 [C_n^{(\lambda,\mu)}(x)]^2 \tilde{w}_{\lambda,\mu}(x)dx = \frac{\lambda+\mu+1}{n+\lambda+\mu+1} C_n^{(\lambda,\mu)}(1),$$

and $\widetilde{w}_{\lambda,\mu} = (\Gamma(\lambda+\mu+1)/(\Gamma(\lambda+\frac{1}{2})\Gamma(\mu+\frac{1}{2})))w_{\lambda,\mu}$ is the normalized weight function. It shows, in particular, that $P^{(\lambda,\mu)}(r;x,y)$ is strict positive. Let $K^{(\tau,\kappa)}(r;x,y)$ be defined by

$$K^{(\tau,\kappa)}(r;x,y) = \frac{\Gamma(\tau_1+\frac{1}{2})}{\Gamma(\kappa_1+\frac{1}{2})}\sum_{n=0}^{\infty}\frac{\Gamma(\kappa_1+\frac{n+1}{2})}{\Gamma(\tau_1+\frac{n+1}{2})}\frac{n+\kappa_1+\kappa_2+1}{\kappa_1+\kappa_2+1}\frac{C_n^{(\tau_2,\tau_1)}(x)C_n^{(\kappa_2,\kappa_1)}(y)}{C_n^{(\tau_2,\tau_1)}(1)}r^n, \tag{4.2}$$

where $0 \le |r| < 1$. The orthogonality of $C_n^{(\kappa_2,\kappa_1)}$ shows that

$$\int_{-1}^{1} K^{(\tau,\kappa)}(r;x,y)\widetilde{w}_{\kappa_2,\kappa_1}(y)dy = 1.$$

By (1.1), the above equation is equivalent to

$$K^{(\tau,\kappa)}(r;x,y) = \frac{\Gamma(\tau_2+\frac{1}{2})}{\Gamma(\tau_1-\kappa_1)\Gamma(\tau_2-\kappa_2)\Gamma(\kappa_2+\frac{1}{2})}\int_{-1}^{1}\int_{-1}^{1} P^{(\kappa_2,\kappa_1)}(rv(t_1,t_2,x);x,y)$$
$$\times |t_1|^{2\kappa_1}|t_2|^{2\kappa_2}(1-t_1^2)^{\tau_1-\kappa_1-1}(1-t_2^2)^{\tau_2-\kappa_2-1}dt_1dt_2,$$

where $v(t_1,t_2,x) = t_1^2\cos^2\theta + t_2^2\sin^2\theta$ and $x = \cos\theta$. Consequently, since $|v(t_1,t_2,\theta)| \le 1$, it follows that $K^{(\tau,\kappa)}(r;x,y)$ is positive. Furthermore, by (4.2),

$$r^n\frac{C_n^{(\tau_2,\tau_1)}(x)}{\Gamma(\tau_1+\frac{n+1}{2})C_n^{(\tau_2,\tau_1)}(1)} = \int_{-1}^{1}\frac{C_n^{(\kappa_2,\kappa_1)}(y)}{\Gamma(\kappa_1+\frac{n+1}{2})C_n^{(\kappa_2,\kappa_1)}(1)}K^{(\kappa_2,\kappa_1)}(r;x,y)\widetilde{w}_{\kappa_2,\kappa_1}(y)dy.$$

Taking the limit $r \to 1$ finishes the proof. □

If $\tau_1 = 0$ and $\kappa_1 = 0$, then the theorem becomes the classical result for the Gegenbauer polynomials, see [2, Theorem 3.3, p. 25]. Taking n to be even in Theorem 4.1 and using the relation (1.3) gives the following:

Theorem 4.2. *Let $\gamma > \alpha > -1$ and $\delta > \beta > -1$ and $-1 \le x \le 1$. Then there exists a measure $d\mu_x(y)$ such that*

$$\frac{P_n^{(\gamma,\delta)}(y)}{P_n^{(\gamma,\delta)}(1)P_n^{(\delta,\gamma)}(1)} = \int_{-1}^{1}\frac{P_n^{(\alpha,\beta)}(y)}{P_n^{(\alpha,\beta)}(1)P_n^{(\beta,\alpha)}(1)}d\mu_x(y),$$

where $\int_{-1}^{1} d\mu_x(y) = 1$, $d\mu_x(y)$ is independent of n and $d\mu_x(y) > 0$ for $-1 < x < 1$ and $-1 < y < 1$.

In particular, if $\gamma = \alpha$, then $P_k^{(\alpha,\beta)}(1) = P_k^{(\gamma,\delta)}(1)$ can be removed from the denominator, and this becomes a special case of Theorem 3.4 of [2]; notice that the indices of the Jacobi polynomials in the numerator and denominator are reversed. The case $\delta = \beta$ gives its counterpart in which the indices of the Jacobi polynomials in the numerator and denominator are the same.

As an immediate consequence of Theorem 4.1, we state the following result:

Theorem 4.3. *If $\tau_1 > \kappa_1 > -1/2$ and $\tau_2 > \kappa_2 > -1/2$ and*

$$f(x) = \sum_{j=0}^{n} a_j\frac{C_j^{(\kappa_2,\kappa_1)}(x)}{(\kappa_1+\frac{1}{2})_{\frac{j}{2}}C_j^{(\kappa_2,\kappa_1)}(1)} \ge 0,$$

then

$$g(y) = \sum_{j=0}^{n} a_j \frac{C_j^{(\tau_2,\tau_1)}(y)}{(\tau_1+\frac{1}{2})_{\frac{j}{2}} C_j^{(\tau_2,\tau_1)}(1)} > 0, \quad -1 < y < 1,$$

unless $a_j \equiv 0,\ j = 0, 1, \dots, n$.

Clearly, the result can be stated for infinite series as long as the absolute convergence of the series is assumed. If $\tau_1 = \kappa_1 = 0$, then this theorem reduces to Theorem 3.2 in [2]. Since it is known [2, (1.22), p. 24] that

$$\sum_{k=0}^{n} \frac{C_k^{1/2}(x)}{C_k^{1/2}(1)} = \sum_{k=0}^{n} P_k^{(0,0)}(x) > 0, \qquad -1 < x \le 1, \tag{4.3}$$

and $C_n^{(\lambda,0)} = C_n^{\lambda}$, the proposition gives the following:

Corollary 4.4. *If* $\tau_2 \ge 1/2$ *and* $\tau_1 \ge 0$, *then*

$$\sum_{k=0}^{n} \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\tau_1 + \frac{k+1}{2})} \frac{C_k^{(\tau_2,\tau_1)}(x)}{C_k^{(\tau_2,\tau_1)}(1)} > 0, \qquad -1 < x \le 1,$$

and the inequality holds for $-1 \le x \le 1$ *if* $\tau_1 > 0$.

Proof. For $-1 < x < 1$ the inequality follows from the proposition and (4.3) with $\kappa_1 = 0$ and $\kappa_2 = 1/2$. At $x = 1$ the inequality is trivially positive. For $\tau_1 > 0$, the inequality at $x = -1$ becomes

$$\sum_{k=0}^{n} (-1)^k \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\tau_1 + \frac{k+1}{2})} = \frac{1}{\Gamma(\tau_1)} \sum_{k=0}^{n} (-1)^k \int_0^1 t^{\tau_1 - 1}(1-t)^{k/2} dt,$$

which is easily seen to be positive upon summing up the geometric sequence. □

If $\tau_1 = 0$, then the corollary becomes the classical result for the Gegenbauer polynomials, see [2, (3.35), p. 25]. Furthermore, the following result follows from Theorem 4.2, or from taking n to be even in Theorem 4.3 and using the relation (1.3):

Theorem 4.5. *If* $\gamma > \alpha > -1$, $\delta > \beta > 1$, *and*

$$f(x) = \sum_{k=0}^{n} a_k \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\alpha,\beta)}(1) P_k^{(\beta,\alpha)}(1)} \ge 0, \quad -1 < x < 1,$$

then

$$g(y) = \sum_{k=0}^{n} a_k \frac{P_k^{(\gamma,\delta)}(y)}{P_k^{(\gamma,\delta)}(1) P_k^{(\delta,\gamma)}(1)} > 0, \quad -1 < y < 1,$$

unless $a_k \equiv 0,\ k = 0, 1, \dots, n$.

Again, if $\gamma = \alpha$, then $P_k^{(\alpha,\beta)}(1) = P_k^{(\gamma,\delta)}(1)$ can be removed from the denominator and this becomes a special case of Theorem 3.5 of [2]. The case $\delta = \beta$ gives its counterpart in which the indices of the Jacobi polynomials in the numerator and denominator are the same. As an application of the above theorem we get the following:

Corollary 4.6. *If* $\alpha, \beta > 0$ *then*

$$\sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\alpha,\beta)}(1) P_k^{(\beta,\alpha)}(1)} > 0, \qquad -1 \le x \le 1.$$

Proof. Since $P_k^{(0,0)}(1) = 1$, the inequality follows form the theorem and (4.3) for $-1 < x < 1$. At $x = 1$ the positivity is trivial, and at $x = -1$, using the fact that $P_k^{(\alpha,\beta)}(-1) = (-1)^k P_k^{(\alpha,\beta)}(1)$, the sum becomes

$$\sum_{k=0}^{n} \frac{(-1)^k}{P_k^{(\alpha,\beta)}(1)} = \sum_{k=0}^{n} (-1)^k / \binom{k+\alpha}{k},$$

which is positive since $\{1/\binom{k+\alpha}{k}\}$ is a decreasing sequence. □

We note that using the formula $P_k^{(\alpha,\beta)}(-x) = (-1)^k P_k^{(\alpha,\beta)}(x)$, the above formula with x replaced by $-x$ also gives

$$\sum_{k=0}^{n} \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\alpha,\beta)}(1) P_k^{(\alpha,\beta)}(-1)} > 0, \qquad -1 \le x \le 1.$$

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