Special functions (Hong Kong, 1999), 423–430, World Sci. Publishing, River Edge, NJ, 2000

A Product Formula for Jacobi Polynomials

Yuan Xu
Department of Mathematics, University of Oregon
Eugene, OR 97403-1222
E-mail: yuan@math.uoregon.edu

We prove a new product formula for the Jacobi polynomials, which writes the product of $P_n^{(\alpha,\beta)}(\cos 4\theta)$ as an integral in terms of the Gegenbauer polynomial $C_{4n}^{(2\alpha+2\beta)}$.

1 Introduction

The product formula of Jacobi polynomials discovered by Koornwinder has found many applications, we refer to [1] for some of them and for the historical account. This formula expresses the product of two Jacobi polynomials as an integral of the Jacobi polynomial of the same index. There is another product formula of Jacobi polynomials due to Dijksma and Koornwinder [2] which expresses the product as an integral of a Gegenbauer polynomial (see (1) below). This "wrong" product formula has been used recently in [6] to study the orthogonal polynomials associated to a family of product weight function on the sphere in $\mathbb{R}^d$, a situation that serves as an example for Dunkl's theory of h-harmonics associated with the reflection groups.

The purpose of this note is to prove yet another product formula for Jacobi polynomials. The new formula is suggested by the intertwining operator in the theory of h-harmonics associated to the dihedral group D_4; it can be realized as an integral formula of the intertwining operator for invariant polynomials. The intertwining operator plays a very important role in the theory of h-harmonics associated with reflection groups. An integral formula of this operator will help us solve several outstanding problems, such as constructing an explicit orthonormal basis and proving sharp results in Fourier expansion of h-harmonic series. Although the group D_4 is a special case whose associated h-harmonics are merely two dimensional, it is the simplest example of the non-commutative reflection groups. We hope that a complete understanding of this case will help us gain insight into more general cases.

2 Product Formula

We prove the new product formula in this section, and discuss the connection to h-harmonics in Section 3.

We use the standard notation for the Jacobi polynomial $P_n^{(\alpha,\beta)}$ and the Gegenbauer polynomial $C_n^{(\lambda)}$. Our starting point is a product formula for Jacobi polynomials due to Dijksma and Koornwinder [2], which states that for $\lambda > -1/2$, $\mu > -1/2$,

$$P_n^{(\lambda-1/2,\mu-1/2)}(\cos 2\theta)P_n^{(\lambda-1/2,\mu-1/2)}(\cos 2\phi) = A_n^{(\lambda,\mu)}\times$$
$$\int_{-1}^{1}\int_{-1}^{1} C_{2n}^{(\lambda+\mu)}(t\cos\theta\cos\phi + s\sin\theta\sin\phi)(1-t^2)^{\lambda-1}(1-s^2)^{\mu-1}dsdt, \quad (1)$$

where

$$A_n^{(\lambda,\mu)} = \frac{\Gamma(\lambda+\mu)\Gamma(n+\lambda+1/2)\Gamma(n+\mu+1/2)}{\pi\Gamma(n+1)\Gamma(n+\lambda+\mu-1)\Gamma(\lambda)\Gamma(\mu)}.$$

The new product formula represents $P_n^{(\lambda,\mu)}(\cos 4\theta)P_n^{(\lambda,\mu)}(\cos 4\phi)$ by an integral operator against $C_{4n}^{(2\lambda+2\mu)}$ whose argument is linear in $\cos\theta$, $\sin\theta$, $\cos\phi$, and $\sin\phi$. More specifically, we have the following formula.

Theorem 1. *Let* $\mathbf{x} = (\cos\theta, \sin\theta)$ *and* $\mathbf{y} = (\cos\phi, \sin\phi)$. *If* $\lambda + \mu \geq 1$, *then for all* $n \geq 0$

$$P_n^{(\lambda-1/2,\mu-1/2)}(\cos 4\theta)P_n^{(\lambda-1/2,\mu-1/2)}(\cos 4\phi)$$
$$= A_n^{(\lambda,\mu)}(\lambda+\mu-1)\pi^{-1}\int_{B^3}(1-\|\mathbf{u}\|^2)^{\lambda+\mu-2}d\mathbf{u}$$
$$\times\Big\{\int_{-1}^{1}\int_{-1}^{t} C_{4n}^{(2\lambda+2\mu)}\Big(\mathbf{x}^T\begin{bmatrix}\sqrt{\frac{1+s}{2}}u_2+\sqrt{\frac{t-s}{2}}u_3 & \sqrt{\frac{1-t}{2}}u_1\\ -\sqrt{\frac{1-t}{2}}u_1 & \sqrt{\frac{1+s}{2}}u_2+\sqrt{\frac{t-s}{2}}u_3\end{bmatrix}\mathbf{y}\Big)$$
$$\times\Psi(t,s)dsdt$$
$$+\int_{-1}^{1}\int_{t}^{1} C_{4n}^{(2\lambda+2\mu)}\Big(\mathbf{x}^T\begin{bmatrix}\sqrt{\frac{1+t}{2}}u_1 & \sqrt{\frac{1-s}{2}}u_2+\sqrt{\frac{s-t}{2}}u_3\\ -\sqrt{\frac{1-s}{2}}u_2+\sqrt{\frac{s-t}{2}}u_3 & \sqrt{\frac{1+t}{2}}u_1\end{bmatrix}\mathbf{y}\Big)$$
$$\times\Psi(t,s)dsdt\Big\}, \quad (2)$$

where $\Psi(t,s) = (1-t^2)^{\mu-1}(1-s^2)^{\lambda-1}$ *and* B^3 *is the unit ball of* $\mathbb{R}^3$, $\mathbf{u} = (u_1,u_2,u_3) \in B^3$.

Proof. From (1) we immediately have

$$P_n^{(\lambda-1/2,\mu-1/2)}(\cos 4\theta)P_n^{(\lambda-1/2,\mu-1/2)}(\cos 4\phi) = A_n^{(\lambda,\mu)}\times$$

$$\int_{-1}^{1}\int_{-1}^{1} C_{2n}^{(\lambda+\mu)}(t\cos 2\theta\cos 2\phi + s\sin 2\theta\sin 2\phi)(1-t^2)^{\lambda-1}(1-s^2)^{\mu-1}dsdt. \quad (3)$$

In order to have the desired form in the right hand side, we use the following two formulae. The first one is a formula on Gegenbauer polynomials,

$$C_n^{(\lambda)}(z) = c_\lambda \int_{-1}^{1} C_{2n}^{(2\lambda)}\left(\sqrt{\tfrac{1+z}{2}}u\right)(1-u^2)^{\lambda-1}du, \qquad (4)$$

where $c_\lambda^{-1} = \int_{-1}^{1}(1-s^2)^{\lambda-1}ds$, which follows from setting $\phi = 0$ and $\lambda = \mu$ in (1), then using the fact that $C_n^{(\lambda)}(s) = \text{const}\ P_n^{(\lambda-1/2,\lambda-1/2)}(s)$. The constant is determined by the fact that $C_n^{(\lambda)}(-1) = C_{2n}^{(2\lambda)}(0)$. If we use the formula (4) to replace $C_{2n}^{(\lambda+\mu)}$ in (3), we will have the correct Gegenbauer polynomial that appears in the right hand side of (2),

$$P_n^{(\lambda-1/2,\mu-1/2)}(\cos 4\theta)P_n^{(\lambda-1/2,\mu-1/2)}(\cos 4\phi) = A_n^{(\lambda,\mu)}c_{\lambda+\mu}\int_{-1}^{1}\int_{-1}^{1}$$
$$\left[\int_{-1}^{1} C_{4n}^{(2\lambda+2\mu)}\left(\sqrt{\tfrac{1+z}{2}}u\right)(1-u^2)^{\lambda+\mu-1}du\right](1-t^2)^{\lambda-1}(1-s^2)^{\mu-1}dsdt, \quad (5)$$

where $z = t\cos 2\theta\cos 2\phi + s\sin 2\theta\sin 2\phi$. For $-1 \le t, s \le 1$ we can write $1+z$ as a sum of positive terms as follows,

$$1+z = (1-t)\sin^2(\theta-\phi) + (1+s)\cos^2(\theta-\phi) + (t-s)\cos^2(\theta+\phi), \quad t > s$$

$$1+z = (1-s)\sin^2(\theta-\phi) + (1+t)\cos^2(\theta-\phi) + (s-t)\sin^2(\theta+\phi), \quad t < s,$$

which allows us to identify $\sqrt{(1+z)/2}$ as the norm of a vector in $\mathbb{R}^3$. More precisely, we have

$$\frac{1+z}{2} = \left\|\left(\sqrt{\tfrac{1-t}{2}}\sin(\theta-\phi), \sqrt{\tfrac{1+s}{2}}\cos(\theta-\phi), \sqrt{\tfrac{t-s}{2}}\cos(\theta+\phi)\right)\right\|^2, \quad t > s$$
$$\frac{1+z}{2} = \left\|\left(\sqrt{\tfrac{1-s}{2}}\sin(\theta-\phi), \sqrt{\tfrac{1+t}{2}}\cos(\theta-\phi), \tfrac{s-t}{2}\sin(\theta+\phi)\right)\right\|^2, \quad t < s,$$

where $\|\cdot\|$ denote the Euclidean norm of $\mathbb{R}^3$. We now need our second formula, which holds for $g : \mathbb{R} \mapsto \mathbb{R}$,

$$\int_{-1}^{1} g(s\|\mathbf{x}\|)(1-s^2)^\gamma ds = \frac{\gamma}{\pi}\int_{B^3} g(\mathbf{x}\cdot\mathbf{u})(1-\|\mathbf{u}\|^2)^{\gamma-1}d\mathbf{u}, \quad \gamma > 0, \qquad (6)$$

where $\mathbf{x}\cdot\mathbf{u}$ is the inner product of $\mathbf{x},\mathbf{u}\in\mathbb{R}^3$. This formula is proved by using the polar coordinates $\mathbf{u}=r\mathbf{u}'$, $\mathbf{u}'\in S^2$, and changing variables several times as follows,

$$\begin{aligned}\int_{B^3} g(\mathbf{x}\cdot\mathbf{u})(1-\|\mathbf{u}\|^2)^{\gamma-1}d\mathbf{u} &= \int_0^1 r^2\int_{S^2} g(r\mathbf{x}\cdot\mathbf{u}')d\omega(1-r^2)^{\gamma-1}dr\\ &= 2\pi\int_0^1 r^2\int_{-1}^1 g(r\|\mathbf{x}\|t)dt(1-r^2)^{\gamma-1}dr\\ &= 2\pi\int_{-1}^1 g(\|\mathbf{x}\|s)ds\int_{\|s\|}^1 r(1-r^2)^{\gamma-1}drdt\\ &= \frac{\pi}{\gamma}\int_{-1}^1 g(\|\mathbf{x}\|s)(1-s^2)^{\gamma}ds.\end{aligned}$$

Substituting (6) into (5) and then using (5) with $\mathbf{x}=(1+z)/2$ and $g=C_{4n}^{(2\lambda+2\mu)}$, we conclude that

$$\begin{aligned}&P_n^{(\lambda-1/2,\mu-1/2)}(\cos 4\theta)P_n^{(\lambda-1/2,\mu-1/2)}(\cos 4\phi) = A_n^{(\lambda,\mu)}\frac{\lambda+\mu-1}{\pi}\\ &\times\Big\{\int_{-1}^1\int_{-1}^t\int_{B^3} C_{4n}^{(2\lambda+2\mu)}\Big(\sqrt{\tfrac{1-t}{2}}\sin(\theta-\phi)u_1+\sqrt{\tfrac{1+s}{2}}\cos(\theta-\phi)u_2\\ &\qquad+\sqrt{\tfrac{t-s}{2}}\cos(\theta+\phi)u_3\Big)(1-\|\mathbf{u}\|^2)^{\lambda+\mu-2}d\mathbf{u}\Psi(t,s)dsdt\\ &+\int_{-1}^1\int_t^1\int_{B^3} C_{4n}^{(2\lambda+2\mu)}\Big(\sqrt{\tfrac{1+t}{2}}\cos(\theta-\phi)u_1+\sqrt{\tfrac{1-s}{2}}\sin(\theta-\phi)u_2\\ &\qquad+\sqrt{\tfrac{s-t}{2}}\sin(\theta+\phi)u_3\Big)(1-\|\mathbf{u}\|^2)^{\lambda+\mu-2}d\mathbf{u}\Psi(t,s)dsdt,\Big\}\end{aligned}$$

which gives the desired form.

The formula looks rather complicated, we make no claim that this is the final form. In fact, it is our hope that an easier formula in a form that may be more suitable for finding the integral formula of the intertwining operator can be found. The relation to the intertwining operator is explained in the following section.

3 Intertwining operator for the h-harmonics

We recall a few facts from Dunkl's theory of h-harmonics (cf. [3,4,5] and the references there), and we restrict the discussion to the the weight function

$$h_{\lambda,\mu}(\mathbf{x}) = |2x_1x_2|^{\lambda}(x_1^2-x_2^2)^{\mu}, \qquad \mathbf{x}=(x_1,x_2),$$

associated with the dihedral group D_4, although most of the statements given below hold for general reflection groups. Let $\mathcal{D}_i$, $i = 1, 2$, denote Dunkl's differential-difference operators associated with $h_{\lambda,\mu}$,

$$\mathcal{D}_1 f(\mathbf{x}) = \partial_1 f(\mathbf{x}) + \lambda \frac{f(\mathbf{x}) - f(\mathbf{x}\sigma_1)}{x_1} + \mu \Big[\frac{f(\mathbf{x}) - f(\mathbf{x}\tau_1)}{x_1 - x_2} + \frac{f(\mathbf{x}) - f(\mathbf{x}\tau_2)}{x_1 + x_2} \Big]$$
$$\mathcal{D}_2 f(\mathbf{x}) = \partial_2 f(\mathbf{x}) + \lambda \frac{f(\mathbf{x}) - f(\mathbf{x}\sigma_2)}{x_2} + \mu \Big[\frac{f(\mathbf{x}) - f(\mathbf{x}\tau_1)}{x_1 - x_2} + \frac{f(\mathbf{x}) - f(\mathbf{x}\tau_2)}{x_1 + x_2} \Big],$$

where ∂_i denote the ordinary partial derivatives, $\mathbf{x}\sigma_1 = (-x_1, x_2)$, $\mathbf{x}\sigma_2 = (x_1, -x_2)$, $\mathbf{x}\tau_1 = (x_2, x_1)$ and $\mathbf{x}\tau_2 = (-x_2, -x_1)$. It is a remarkable fact that Dunkl's operators commute. The h-harmonics are homogeneous polynomials P of degree n that satisfy the equation

$$\Delta_h P = (\mathcal{D}_1^2 + \mathcal{D}_2^2) P = 0,$$

and they are orthogonal with respect to $h^2_{\lambda,\mu} d\omega$ on S^1. Let $\mathcal{H}_n^h$ be the space of h-harmonic polynomials of degree n. An orthonormal basis for $\mathcal{H}_{2n}^h$ is given by

$$Y_{2n}^{(1,h)}(\mathbf{x}) = r^{2n} D_n^{(\lambda,\mu)}(\cos 2\theta),$$
$$Y_{2n}^{(2,h)}(\mathbf{x}) = r^{2n} \Big(\frac{\lambda + \mu + 1}{\lambda + 1/2} \Big)^{1/2} \sin 2\theta D_{n-1}^{(\lambda+1,\mu)}(\cos 2\theta),$$

where we use the polar coordinates $x_1 = r\cos\theta$ and $x_2 = r\sin\theta$, and the polynomial $D_n^{(\lambda,\mu)}$ is the orthonormal polynomial of degree n with respect to the weight function

$$w^{(\lambda,\mu)}(x) = w_{\lambda,\mu}(1 - x^2)^{\lambda - \frac{1}{2}} |x|^{2\mu}, \qquad -1 \le x \le 1, \qquad \lambda, \mu > -1/2,$$

where $w_{\lambda,\mu}$ is the normalization constant so that the integral of $w^{(\lambda,\mu)}$ on $[-1, 1]$ is 1. In terms of the classical Jacobi polynomials $P_n^{(\alpha,\beta)}$ we have ([6])

$$D_{2n}^{(\lambda,\mu)}(\cos\theta) = c_n(\lambda,\mu) P_n^{(\lambda-\frac{1}{2},\mu-\frac{1}{2})}(\cos 2\theta),$$
$$D_{2n+1}^{(\lambda,\mu)}(\cos\theta) = c_n(\lambda,\mu+1) \Big(\frac{\lambda+\mu+1}{\mu+1/2} \Big)^{1/2} x P_n^{(\lambda-\frac{1}{2},\mu+\frac{1}{2})}(\cos 2\theta),$$

where $c_n(\lambda,\mu)$ is a normalization constant,

$$c_n(\lambda,\mu) = \Big(\frac{\Gamma(\mu+\frac{1}{2})\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+\mu+1)} \Big)^{1/2} \Big(\frac{(2n+\lambda+\mu)\Gamma(n+\lambda+\mu)\Gamma(n+1)}{\Gamma(n+\mu+\frac{1}{2})\Gamma(n+\lambda+\frac{1}{2})} \Big)^{1/2}.$$

We note that if $\mu = 0$, then $D_n^{(\lambda,0)} = \text{const } C_n^{(\lambda)}$ is the orthonormal Gegenbauer polynomial.

The intertwining operator V acts between the commuting algebra generated by the partial differentials and that generated by Dunkl's operators, it allows us to transform certain properties of ordinary harmonics to h-harmonics. The operator V is a linear operator uniquely defined by ([5])

$$V\mathcal{P}_n \subset \mathcal{P}_n, \qquad V1 = 1, \qquad \mathcal{D}_i V = V\partial_i, \qquad i = 1, 2, \tag{7}$$

where $\mathcal{P}_n$ denotes the class of homogeneous polynomials of degree n. No closed formula of V is known at the moment. A general formula for the reproducing kernel P_n^h of $\mathcal{H}_n^h$ states that ([5,7])

$$\begin{aligned} P_n^h(\mathbf{x},\mathbf{y}) :&= Y_n^{(1,h)}(\mathbf{x})Y_n^{(1,h)}(\mathbf{y}) + Y_n^{(2,h)}(\mathbf{x})Y_n^{(2,h)}(\mathbf{y}) \\ &= \frac{n+2\lambda+2\mu}{2\lambda+2\mu}[VC_n^{(2\lambda+2\mu)}(\langle\cdot,\mathbf{y}\rangle)](\mathbf{x}), \qquad |\mathbf{y}| = |\mathbf{x}| = 1, \end{aligned} \tag{8}$$

where $\mathbf{x} = r(\cos\theta, \sin\theta)$ and $\mathbf{y} = r(\cos\phi, \sin\phi)$. In particular, by the definition of h-harmonics in this case, we have

$$\begin{aligned} P_{2n}^h(\mathbf{x},\mathbf{y}) = \ & D_n^{(\lambda,\mu)}(\cos 2\theta)D_n^{(\lambda,\mu)}(\cos 2\phi) \\ & + \frac{\lambda+\mu+1}{\lambda+\frac{1}{2}} \sin 2\theta \sin 2\phi D_{n-1}^{(\lambda+1,\mu)}(\cos 2\theta)D_{n-1}^{(\lambda+1,\mu)}(\cos 2\phi). \end{aligned}$$

It is this formula that suggests the product formula in Proposition 1. In fact, the following formula is proved in [6],

$$\begin{aligned} & D_n^{(\lambda,\mu)}(\cos\theta)D_n^{(\lambda,\mu)}(\cos\phi) \\ & \qquad + \frac{\lambda+\mu+1}{\lambda+\frac{1}{2}} \sin\theta \sin\phi D_{n-1}^{(\lambda+1,\mu)}(\cos\theta)D_{n-1}^{(\lambda+1,\mu)}(\cos\phi) \\ & = \frac{n+\lambda+\mu}{\lambda+\mu} c_\lambda c_\mu \int_{-1}^{1}\int_{-1}^{1} C_n^{(\lambda+\mu)}(\cos\theta\cos\phi\, t + \sin\theta\sin\phi\, s)\Phi(t,s)dsdt, \end{aligned}$$

where $\Phi(t,s) = (1+t)(1-t^2)^{\mu-1}(1+s)(1-s^2)^{\lambda-1}$, from which we can follow the proof of the Proposition 1 to conclude that

$$P_{2n}^h(\mathbf{x},\mathbf{y}) = c_\lambda c_\mu(\lambda+\mu-1)\pi^{-1}\int_{B^3}(1-\|\mathbf{u}\|^2)^{\lambda+\mu-2}d\mathbf{u}$$

$$\Big\{\int_{-1}^{1}\int_{-1}^{t} C_{2n}^{(2\lambda+2\mu)}\Big(\mathbf{x}^T \begin{bmatrix} \sqrt{\frac{1+s}{2}}u_2 + \sqrt{\frac{t-s}{2}}u_3 & \sqrt{\frac{1-t}{2}}u_1 \\ -\sqrt{\frac{1-t}{2}}u_1 & \sqrt{\frac{1+s}{2}}u_2 + \sqrt{\frac{t-s}{2}}u_3 \end{bmatrix} \mathbf{y}\Big)$$

$$\times \Phi(t,s) ds dt$$

$$+ \int_{-1}^{1}\int_{t}^{1} C_{2n}^{(2\lambda+2\mu)}\Big(\mathbf{x}^T \begin{bmatrix} \sqrt{\frac{1+t}{2}}u_1 & \sqrt{\frac{1-s}{2}}u_2 + \sqrt{\frac{s-t}{2}}u_3 \\ -\sqrt{\frac{1-s}{2}}u_2 + \sqrt{\frac{s-t}{2}}u_3 & \sqrt{\frac{1+t}{2}}u_1 \end{bmatrix} \mathbf{y}\Big) \times \Phi(t,s) ds dt\Big\}.$$

Since this formula holds for all n and $C_{2n}^{(\lambda)}$ is an even function, we conclude from (2) that

Theorem 2. *If $\lambda+\mu \geq 3/2$ is an half integer, then for all $n \geq 0$,*

$$[V(\langle\cdot,\mathbf{y}\rangle)^{2n}](\mathbf{x}) = c_\lambda c_\mu (\lambda+\mu-1)\pi^{-1}\int_{B^3}(1-\|\mathbf{u}\|^2)^{\lambda+\mu-2} d\mathbf{u}$$

$$\Big\{\int_{-1}^{1}\int_{-1}^{t}\Big(\mathbf{x}^T \begin{bmatrix} \sqrt{\frac{1+s}{2}}u_2 + \sqrt{\frac{t-s}{2}}u_3 & \sqrt{\frac{1-t}{2}}u_1 \\ -\sqrt{\frac{1-t}{2}}u_1 & \sqrt{\frac{1+s}{2}}u_2 + \sqrt{\frac{t-s}{2}}u_3 \end{bmatrix} \mathbf{y}\Big)^{2n} \Phi(t,s) ds dt$$

$$+ \int_{-1}^{1}\int_{t}^{1}\Big(\mathbf{x}^T \begin{bmatrix} \sqrt{\frac{1+t}{2}}u_1 & \sqrt{\frac{1-s}{2}}u_2 + \sqrt{\frac{s-t}{2}}u_3 \\ -\sqrt{\frac{1-s}{2}}u_2 + \sqrt{\frac{s-t}{2}}u_3 & \sqrt{\frac{1+t}{2}}u_1 \end{bmatrix} \mathbf{y}\Big)^{2n} \times \Phi(t,s) ds dt\Big\}.$$

This formula suggests a possible compact formula for V, but the obvious choice suggested by the above formula and (8) fails to verify the definition (7). In order to get a formula for V, we may need to know a formula for $[V(\langle\cdot,\mathbf{y}\rangle)^{2n+1}](\mathbf{x})$. However, if we follow the proof above, we will need a product formula for $Y_{2n+1}^{(i,h)}$, which can be difficult. For example,

$$Y_{2n+1}^{(1,h)}(\mathbf{x}) = a_n r^{2n+1}\cos\theta\big[D_n^{(\lambda,\mu)}(\cos 2\theta) - b_n \cos^2\theta D_{n-1}^{(\lambda+1,\mu)}(\cos 2\theta)\big],$$

where a_n and b_n are constants and their formulae are different for n being even and odd. The formula for $Y_{2n+1}^{(2,h)}$ takes a similar form. See [4] for explicit formulae given in terms of Jacobi polynomials.

In conclusion, the product formula (2) in Theorem 1 can be viewed as the restriction of the formula (7) on the h-harmonic polynomials invariant under the group. It should be pointed out that we consider it likely that the formula (2) is not the most convenient form. There does not seem to be a convincing reason why the integral form is a sum of two separate parts. Perhaps the integral should be six-fold, absorbing the two terms in the additional integral. Nevertheless, this seems to be the first attempt in finding such a formula. It is our hope that this note will help to draw the attention of experts to this problem; the product formula of the Jacobi polynomials is of independent interests regardless its potential application in the study of h-harmonics.

References

1. R. Askey, Orthogonal polynomials and special functions, SIAM, Philadelphia, 1975.
2. A. Dijksma and T. Koornwinder, Spherical harmonics and the product of two Jacobi polynomials, Indag. Math., **33** (1971), pp. 191-196.
3. C. Dunkl, Reflection groups and orthogonal polynomials on the sphere, *Math. Z.*, **197** (1988), 33-60.
4. C. Dunkl, Differential-difference operators associated to reflection groups, *Trans. Amer. Math. Soc.*, **311** (1989), 167-183.
5. C. Dunkl, Integral kernels with reflection group invariance, *Canadian J. Math.*, **43** (1991), 1213-1227.
6. Y. Xu, Orthogonal polynomials for a family of product weight functions on the spheres, *Canadian J. Math.*, **49** (1997), 175-192.
7. Y. Xu, Integration of the intertwining operator for h-harmonic polynomials associated to reflection groups, *Proc. Amer. Math. Soc.*, **125** (1997), 2963-2973.