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POLYNOMIAL INTERPOLATION IN SEVERAL VARIABLES, CUBATURE FORMULAE, AND IDEALS

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Abstract. We discuss polynomial interpolation in several variables from a polynomial ideal point of view. One of the result states that if I is a real polynomial ideal with real variety and if its codimension is equal to the cardinality of its variety, then for each monomial order there is a unique polynomial that interpolates on the points in the variety. The result is motivated by the problem of constructing cubature formulae, and it leads to a theorem on cubature formula which can be considered as an extension of Gaussian quadrature formulae to several variables.

1. Introduction

For functions of one variable, polynomial interpolation and quadrature formula are closely related. Integrating the Lagrange interpolating polynomial based on n distinct points on the real line leads to a quadrature formula of degree at least $n-1$; the interpolating points are the nodes of the quadrature formula. Moreover, Gaussian quadrature formulae can be obtained this way by taking the interpolating points as zeros of orthogonal polynomials. In fact, a Gaussian quadrature formula of degree $2n-1$ exists if and only if the nodes are the zeros of the orthogonal polynomial of degree n. The purpose of this paper is to discuss extensions of such results to several variables.

Let $\Pi^d = R[x_1, \ldots, x_d]$ be the space of polynomials in d real variables, and let Π_n^d be the space of polynomials of degree at most n. It is known that $\dim \Pi_n^d = \binom{n+d}{d}$. Comparing to one variable, polynomial interpolation in several variables is much more complicated. For example, if 6 points are lying on a circle in $\mathbb{R}^2$, then there is no unique interpolating polynomial of degree 2 in two variables, even though $\dim \Pi_2^2 = 6$. Let V be a set of distinct nodes in $\mathbb{R}^d$. One notion of Lagrange interpolation consists of finding a subspace $\mathcal{P} \in \Pi^d$, often depending on V, such that for each f defined at least on V, there is a unique polynomial P in $\mathcal{P}$ such that P agrees with f on V. Following [2], we call such a pair $(V, \mathcal{P})$ *correct*. Many authors have contributed to the study of polynomial interpolation in several variables. We refer to [1 - 4, 6, 8, 16] and the references therein. The above notion of interpolation was introduced in [2], where the definition works for

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more general situation in which the interpolation data are specified by linear maps, in contrast to point evaluations. We will restrict to the Lagrange interpolation in the present paper. We consider the situation that V is the affine variety of a polynomial ideal I, and will show that a unique interpolating polynomial exists if the codimension of I is equal to the cardinality of the variety. This consideration is motivated by the problem of constructing cubature formulae.

Let $\mathcal{L}$ be a square positive linear functional defined on the polynomial space Π^d, such as those given by $\mathcal{L}(f) = \int_{\mathbb{R}^d} f(\mathbf{x})W(\mathbf{x})d\mathbf{x}$, where W is a nonnegative weight function with finite moments. A cubature formula of degree s ($s = 2n-1$ or $s = 2n$) with respect to $\mathcal{L}$ is a linear functional

$$\mathcal{L}_n(f) := \sum_{k=1}^{N} \lambda_k f(\mathbf{x}_k) \ , \quad \mathbf{x}_k \in \mathbb{R}^d, \quad \lambda_k \in \mathbb{R}, \tag{1.1}$$

where the points $\mathbf{x}_1, \dots, \mathbf{x}_N$ are assumed to be distinct, such that $\mathcal{L}_n(f) = \mathcal{L}(f)$ for all $f \in \Pi^d_s$ and there exists at least one polynomial $f^* \in \Pi^d_{s+1}$ such that $\mathcal{L}_n(f^*) \neq \mathcal{L}(f)$. The points $\mathbf{x}_k$ are called *nodes* and λ_k are called *weights* of the cubature formula. We are interested in the case that the nodes of the formula (1.1) are common zeros of certain orthogonal polynomials of degree n with respect to $\mathcal{L}$. It is known that for a cubature formula of degree $2n-1$ to exist, it is necessary that $N \geq \dim \Pi^d_{n-1}$ (cf. [18]). The cubature formulae that attain this lower bound are called Gaussian cubature formulae. Fixing an order of monomials, we can use the Gram-Schmidt process to generate a sequence of orthonormal polynomials, denoted by $\{P^n_k\}$, with respect to the inner product defined by $\langle f, g\rangle = \mathcal{L}(fg)$, where the superscript n means that P^n_k is of degree n and the subindex k satisfies $1 \leq k \leq r^d_n$, where $r^d_n = \binom{n+d-1}{n}$ is the dimension of the subspace of homogeneous polynomials of degree n. It is proved by Mysovskikh [12] that a Gaussian cubature formula exists if and only if its $N = \dim \Pi^d_{n-1}$ nodes are common zeros of $\mathcal{V}^d_n$, or common zeros of $\mathbb{P}_n$, where we use the notation $\mathbb{P}_n = (P^n_1, \dots, P^n_{r^d_n})$, which we interpret as either a set or a column vector. However, Gaussian cubature formulae exist only rarely. In [9], Möller gave an improved lower bound which takes the form $N \geq \dim \Pi^2_{n-1} + [n/2]$ for $d = 2$ and $\mathcal{L}$ being centrally symmetric; moreover, he showed that the nodes of cubature formulae attain his improved lower bound are common zeros of a subset of orthogonal polynomials of degree n. In [19] we studied more general situation that subset contains also quasi-orthogonal polynomials, discussing essentially cubature formulae with minimal or near minimal nodes. We refer to [9 - 14, 19, 20] for results and detail discussions in this direction. Recently, in [21], we have studied cubature formulae from polynomial ideal and variety point of view; the main result shows that a cubature formula of degree $2n-1$ exists if its nodes form an affine variety of an ideal I generated by a basis of $(2n-1)$-orthogonal polynomials, and the codimension of the ideal is equal to the size of the real variety. The result can be applied to Gaussian cubature formulae and those mentioned in the above, it can also be applied to the usual product type cubature formulae. In the present paper we will prove this result using polynomial interpolation and polynomial ideal theory. This proof bears a close similarity to that of Gaussian quadrature formula in one variable.

The paper is organized as follows. In Section 2 we state and prove the main results on interpolation and cubature formulae, and we also include a couple of examples to illustrate our results. To avoid distraction from the main development, some further discussions are deferred to Section 3.

2. Interpolation and cubature formulae

We start with notation and definition of ideal and variety, see [5]. We write the monomials $\mathbf{x}^\alpha = x_1^{\alpha_1}\cdots x_d^{\alpha_d}$ for $\mathbf{x} \in \mathbb{R}^d$ and $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ and use the usual multiindex notation. The polynomial $\mathbf{x}^\alpha$ is of (total) degree $|\alpha| = \alpha_1 + \cdots + \alpha_d$. To order the monomials in $\Pi^d = \mathbb{R}[x_1, \dots, x_d]$, we use either graded lexicographic order or the graded reverse lexicographic order (see, for example, [5, Chapt. 2]).

Let I be a polynomial ideal in Π^d. The codimension of I is denoted by $\operatorname{codim} I$, that is,

$$\operatorname{codim} I = \dim \Pi^d / I.$$

If there are polynomials $f_1, \dots, f_M$ in Π^d such that every $f \in I$ can be written as

$$f = a_1 f_1 + \cdots + a_M f_M, \qquad a_j \in \Pi^d, \tag{2.1}$$

we say that I is generated by the basis $f_1, \dots, f_M$, and we write $I = \langle f_1, \dots, f_M\rangle$. According to the Hilbert basis theorem, every polynomial ideal has a finite basis. For a fixed monomial order, we denote by $LT(f)$ the leading monomial term for any polynomial $f \in \Pi^d$; that is, if $f = \sum c_\alpha \mathbf{x}^\alpha$, then $LT(f) = c_\beta \mathbf{x}^\beta$, where $\mathbf{x}^\beta$ is the leading monomial among all monomials appeared in f. For an ideal I in Π^d other than $\{0\}$, we denote by $LT(I)$ the leading terms of I, that is,

$$LT(I) = \{c\mathbf{x}^\alpha | \text{there exists } f \in I \text{ with } \quad LT(f) = c\mathbf{x}^\alpha\}.$$

We further denote the ideal of leading terms $\langle LT(I)\rangle$, which is the ideal generated by the leading terms of $LT(f)$ for all $f \in I \setminus \{0\}$. A set $\{g_1, \dots, g_t\}$ is called a Gröbner basis of I if

$$\langle LT(g_1), \dots, LT(g_t)\rangle = \langle LT(I)\rangle.$$

It is known that every polynomial ideal has a Gröbner basis. Such a basis enjoys many interesting properties that have important applications. For example, it is used to prove the following theorem which is crucial for our purpose (see [5, Chapt 5]).

Theorem 2.1. *Fix a monomial ordering on Π^d and let $I \in \Pi^d$ be an ideal. Then there is an isomorphism between Π^d/I and the space*

$$\mathcal{S}_I := \operatorname{span}\{\mathbf{x}^\alpha | \mathbf{x}^\alpha \notin \langle LT(I)\rangle\}.$$

More precisely, every $f \in \Pi^d$ is congruent modulo I to a unique polynomial $r \in \mathcal{S}_I$.

In fact, the polynomial r is the remainder of f on division by I. For $I = \langle f_1, \dots, f_M\rangle$ and a fixed monomial order, the division algorithm states that for every $f \in \Pi^d$, there exist p_i and r in Π^d such that $f = \sum p_i f_i + r$, where $r \in \mathcal{S}_I$ and no term of r is divisible by any of $LT(f_1), \dots, LT(f_M)$. Moreover, since we are using the graded monomial order, we have that the degree $p_i f_i$ is at most the degree f (in general, the multidegree of $p_i f_i$ is no greater than the multidegree of f). The remainder polynomial r is unique if the basis $f_1, \dots, f_M$ is a Gröbner basis.

For an ideal $I \in \Pi^d$, we denote by $V = V(I)$ its affine variety. We consider the case that V is a finite set of distinct points in $\mathbb{R}^d$. The cardinality of V is denoted by $|V|$, it is equal to the number of distinct elements in V. It is known that

$$|V| \le \operatorname{codim} I. \tag{2.2}$$

For a finite set V in $\mathbb{R}^d$, we also consider the ideal $I(V)$ associated to V, that is,

$$I(V) = \{P \in \Pi^d | P(\mathbf{x}) = 0, \quad \forall \mathbf{x} \in V\},$$

which is the largest ideal having V as its variety. For a given ideal I, we have $I \subset I(V(I))$ in general.

We are now ready to prove our first result on polynomial interpolation.

Theorem 2.2. *Let I be a polynomial ideal in Π^d and V be its affine variety. Assume that I has finite codimension and V is a subset of $\mathbb{R}^d$. If $|V| = \operatorname{codim} I$, then there is a unique interpolation polynomial in $\mathcal{S}_I$ based on the points in the variety.*

Proof. Using the isomorphism between Π^d/I and the space S_I, we show that there is a unique interpolation polynomial in S_I. Let $\Lambda_I = \{\alpha | \mathbf{x}^\alpha \in S_I\}$ so that we can write any polynomial P in $\mathcal{S}_I$ as $P(\mathbf{x}) = \sum_{\alpha \in \Lambda_I} c_\alpha \mathbf{x}^\alpha$. We define a column vector $\Phi(\mathbf{x}) = [\mathbf{x}^\alpha]_{\alpha \in \Lambda_I}$, where the elements are arranged according to our fixed monomial order. It is sufficient to prove that the matrix $[\Phi(\mathbf{x}_1)|\cdots|\Phi(\mathbf{x}_N)]$, $N = |V|$, is invertible, where $\mathbf{x}_1, \dots, \mathbf{x}_N$ are distinct points in V. Suppose it is not, and assume that the matrix has rank $r < N$. We may assume that the first $r-1$ columns of the matrix are linearly independent and there exist scalars $a_1, a_2, \dots, a_r$, not all zero, such that $\sum_{k=1}^r a_k \Phi(\mathbf{x}_k) = 0$. In terms of the components of the vector Φ, this shows that

$$\sum_{k=1}^r a_k \mathbf{x}_k^\alpha = 0, \qquad \alpha \in \Lambda_I.$$

Moreover, there are at least two a_i's that are not zero; we may assume that a_1 and a_r are not zero. On the other hand, we clearly have $\sum_{k=1}^r a_k P(\mathbf{x}_k) = 0$ for any $P \in I$. By the isomorphism, or the division algorithm, for any $\mathbf{x}^\alpha$ there exist $P \in I$ and $q \in \mathcal{S}_I$ such that $\mathbf{x}^\alpha = P + q$. Hence, it follows that $\sum_{k=1}^r a_k \mathbf{x}_k^\alpha = 0$ for all α. In particular, this shows that

$$\sum_{k=1}^r a_k x_{k,i} \mathbf{x}_k^\alpha = 0, \qquad \alpha \in \Lambda_I, \qquad 1 \le i \le d,$$

where we write $\mathbf{x}_k = (x_{k,1}, \dots, x_{k,d})$. Together, the two displayed formulae yield that

$$x_{r,i} \sum_{k=1}^r a_k \mathbf{x}_k^\alpha - \sum_{k=1}^r a_k x_{k,i} \mathbf{x}_k^\alpha = \sum_{k=1}^{r-1} a_k (x_{k,i} - x_{r,i}) \mathbf{x}_k^\alpha = 0,$$

for any $\alpha \in \Lambda_I$ and $1 \le i \le d$. That is, we have $\sum_{k=1}^{r-1} a_k(x_{k,i} - x_{r,i})\Phi(\mathbf{x}_k) = 0$. Since $\Phi(\mathbf{x}_1), \ldots, \Phi(\mathbf{x}_{r-1})$ are linearly independent, we conclude that $a_k(x_{k,i} - x_{r,i}) = 0$ for $1 \le k \le r-1$ and $1 \le i \le d$. Since $a_1 \ne 0$, it follows that $x_{1,i} = x_{r,i}$ for $1 \le i \le d$, that is, $\mathbf{x}_1 = \mathbf{x}_d$, which contradicts the fact that $\mathbf{x}_k$ are distinct. □

In other words, the theorem states that $(V, \mathcal{S}_I)$ is correct. The uniqueness of the interpolating polynomial is relative to $\mathcal{S}_I$. The space $\mathcal{S}_I$ depends on the monomial ordering. Different orderings will lead to different $\mathcal{S}_I$ and to different interpolating polynomials. We emphasis that we are dealing with *real* polynomial ring $\Pi^d = \mathbb{R}[x_1, \ldots, x_d]$ with *real* variety which makes the problem harder. If we work with the complex case, $\mathbb{C}[x_1, \ldots, x_d]$, then the condition $\operatorname{codim} I = |V|$ holds as long as all points in V are simple, or that I is a radical ideal.

In the usual problem of polynomial interpolation, we are given a set V of distinct points in $\mathbb{R}^d$ which are the interpolation nodes, and look for a subspace of polynomials $\mathcal{P}$ so that the $(V, \mathcal{P})$ is correct. The following theorem shows that if we define $I = I(V)$ in such a case, then the usual interpolation problem is solved by Theorem 2.2 as well.

Theorem 2.3. *Let I be a polynomial ideal with finite codimension and let $V = V(I)$ be its affine variety. Then* $\operatorname{codim} I = |V|$ *if and only if $I = I(V)$.*

Proof. It is known that $V = V(I(V))$ ([5, p. 175]), and $\operatorname{codim} I \ge \operatorname{codim} I(V)$ since $I \subset I(V)$. If $\operatorname{codim} I = |V|$, then by (2.2) we have the opposite inequality

$$\operatorname{codim} I = |V| = |V(I(V))| \le \operatorname{codim} I(V),$$

which implies that $\operatorname{codim} I = \operatorname{codim} I(V)$. Hence, $I = I(V)$. On the other hand, if $I = I(V)$, then in order to prove that $\operatorname{codim} I = |V|$, it is sufficient to show that $\operatorname{codim} I(V) = |V|$. The proof of this fact is rather standard, we give a version here since the proof itself is useful in our discussion below. Let $\mathbf{x}_1, \ldots, \mathbf{x}_N$ be the distinct points in V, $N = |V|$. For each $\mathbf{x}_i$ we construct a polynomial f_i such that $f_i(\mathbf{x}_i) = 1$ and $f(\mathbf{x}_j) = 0$ for $j \ne i$. Such a polynomial can be constructed in many different ways if we do not restrict the degree of polynomials; for example, it can be obtained by multiplying linear polynomials that passes through $\mathbf{x}_j$ but not $\mathbf{x}_i$ and then normalizing. We identify f_i with its equivalent class $[f_i] \in \Pi^d/I$. Since $f_1, \ldots, f_N$ are evidently linearly independent, we have $|V| = N \le \dim \Pi^d/I = \operatorname{codim} I$. To prove the equality holds, it suffices to show that $f_1, \ldots, f_N$ spans Π^d/I. Let $g \in \Pi^d/I$. Consider $h := g - \sum_{i=1}^N g(\mathbf{x}_i) f_i$. By the definition of f_i, we have $h(\mathbf{x}_i) = 0$, $\mathbf{x}_i \in V$, which implies that $h \in I = I(V)$. Hence, it follows that $g = \sum_{i=1}^N g(\mathbf{x}_i) f_i$ congruent modulo I, which shows that $f_1, \ldots, f_N$ spans Π^d/I. □

Corollary 2.4. *Let $V = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ be a set of distinct points in $\mathbb{R}^d$. Let $I = I(V)$. Then the interpolation problem $(V, \mathcal{S}_I)$ is correct.*

We note that this result is more or less known. Indeed, there are several approaches of constructing interpolating polynomials, each leads to a polynomial subspace $\mathcal{P}$ for which $(V, \mathcal{P})$ is correct. The problem has been extensively studied by de Boor and Ron (see [2-4]); the space $\mathcal{P}$ they constructed enjoys many interesting properties. They also provide an algorithm for constructing interpolating polynomials. Another algorithm is given in Sauer [16], and he also studied the connection between interpolating polynomials and the Gröbner bases ([17]).

Let us now apply this result to the study of cubature formulae. Let $\mathcal{L}$ be a square positive linear functional and let $\mathcal{L}_n$ be a cubature formula of degree s, $s = 2n-1$ or $s = 2n$, as in (1.1). We start with a result due to Sobolev, see [13].

Theorem 2.5. *The points in $V = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$ are the nodes of a cubature formula of degree s if, and only if, $\mathcal{L}(f) = 0$ for each $f \in I(V) \cap \Pi_s^d$.*

Proof. If the points in V are the nodes of a cubature formula, denoted by $\mathcal{L}_n(f)$, of degree s, then $\mathcal{L}(f) = \mathcal{L}_n(f)$ for all $f \in \Pi_s^d$. Since $\mathcal{L}_n(f) = 0$ for all $f \in I(V)$, we conclude that $\mathcal{L}(f) = 0$ for all $f \in I(V) \cap \Pi_s^d$.

On the other hand, assume $\mathcal{L}(f) = 0$ for each $f \in I(V) \cap \Pi_s^d$. Let $I = I(V)$. By Corollary 2.4, there is a unique interpolation polynomial P_f in $\mathcal{S}_I$. Since P_f is a linear operator, we can write it as

$$P_f(\mathbf{x}) = \sum_{i=1}^{N} f(\mathbf{x}_i)\ell_i(\mathbf{x}), \qquad \ell_i \in \mathcal{S}_I.$$

Applying $\mathcal{L}$ to P_f we get a cubature formula $\mathcal{L}_n(f) = \sum \Lambda_k f(\mathbf{x}_k)$, which is exact for all polynomials in $\mathcal{S}_I$. For each $f \in I(V) \cap \Pi_s^d$ we have by assumption that $\mathcal{L}(f) = 0 = \mathcal{L}_n(f)$. Let $f \in \Pi_s^d$ and let r be the remainder of f divided by I as in Theorem 2.1. That is, $f = \sum a_i f_i + r$, where $f_1, \ldots, f_M$ is a basis of I and $a_i \in \Pi^d$, $r \in \mathcal{S}_I$; moreover, the degree of $a_i f_i$ is at most the degree of f by the division algorithm and the graded order that we choose, that is, $a_i f_i \in I(V) \cap \Pi_s^d$. Hence, we have that

$$\mathcal{L}(f) = \sum_i \mathcal{L}(a_i p_i) + \mathcal{L}(r) = 0 + \mathcal{L}_n(r) = \sum_i \mathcal{L}_n(a_i p_i) + \mathcal{L}_n(r) = \mathcal{L}_n(f);$$

for every $f \in \Pi_s^d$, which completes the proof. $\square$

As in the case of one variable, if we choose V as common zeros of a subset of orthogonal polynomials, then we may end up with a cubature formula of degree $2n-1$. To be more precise, we need one more definition. As in the introduction, we denote by $\{P_k^n\}$ a sequence of orthonormal polynomials with respect to the inner product defined by $\langle f, g\rangle = \mathcal{L}(fg)$, where $1 \le k \le r_n^d$, the index n indicates that P_k^n is of degree n. A polynomial P is called *m-orthogonal* if $\mathcal{L}(Pg) = 0$ for all $g \in \Pi^d$ such that $Pg \in \Pi_m^d$. The notion of m-orthogonal was introduced by Möller ([9]). An m-orthogonal polynomial P of degree s is the same as a quasi-orthogonal polynomial of degree s and order $2s - m - 1$, which can be written in terms of orthogonal polynomials $\{P_k^n\}$ as $P = \mathbf{a}_s^T \mathbb{P}_s + \ldots + \mathbf{a}_{m-s+1}^T \mathbb{P}_{m-s+1}$, where $\mathbb{P}_n = (P_1^n, \ldots, P_{r_n^d}^n)$ is a column vector and $\mathbf{a}_n$ are scalar vectors. (Since the elements of $\mathbb{P}_n$ contains a basis for all polynomials of degree n that are orthogonal to polynomials in Π_{n-1}^d, each orthogonal polynomial can be written as $\mathbf{a}^T \mathbb{P}_n$ for some $\mathbf{a}$.) For example, each orthogonal polynomial $\mathbf{a}^T \mathbb{P}_n$ is $(2n-1)$-orthogonal. Other $(2n-1)$-orthogonal polynomials are quasi-orthogonal polynomials of degree $n+1$

and order 2, degree $n+2$ and order 4, etc. (see [19]). The orthogonality shows that the multiplication by homogeneous linear polynomials acts as a mapping that takes an s-orthogonal polynomial of degree n to an s-orthogonal polynomial of degree $n+1$.

Let I be an ideal generated by a $(2n-1)$-orthogonal basis $Q=\{q_1,\dots,q_r\}$. We define a sequence of linear spaces J_m, $m\ge n$, by

$$J_n=\operatorname{span}\{Q\cap\Pi^d_n, 1\le i\le d\},\qquad J_m=\operatorname{span}\{Q\cap\Pi^d_m, x_iJ_{m-1}, 1\le i\le d\},\quad m\ge n.$$

Then we define a linear space $\tilde I$ associated to I by

$$\tilde I=\operatorname{span}\{J_m, m\ge n\}.$$

If a cubature formula of degree $2n-1$ exists, then its nodes form a variety of an ideal generated by $(2n-1)$-orthogonal polynomials; moreover, we showed in [21] that it is necessary that $I=\tilde I$, provided that $\Pi^d/I=\Pi^d_{2n-1}/(\Pi^d_{2n-1}\cap I)$. We now state our result on cubature formulae of degree $2n-1$.

Theorem 2.6. *Let $\mathcal{L}$ be a square positive linear functional. Let I be an ideal generated by a set of $(2n-1)$-orthogonal polynomials. Assume that I has finite codimension and the variety V of I is a subset of $\mathbb{R}^d$. If $\operatorname{codim}\tilde I=|V|$, then there is a cubature formula of degree $2n-1$.*

Proof. It is evident that J is a subset of I, so that $\operatorname{codim} I\le\operatorname{codim}\tilde I$. By the assumption $\operatorname{codim}\tilde I=|V|$ and the fact that $|V|\le\operatorname{codim} I$, it follows that $\operatorname{codim}\tilde I=|V|\le\operatorname{codim} I\le\operatorname{codim}\tilde I$. Consequently, we have $\operatorname{codim} I=|V|$ and that $I=\tilde I$. The proof then follows the same line as in the previous theorem. We again have that $\mathcal{L}_n(f)$ comes from applying $\mathcal{L}$ on the interpolating polynomial, whose existence is shown in Theorem 2.1, and we know that $\mathcal{L}_n(f)=\mathcal{L}(f)$ for $f\in\mathcal{S}_I$ by the uniqueness of the interpolating polynomial. Write $I=\langle f_1,\dots,f_M\rangle$, where $\{f_1,\dots,f_M\}$ is the $(2n-1)$-orthogonal basis. By Theorem 2.4 and 2.5, we only need to show that $\mathcal{L}_n(f)=0$ for $f\in I\cap\Pi^d_{2n-1}$. Since $I=\tilde I$, it follows from the definition of $\tilde I$ that $f\in I\cap\Pi^d_{2n-1}$ can be written as $f=\sum_i a_if_i$, where $a_i\in\Pi^d$, and the degree of a_ip_i is no greater than the degree of f. Since $a_if_i\in\Pi^d_{2n-1}$, the $(2n-1)$-orthogonality of the basis implies that $\mathcal{L}(a_if_i)=0$ for each i, which proves the theorem. $\square$

This theorem was stated in [21] in a somewhat different form with a somewhat more complicated proof. The main difference between the proofs, however, lies in the fact that a basis of quasi-orthogonal polynomials is used in both the interpolating part and in the cubature formula part in [21].

In the theorem, the ideal I is given and $V=V(I)$ is the affine variety of I. If $V=\{\mathbf{x}_1,\dots,\mathbf{x}_N\}$ is given and $I=I(V)$, then such a theorem has been proved by Möller, see [11], under the assumption that the $(2n-1)$-basis of I is also an H-basis. A basis $\{f_1,\dots,f_M\}$ of an ideal I is called an H-basis, if the expansion (2.1) of $f\in I$ satisfies the property that the degree of a_jf_j is less than or equal to the degree of f for every $f\in I$. In particular, a Gröbner basis is an H-basis. In general, it is not easy to verify that a given basis is an H-basis. In our theorem, our condition implies that $I=\tilde I$, which shows that the $(2n-1)$ basis in the theorem is indeed an H-basis. The condition $\operatorname{codim} I = |V|$ is crucial in our theorem. In the complex field, this condition holds if V contains only simple zeros. In Example 2.2 below, we will see a situation where two polynomials have a large amount of common zeros, but not enough to match the $\operatorname{codim} \tilde{I}$; our theorem suggests that we add more polynomial to enlarge I. The statement of Theorem 2.6 can be compared with another theorem due to Möller ([9], see [13]), which states that if I has an s-orthogonal H-basis, then it generates a *generalized cubature formula* of degree s, where a generalized cubature formula has certain linear functionals in place of the usual point evaluations.

In the simplest case, the ideal I is generated by a set of orthogonal polynomials. Instead of writing individual orthogonal polynomials, we shall write the set as $U^T\mathbb{P}_n$, where U^T is a matrix of size $r_n^d \times \tau$ which has full rank. We take $U^T\mathbb{P}_n$ either as a set or as a column vector. Since the vectors $\mathbb{P}_n$ satisfy a three-term relation

$$x_i\mathbb{P}_n = A_{n,i}\mathbb{P}_{n+1} + B_{n,i}\mathbb{P}_n + A_{n-1,i}\mathbb{P}_{n-1}, \quad 1 \le i \le d,$$

where $A_{n,i}$ and $B_{n,i}$ are proper matrices, the dimension of J_m can be computed as rank of a proper matrix. Indeed, it is not difficult to see that $\dim J_n = \operatorname{rank} U$ and for $m > n$

$$\dim J_m = \operatorname{rank} U_m, \qquad U_m = ((A_{m,1}U_{m-1})^T, \ldots, (A_{m,d}U_{m-d})^T)^T.$$

The $\operatorname{codim} \tilde{I}$ is equal to $\sum (r_m^d - \dim J_m)$, where the sum is over m from n to the first integer p such that $\dim J_p = r_p^d$.

As an application of the above theorem, let us consider the case when U is an identity matrix, that is, $U^T\mathbb{P}_n = \mathbb{P}_n$. In this case, $m = 1$ and Theorem 2.6 shows that if $\mathbb{P}_n$ has $|V| = \dim \Pi_{n-1}^d$ real common zeros, then there is a cubature formula. This is the result of Mysovskikh [12] on Gaussian cubature formula, the condition is in fact both sufficient and necessary. It is an extreme case of Theorem 2.6 since the basis contains maximal number of orthogonal polynomials. Another extreme case is when $U^T\mathbb{P}_n$ contains only d polynomials; for $d = 2$, it means that $U^T\mathbb{P}_n$ contains only 2 polynomials, which is as few as it can possibly be while still having enough distinct common zeros. It turns out that such cases cover the usual product type cubature formulae. We treat one such case as the following example.

Example 2.1. (Product cubature formulae). Let us consider the product weight function $W(\mathbf{x}) = w_1(x_1)\cdots w_d(x_d)$, where each w_i is a weight function defined on $[-1, 1]$. Let $p_n(w_i; \cdot)$ denote the orthonormal polynomial of degree n associated with w_i. It is known that a basis of orthonormal polynomials with respect to W consists of

$$P_{\mathbf{k}}(\mathbf{x}) = p_{k_1}(x_1)\cdots p_{k_d}(x_d), \qquad \mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d.$$

In particular, the polynomials $P_{n\mathbf{e}_i}(\mathbf{x}) = p_n(w_i, x_i)$ are orthonormal polynomials of degree n with respect to W, where $\mathbf{e}_i$ form the standard Euclidean basis of $\mathbb{R}^d$. We consider the case that $U^T\mathbb{P}_n = (P_{n\mathbf{e}_1}^n, \ldots, P_{n\mathbf{e}_d}^n)$. Let $x_{1,i}, \ldots, x_{d,i}$ be the zeros of $p_n(w_i)$. Then, the common zeros of $U^T\mathbb{P}_n$ are $(x_{i_1,1}, \ldots, x_{i_d,n})$ for $1 \le i_1, \ldots, i_d \le n$. Hence, we have $|V| = n^d$. On the other hand, since $P_{n\mathbf{e}_i}(\mathbf{x}) = \gamma_{n,i}x_i^n + \ldots$, it follows that

$$\dim J_{n+j} = \#\{\alpha \in \mathbb{N}^d : \exists \alpha_i \ge n, \text{ and } |\alpha| = n + j\},$$

from which and the fact that $r_n^d = \#\{\alpha \in \mathbb{N}^d : |\alpha| = n+j\}$ we have that

$$r_{n+j}^d - \dim J_{n+j} = \#\{\alpha \in \mathbb{N}^d : 0 \le \alpha_i \le n-1,\ 1 \le i \le d,\ |\alpha| = n+j\}.$$

Hence, it follow that $\operatorname{codim} \tilde{I} = \#\{\alpha \in \mathbb{N}^d : 0 \le \alpha_i \le n-1,\ 1 \le i \le d\} = n^d$. By Theorem 2.6, there is a cubature formula based on the common zeros of $U^T \mathbb{P}_n$, that is, on $(x_{i_1,1}, \ldots, x_{i_d,d})$ for $1 \le i_1, \ldots, i_d \le n$. This formula is the standard product formula given by

$$\int_{[-1,1]^d} f(\mathbf{x}) W(\mathbf{x}) d\mathbf{x} = \sum_{i_1=1}^{n} \cdots \sum_{i_d=1}^{n} \lambda_{i_1,1} \cdots \lambda_{i_d,d} f(x_{i_1,1}, \ldots, x_{i_d,d}),$$

where $\lambda_{1,i}, \ldots \lambda_{n,i}$ are the weights of Gaussian quadrature formula associated with the weight function w_i. $\square$

The example shows that Theorem 2.6 applies to many existing examples, ranging from Gaussian cubature formulae in one end, which has the smallest number of nodes among all possible cubature formulae, to product type formulae in the other. We refer to [21] for further discussion and additional examples. In particular, it is shown there that Theorem 2.6 can also be applied to product type cubature formula on the balls and on the simplices, and one example given in [14] deals with cubature formulae of degree 9 on the unit disk for which the corresponding ideals have more than 2 generators. The following example explains the importance of the condition $|V| = \operatorname{codim} \tilde{I}$.

Example 2.2. . We consider a cubature formula of degree 9 for the Lebesgue measure (unit weight function) on the unit disk of $\mathbb{R}^2$. This formula was discovered by Mysovskikh (see [14, formula 29 for the ball]). It uses 19 nodes which was shown to be the common zeros of two orthogonal polynomials in [14]. We give a basis of orthonormal polynomials with respect to the Lebegue measure as

$$P_k^n(x,y) = A_k^n C_{n-k}^{(k+1)}(x)(1-x^2)^{k/2} C_k^{(1/2)}(y(1-x^2)^{-1/2}), \qquad 0 \le k \le n, \tag{2.3}$$

where $C_k^{(\lambda)}$ denotes the Gegenbauer polynomial of degree k, orthogonal with respect to $(1-t^2)^{\lambda-1/2}$ on $[-1,1]$, and A_k^n are normalization constants (cf. [7]). Then the two polynomials given in [14] can be written as

$$\begin{aligned} Q_1^5(x,y) =& 3P_1^5(x,y) + 2\sqrt{14} P_3^5(x,y) + 5\sqrt{10} P_5^5(x,y), \\ Q_2^5(x,y) =& \frac{4+\sqrt{6}}{60} P_0^5(x,y) - \frac{11\sqrt{14}+13\sqrt{21}}{840} P_2^5(x,y) + \frac{-1+\sqrt{6}}{10\sqrt{42}} P_4^5(x,y). \end{aligned}$$

It is not difficult to verify that in this case the codimension of $\tilde{I}$ for $I = \langle Q_1^5, Q_2^5 \rangle$ is 25, so that $|V| = \operatorname{codim} \tilde{I}$ is not satisfied. Our theorem suggests that there may be other orthogonal polynomials vanishing on V. With the help of *Mathematica*, it is not difficult to find all 19 common zeros of Q_1^5 and Q_2^5 (they are also given in [14]). We then find out one more orthogonal polynomial of degree 5 vanishing on all 19 nodes, it is given by

$$Q_3^5(x,y) = \sqrt{\frac{7}{6}} P_0^5(x,y) + \frac{2}{\sqrt{3}} P_2^5(x,y) + P_4^5(x,y).$$

However, it turns out that the codimension of $\tilde{I}$ for $I = \langle Q_1^5, Q_2^5, Q_3^5 \rangle$ is 21. We have to enlarge our ideal again by a 9-orthogonal polynomial. Using *Mathematica*, we found that an orthogonal polynomial of degree 6 vanishing on all zeros. It is

$$Q_1^6(x,y) = \frac{3(-491+201\sqrt{6})}{4(157\sqrt{2}-129\sqrt{3})} P_1^6(x,y) + \frac{\sqrt{7}(173-72\sqrt{6})}{20(4\sqrt{2}-3\sqrt{3})} P_3^6(x,y) + P_5^6(x,y).$$

Moreover, it is easy to verify that Q_1^6 is not in the ideal $\langle Q_1^5, Q_2^5, Q_3^5 \rangle$. Adding it to the ideal and checking the codimension, we finally conclude that $\operatorname{codim} \tilde{I} = |V|$ and that the cubature formula is generated by the ideal $\langle Q_1^5, Q_2^5, Q_3^5, Q_1^6 \rangle$. $\square$

3. Further discussions

1. It is shown in [21] that if I is a generating ideal of a cubature formula of degree $2n-1$ and $\Pi^d/I = \Pi^d_{2n-1}/(I \cap \Pi^d_{2n-1})$, then I necessarily has a basis consisting of $(2n-1)$-orthogonal polynomials and $\dim \tilde{I} = |V|$. Hence, Theorem 2.6 offers a characterization of cubature formulae of degree $2n-1$. Similar characterization can be derived for cubature formulae of even degree. The essential question, however, is how to find subsets of $(2n-1)$-orthogonal polynomials that have large number of real common zeros. One necessary condition is derived in [21], which we state below. We need to introduce the notation of leading coefficient of orthogonal polynomials in several variables. Let $\mathbf{x}^n = (\mathbf{x}^\alpha)_{|\alpha|=n}$ be a column vector of all monomials or degree n, arranged according to our fixed monomial order. The column vector of orthogonal polynomials $\mathbb{P}_n$ can be written in terms of $\mathbf{x}^n$ as

$$\mathbb{P}_n = G_n \mathbf{x}^n + G_{n,1} \mathbf{x}^{n-1} + \dots,$$

where $G_{n,k}$ are proper matrices and G_n is call the leading coefficient matrix of $\mathbb{P}_n$. It is known that G_n is invertible (see [20]). We have

Theorem 3.1. *Let $d = 2$ and let I be an ideal generated by a set of $(2n-1)$-orthogonal polynomials, among which the orthogonal polynomials of degree n are given by $U^T \mathbb{P}_n$. Assume that I generates a cubature formula of degree $2n-1$. Then there is a Hankl matrix $H = (h_{i+j})$ such that the matrix $I + G_n H G_n^T$ is nonnegative definite and U satisfies $(I + G_n H G_n)U = 0$.*

We refer to [21] for the proof of this theorem. For each given linear functional $\mathcal{L}$, the matrices G_n can be computed explicitly. Moreover, the set of G_n for all n determines the orthogonality of orthogonal polynomials and, thus, captures the essence of $\mathcal{L}$. The necessary condition in the theorem can be used to narrow down the choice of U significantly. For example, in one example on formulae of degree 9 in [21], the condition narrows down the choice of U to a class of matrices determined by only two parameters. On the other hand, however, there is no practical sufficient condition known at this point.

2. We discuss Theorem 2.2, which shows that the interpolation problem (V, S_I) is correct if $|V| = \operatorname{codim} I$, and Corollary 2.4, which gives the same result when $V = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$, where x_i are distinct real points. In the latter case that V is given, by Theorem 2.3, we see that Theorem 2.2 is equivalent to Corollary 2.4. As we point out before that the result in Corollary 2.4 is more or less known. In fact, another proof can be given as follows.

Alternative proof of Corollary 2.4. For a given V, we can find an interpolation polynomial P_f easily if we consider all polynomials (P_f may have a degree much higher than necessary, as in the proof of Theorem 2.3). Let $I = I(V)$. Construct a Gröbner basis $G = \{g_1, \ldots, g_M\}$ for the ideal I. Then use the division algorithm to find the unique remainder r_f of P_f with respect to the ideal I. The basic theorem on the division algorithm with the Gröbner basis shows that the polynomial r_f is the unique polynomial in $\mathcal{S}_I$ that interpolates f. $\square$

The connection between interpolation polynomials and Gröbner basis has been studied in [17], where an algorithm is given which computes a Gröbner basis using interpolating polynomials. The proof above offers an algorithm for computing the interpolating polynomial in S_I using the division algorithm, which seems to be especially suitable for the situation that the nodes form an affine variety of a given polynomial ideal. Indeed, the first step that constructs an interpolating polynomial without restriction on the polynomial degree poses no difficulty. The second step is to compute a Gröbner basis, which can be carried out using Buckenberg's algorithm or one of its refinement, if we are given I with $V = V(I)$ or if we can find $I = I(V)$ easily. The third step is using division algorithm to find the unique interpolation polynomial in S_I, which is a standard procedure. The main part is to find a Gröbner basis. We explain the procedure by a simple example.

Example 3.1. We consider the case of a cubature formula of degree 5 with 7 nodes on the unit disk of $\mathbb{R}^2$. This formula is a minimal one, meaning that it has the smallest number of nodes among all formulae of degree 5. Its nodes are given by $(0,0)$, $(\pm\sqrt{2/3},,0)$, and $(\pm\sqrt{1/6}, \pm\sqrt{1/2})$, and they form the affine variety of the ideal

$$I = \langle P_0^3(x,y) + P_2^3(x,y), P_1^3(x,y), P_3^3(x,y) \rangle,$$

where P_i^3 are the orthonormal polynomials of degree 3 given by (2.3). Using *Mathematica*, we found that a Gröbner basis of I is given by

$$G_1(x,y) = y(-1+2y^2), \quad G_2(x,y) = (-1+6x^2)y, \quad G_3(x,y) = x(-2+3x^2+3y^2),$$

the default monomial order of *Mathematica* is the Lex order. From the formulae of the basis, it follows that $LT(I)$ contains y^3, x^2y, x^3. So $S_I = \operatorname{span}\{1, x, y, x^2, xy, y^2, xy^2\}$. From the given nodes, we can easily construct the fundamental interpolating polynomials, if we allow the polynomials to have higher degree. For example, it is easy to see that $p_1(x,y) = 3(y^2 - 1/2)(x^2 - 2/3)$ is the fundamental polynomial of $(0,0)$, that is, it takes value 1 at $(0,0)$ and value 0 at all other 6 points. Likewise, the polynomial $p_2(x,y) = -(3/2)x(y^2-1/2)(x+(2/3)^{1/2})$ is the fundamental polynomial of the point $(\sqrt{2/3},0)$, etc. Using the command PolynomialReduce in *Mathematica*, we can then find the polynomials $r_i = p_i \bmod I$, which are the unique fundamental polynomials in S_I. Thus, for example, the fundamental polynomial of the point $(0,0)$ is $r_0(x,y) = (2-3x^2-3y^2)/2$ and the fundamental polynomial of the point $(\sqrt{2/3},0)$ is $r_2(x,y) = (\sqrt{6}x+3x^2-y^2-2\sqrt{6}xy^2)/4$, etc. $\square$

Incidentally, in this case, the Gröbner basis are also orthogonal polynomials . In fact, we have $G_1 = P_3^3/(2\sqrt{5}) - \sqrt{5}P_1^3/20$, $G_2 = \sqrt{5}P_1^3/4$, $G_3 = (P_0^3+P_2^3)/4$. In general, however, this is not true. For example, in the case of Example 2.2 in the previous section, the ideal is generated by four 9-orthogonal polynomials. We found that the Gröbner basis with respect to the Lex order also contains 4 elements, but their total degrees are 7, 8, 9 and 11. Moreover, the Gröbner basis corresponds to the degree reverse Lex order contains 11 elements whose degrees are between 5 to 7.

In the interpolation problem related to the cubature formulae, we can integrate the fundamental polynomials to find the cubature weights. That is, the cubature weight λ_i corresponds to the point $\mathbf{x}_i$ can be obtained by $\lambda_i = \mathcal{L}(p_i)$, where p_i is the fundamental interpolating polynomial of the point $\mathbf{x}_i$. For this purpose, however, we do not have to find the interpolating polynomial in S_I. Any fundamental interpolating polynomial of degree less than the maximum precision of the cubature formula will do. In particular, this allows us to take any interpolating polynomial and use the division algorithm with respect to the original basis of I, so that we do not have to compute the Gröbner basis.

References

1. B. Bojanov, H. A. Hakopian, and A. A. Sahakian, *Spline functions and multivariate interpolations*, Mathematics and its Applications, vol. 248, Kluwer Academic Publishers, Dordrecht, 1993.
2. C. de Boor and A. Ron, *On multivariate polynomial interpolation.*, Constr. Approx. **6** (1990), 287–302.
3. C. de Boor and A. Ron, *On polynomial ideals of finite codimension with applications to box spline theory*, J. Math. Anal. and Appl. **158** (1991), 168–193.
4. C. de Boor and A. Ron., *The least solution for the polynomial interpolation problem*, Math. Z. **210** (1992), 347–378.
5. D. Cox, J. Little, and D. O'shea, *Ideals, Varieties, and Algorithms, 2nd ed.*, Springer, Berlin, 1997.
6. M. Gasca, *Multivariate polynomial interpolation.*, Computation of Curves and Surfaces (W. Dahmen, M. Gasca, and C. A. Micchelli, ed.), Kluwer Academic Publishers, Dordrecht, 1990, pp. 215–236.
7. T. Koornwinder, *Two-variable analogues of the classical orthogonal polynomials*, Theory and Application of Special Functions (R. A. Askey, ed.), Academic Press, New York, 1975, pp. 435–495.
8. R. A. Lorentz., *Multivariate Birkhoff Interpolation*, Lecture Notes in Mathematics, vol. 1516, Springer Verlag, 1992.
9. H. M. Möller, *Polynomideale und Kubaturformeln*, Thesis, Univ. Dortmund, 1973.
10. H. M. Möller, *Kubaturformeln mit minimaler Knotenzahl*, Numer. Math. **35** (1976), 185–200.
11. H. M. Möller, *On the construction of cubature formulae with few nodes using Groebner bases*, Numerical Integration (P. Keast and G. Fairweather, ed.), D. Reidel Publishing Company, 1998, pp. 177–192.
12. I. P. Mysovskikh, *Numerical characteristics of orthogonal polynomials in two variables*, Vestnik Leningrad Univ. Math. **3** (1976), 323–332.

13. I. P. Mysovskikh, *The approximation of multiple integrals by using interpolatory cubature formulae*, Quantitative Approximation (R. A. DeVore and K. Scherer, ed.), Academic Press, New York, 1980.
14. I. P. Mysovskikh, *Interpolatory cubature formulas*, "Nauka", Moscow, 1981. (Russian)
15. Radon, *Zur mechanischen Kubatur*, Monatsh. Math. **52** (1948), 286–300.
16. Th. Sauer, *Polynomial interpolation of minimal degree*, Numer. Math. **78** (1997), 59-85.
17. Th. Suaer, *Polynomial interpolation of minimal degree and Gröbner bases*, Gröbner bases and applications (B. Buchberger and F. Winkler, ed.), London Math. Soc. Lecture Notes Seris, vol. 251, Cambridge Univ. Press, 1998, pp. 483–494.
18. A. Stroud, *Approximate calculation of multiple integrals*, Prentice Hall, Englewood Cliffs, NJ, 1971.
19. Y. Xu, *Common zeros of polynomials in several variables and higher dimensional quadrature*, Pitman Research Notes in Mathematics Series, Longman, Essex, 1994.
20. Y. Xu, *On orthogonal polynomials in several variables*, Special functions, q-series, and related topics, Fields Institute Communications, vol. 14, 1997, pp. 247 - 270.
21. Y. Xu, *Cubature formulae and polynomial ideals*, Advances in Appl. Math. (to appear).

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