POLYNOMIAL INTERPOLATION ON THE UNIT SPHERE *

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Abstract. The problem of interpolation at $(n+1)^2$ points on the unit sphere S^2 by spherical polynomials of degree at most n is studied. Many sets of points that admit unique interpolation are given explicitly. The proof is based on a method of factorization of polynomials. A related problem of interpolation by trigonometric polynomials is also solved.

Key words. Interpolation, spherical polynomials, unit sphere

AMS subject classifications. 41A05, 41A63, 65D05

1. Introduction. The purpose of this paper is to study polynomial interpolation on the unit sphere $S^2 = \{x : \|x\| = 1\}$ of $\mathbb{R}^3$, where $\|x\|$ is the Euclidean norm of $\mathbb{R}^3$. Let Π_n^2 denote the space of polynomials of degree at most n in 2 variables and let $\mathcal{P}_n^3$ denote the space of homogeneous polynomials of degree n in 3 variables. The notation $\Pi_n(S^2)$ denotes the space of spherical polynomials of 3 variables; that is, the restriction of polynomials of 3 variables in Π_n^3 on S^2. It is known that

$$\dim \Pi_n(S^2) = (n+1)^2, \qquad n \ge 0.$$

We study the following polynomial interpolation problem on S^2:

Problem 1. Let $X = \{\mathbf{a}_i : 1 \le i \le (n+1)^2\}$ be a set of distinct points on S^2. Find conditions on X such that there is a unique polynomial $T \in \Pi_n(S^2)$ satisfying

$$T(\mathbf{a}_i) = f_i, \qquad \mathbf{a}_i \in X, \quad 1 \le i \le (n+1)^2,$$

for any given data $\{f_i\}$.

If there is a *unique* solution to the interpolation problem, we say that the problem is *poised* and that X solves Problem 1. In terms of a basis of $\Pi_n(S^2)$, the interpolation conditions give linear equations for the coefficients of T. It follows that the problem has a unique solution if and only if $T(\mathbf{a}_i) = 0$ for all $\mathbf{a}_i \in X$ implies $T = 0$, which holds if and only if the determinant of the linear system of equations is non-zero. The determinant is a polynomial of the interpolation points; hence it is non-zero for almost all choices of interpolation points. In other words, Problem 1 is poised for almost all choices of X. On the other hand, given a set of points, it is often difficult to determine if it leads to unique interpolation. The question is related to several other problems; see the discussion below. In many applications, one would like to have poised sets of points explicitly given. The main result of this paper provides families of such points that admit unique interpolation by $\Pi_n(S^2)$ for all n.

We need some basic facts about spherical harmonics (see [3, 10], for example). The harmonic polynomials are homogeneous polynomials Y in $\mathcal{P}_n^3$ that satisfy $\Delta Y = 0$, where Δ is the usual Laplace operator, $\Delta = \partial_1^2 + \partial_2^2 + \partial_3^2$ with $\partial_i = \partial^2/\partial x_i^2$. The spherical harmonics are the restriction of the harmonic polynomials on S^2. Let $\mathcal{H}_n^3$ denote the space of spherical harmonics of degree n in 3 variables. It is known that

$$\dim \mathcal{H}_n^3 = 2n+1, \qquad n \ge 0.$$

*Work partially supported by the National Science Foundation under Grant DMS-0201669

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Let $\{S_{k,n} : 1 \le k \le 2n+1\}$ be an orthonormal basis of $\mathcal{H}_n^3$. The reproducing kernel $Y_n(\mathbf{x},\mathbf{y})$ of $\mathcal{H}_n^3$ is defined by

$$Y_n(\mathbf{x},\mathbf{y}) = \sum_{k=1}^{2n+1} S_{k,n}(\mathbf{x}) S_{k,n}(\mathbf{y}), \qquad \mathbf{x},\mathbf{y} \in S^2.$$

In particular, for each fixed $\mathbf{y} \in S^2$, $Y_n(\mathbf{x},\mathbf{y})$ is a spherical harmonic, the so-called zonal harmonic of degree n. Let $C_n^\lambda(t)$ be the Gegenbauer polynomial of degree n normalized so that $C_n^\lambda(1) = \binom{n+2\lambda-1}{n}$. Then Y_n satisfies

$$Y_n(\mathbf{x},\mathbf{y}) = (2n+1) C_n^{1/2}(\langle \mathbf{x},\mathbf{y}\rangle), \qquad \mathbf{x},\mathbf{y} \in S^2.$$

In $L^2(S^2)$, the space $\Pi_n(S^2)$ admits a unique orthogonal decomposition $\Pi_n(S^2) = \bigoplus_{k=0}^n \mathcal{H}_k^3$ in terms of the spaces of spherical harmonics. The reproducing kernel of $\Pi_n(S^2)$ in $L^2(S^2)$ is given by

$$K_n(\mathbf{x},\mathbf{y}) = \sum_{k=0}^n Y_k(\mathbf{x},\mathbf{y}) = \sum_{k=0}^n (2k+1) C_k^{1/2}(\langle \mathbf{x},\mathbf{y}\rangle).$$

For each fixed $\mathbf{y} \in S^2$, $K_n(\mathbf{x},\mathbf{y})$ is a polynomial in $\Pi_n(S^2)$ that depends only on $\langle \mathbf{x},\mathbf{y}\rangle$.

Apart from its own interest, the interpolation problem is related to other problems concerning spherical polynomials. As an example, let $X = \{\mathbf{a}_i : 1 \le i \le (n+1)^2\}$ be a set of distinct points on S^2; one can ask the question of when $\{K_n(\mathbf{x},\mathbf{a}_i), 1 \le i \le (n+1)^2\}$ is a basis of $\Pi_n(S^2)$. It turns out that this problem is equivalent to Problem 1. The following proposition is folklore and can be proved easily using the formulae for K_n and Y_n given above.

Proposition 1.1. *Let $X = \{\mathbf{a}_i : 1 \le i \le (n+1)^2\}$ be a set of distinct points. Then X is a solution of Problem 1 if and only if $\{K_n(\mathbf{x},\mathbf{a}_i), 1 \le i \le (n+1)^2\}$ forms a basis of $\Pi_n(S^2)$.*

The problem of $\{K_n(\mathbf{x},\mathbf{a}_i)\}$ being a basis has been considered recently in [4] for possible application in wavelets, since such a basis is better localized than the usual orthonormal basis. The interpolation problem is also closely related to constructing cubature formulae on the sphere (see Section 3), and there have been considerable efforts of computing good interpolation points for that purpose; see [11] and the references therein. Recently the polynomial interpolation has also been studied in [6] in connection with radial basis interpolation on the sphere. In [14], we showed that there is a close relation between interpolation on S^d and on the unit ball B^d of $\mathbb{R}^d$. In particular, it shows that many sets of points that allow unique polynomial interpolation on $B^2 = \{(x,y) : x^2+y^2 \le 1\}$ can be used to generate symmetric points on S^2 that solve Problem 1, where symmetry means that the points are symmetric with respect to a coordinate plane. The interpolation on B^2 is studied in [1, 2] recently. The result in [1] leads to a set of symmetric points on S^2 in which points are equidistant points on parallel circles, and each circle has the same number of points. However, the more general result in [2] cannot be used to give points on S^2, since the correspondence between points on B^2 and on S^2 requires that there are $n+1$ points on the boundary circle of B^2.

In the following section, we present a theorem that gives many sets of points that solve Problem 1. The $(n+1)^2$ points all lie on $n+1$ distinct latitudes (parallel circles on S^2), and on each latitude there are an odd number of equidistant points. The number of points need not be same on each latitude and there is no restriction on the distribution of latitudes. The proof relies on a method of factorization, which is closely related to the method used for polynomial interpolation on the unit ball B^2 in [2, 7]. The use of equidistant points allows us to reduce the problem of interpolation on the sphere to an interpolation problem by a family of special trigonometric polynomials, which turns out to be equivalent to a Hermite-Birkhoff interpolation by trigonometric polynomials.

As far as we know, apart from a result that is a simple consequence of Bezout's theorem (see Proposition 2.1 below), these families of points are first examples of poised interpolation points that are given explicitly for all n.

The paper is organized as follows. We prove the factorization theorem in the following section. The main result and various examples are given in Section 3, which also includes a result on cubature formulae. The related trigonometric Hermite-Birkhoff interpolation is discussed in Section 4.

2. Preliminary and Factorization of polynomials. On S^2 it is more convenient to work with spherical coordinates,

$$x = \sin\theta\sin\phi, \quad y = \sin\theta\cos\phi, \quad z = \cos\theta, \qquad 0 \le \phi \le 2\pi, \quad 0 \le \theta \le \pi.$$

For a polynomial $T \in \Pi(S^2)$ we introduce the notation $\widetilde{T}$ defined by

$$\widetilde{T}_n(\theta,\phi) = T_n(\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta), \quad 0 \le \phi \le 2\pi, \quad 0 \le \theta \le \pi.$$

If $X = \{(x_i, y_i, z_i) : 1 \le i \le M\}$ is a set of points on S^2, we also use the notation $\widetilde{X} = \{(\theta_i, \phi_i) : 1 \le i \le M\}$ for the corresponding set in spherical coordinates.

The notation $S^2(a) := \{(x,y,z) : (x,y,z) \in S^2,\ z = a\}$, $-1 < a < 1$, denotes the circle on S^2 resulting from the intersection of S^2 with the plane $z = a$ (called the latitude at $z = a$).

An orthogonal basis of $\mathcal{H}_n^3$ can be given in terms of the Gegenbauer polynomials. In spherical coordinates, define

$$\begin{aligned} Y_{k,n}^{(1)}(x,y,z) &= C_{n-k}^{k+1/2}(\cos\theta)(\sin\theta)^k\cos k\phi, \qquad 0 \le k \le n, \\ Y_{k,n}^{(2)}(x,y,z) &= C_{n-k}^{k+1/2}(\cos\theta)(\sin\theta)^k\sin k\phi, \qquad 1 \le k \le n. \end{aligned}$$

Then $\mathcal{H}_n^3 = \mathrm{span}\{Y_{k,n}^{(1)},\ 0 \le k \le n, \text{ and } Y_{k,n}^{(2)},\ 1 \le k \le n\}$. Since $\Pi_n(S^2) = \bigoplus_{k=0}^n \mathcal{H}_k^3$, every polynomial $T_n \in \Pi_n(S^2)$ can be written as

$$T_n(x,y,z) = \sum_{j=0}^{n}\sum_{k=0}^{j} C_{j-k}^{k+1/2}(\cos\theta)(\sin\theta)^k(a_{k,j}\cos k\phi + b_{k,j}\sin k\phi).$$

Changing the order of sums in the above expression, we see that

$$\widetilde{T}_n(\theta,\phi) = a_0(\cos\theta) + \sum_{k=1}^{n}[a_k(\cos\theta)(\sin\theta)^k\cos k\phi + b_k(\cos\theta)(\sin\theta)^k\sin k\phi], \quad (2.1)$$

where $a_k(t)$ and $b_k(t)$ are polynomials of degree $n-k$ in one variable,

$$a_k(\cos\theta) = \sum_{j=0}^{n-k} a_{k,k+j}C_j^{k+1/2}(\cos\theta) \quad \text{and} \quad b_k(\cos\theta) = \sum_{j=0}^{n-k} b_{k,k+j}C_j^{k+1/2}(\cos\theta).$$

Since C_j^λ is a polynomial of degree exactly j, a_k and b_k are generic polynomials of degree $n-k$. Notice that for a fixed θ, the polynomial $\widetilde{T}(\theta,\phi)$ is a trigonometric polynomial of degree n in ϕ. We will use the fact that the interpolation on $2n+1$ distinct points inside $[0,2\pi)$ by trigonometric polynomials of degree n is unique.

The proof of our main result is based on a factorization of the polynomial that vanishes on the interpolation points. In the simplest case, an analog of such a factorization is akin to the Bezout theorem for algebraic curves, as explained by the following simple result.

PROPOSITION 2.1. *Let $z_0, z_1, \ldots, z_n$ be $n+1$ distinct elements in $(-1,1)$. If X consists of $2k+1$ distinct points on the latitude $S^2(z_k)$ for $0 \le k \le n$, then X solves Problem 1 in $\Pi_n(S^2)$.*

Proof. It is sufficient to prove that if $T_n \in \Pi_n(S^2)$ vanishes on X then T_n is identically zero. Using the expression of $\widetilde{T}_n$ in (2.1) and the fact that T_n vanishes on $2n+1$ points of $S^2(z_n)$, it follows from the uniqueness of the trigonometric interpolation that $a_k(z_n) = 0$ and $b_k(z_n) = 0$, $0 \le k \le n$. Consequently, $a_n(z) = b_n(z) = 0$ and $a_k(z) = (z-z_n)a_k^*(z)$ and $b_k(z) = (z-z_n)b_k^*(z)$ for $0 \le k \le n-1$, so that T_n satisfies a factorization $T_n(x,y,z) = (z-z_n)T_{n-1}(x,y,z)$, where $T_{n-1} \in \Pi_{n-1}(S^2)$. Evidently, we can continue this process for $z_{n-1}, \ldots, z_0$ and conclude that $T_n(x,y,z) = (z-z_n)\ldots(z-z_0)T^*(x,y,z)$. But T_n is a polynomial of degree n, so that $T_n \equiv 0$. ☐

The analog of this proposition holds for higher dimensional sphere, as shown in [6]. That Bezout's theorem can be used to establish the uniqueness of the interpolation is folklore (see, for example, [5]). We present the proposition here since its proof gives a prelude of the factorization that leads to our main result.

The points in the above proposition have little symmetry on S^2, since no two latitudes have the same number of points. It is worthwhile to emphasis that the position of the points on each latitude is completely arbitrary. In our result below, the points on each latitude are equidistant points. In essence, our main result is based on a more general factorization that holds, however, only for equidistant points.

Let $\Theta_{\alpha,m}$ denote a set of $2m+1$ equidistant points,

$$\Theta_{\alpha,m} = \{\theta_j^\alpha : \theta_j^\alpha = (2j+\alpha)\pi/(2m+1), \quad j = 0,1,\ldots,2m, \quad 0 \le \alpha < 2\}. \tag{2.2}$$

Under the mapping $\phi \mapsto e^{i\phi}$, these points can be considered as points on the unit circle. The presence of the number α means that the equidistant points are defined up to a rotation. The following simple fact plays an important role in the development below.

LEMMA 2.2. *Let $n = 2m$ or $n = 2m-1$. For $\phi \in \Theta_{\alpha,m}$,*

$$\begin{aligned}\widetilde{T}_n(\theta,\phi) = a_0(\cos\theta) + \sum_{k=1}^{m} \Big[& \big(a_k(\cos\theta)(\sin\theta)^k + u_{2m-k+1}(\cos\theta)(\sin\theta)^{2m-k+1}\big)\cos k\phi \\ & + \big(b_k(\cos\theta)(\sin\theta)^k + v_{2m-k+1}(\cos\theta)(\sin\theta)^{2m-k+1}\big)\sin k\phi\Big],\end{aligned}$$

where

$$\begin{aligned} u_{2m-k+1}(\cos\theta) &= a_{2m-k+1}(\cos\theta)\cos\alpha\pi + b_{2m-k+1}(\cos\theta)\sin\alpha\pi, \\ v_{2m-k+1}(\cos\theta) &= a_{2m-k+1}(\cos\theta)\sin\alpha\pi - b_{2m-k+1}(\cos\theta)\cos\alpha\pi, \end{aligned}$$

and we assume that $a_{2m}(t) = b_{2m}(t) = 0$ if $n = 2m-1$.

Proof. The proof amounts to using the fact that

$$\begin{aligned}\cos(2m-k+1)\phi &= \cos(\alpha\pi - k\phi) = \cos\alpha\pi\cos k\phi + \sin\alpha\pi\sin k\phi,\\ \sin(2m-k+1)\phi &= \sin(\alpha\pi - k\phi) = \sin\alpha\pi\cos k\phi - \cos\alpha\pi\sin k\phi\end{aligned}$$

for $\phi \in \Theta_{\alpha,m}$, and rewrite $\widetilde{T}(\theta,\phi)$ accordingly. □

In particular, using the uniqueness of trigonometric interpolation, this shows that if $\widetilde{T}_n(\theta_j,\phi_i) = 0$ for $\phi_i \in \Theta_{\alpha,m}$, $0 \le i \le 2m$ then $a_0(\theta_j) = 0$ and

$$\begin{aligned}a_k(\cos\theta_j)(\sin\theta_j)^k + u_{2m-k+1}(\cos\theta_j)(\sin\theta_j)^{2m-k+1} &= 0\\ b_k(\cos\theta_j)(\sin\theta_j)^k + v_{2m-k+1}(\cos\theta_j)(\sin\theta_j)^{2m-k+1} &= 0.\end{aligned} \tag{2.3}$$

Hence, we need to consider the poisedness of interpolation by trigonometric polynomials of the form

$$p(\cos\theta)(\sin\theta)^k + q(\cos\theta)(\sin\theta)^{2m-k+1},$$

where $p(t)$ and $q(t)$ are polynomials of degree $n-k$ and $n-2m+k-1$, respectively. For this purpose, we need a lemma which is elementary but somewhat unexpected. It contains the following elementary formula as its simplest case,

$$\frac{d^5}{dt^5}\left[(1-t^2)^{3/2}(at+b)\right] = 45\frac{a+bt}{(1-t^2)^{7/2}}.$$

It is this simple formula, stumbled upon using a computer algebra system, that leads to the lemma below. We will use the Pochhammer symbol $(a)_n = a(a+1)\dots(a+n-1)$.

Lemma 2.3. *Let m and k be positive integers and $1 \le k \le 2m$. Let $q_{k-1}(t)$ be an algebraic polynomial of degree $k-1$. Then*

$$(1-t^2)^{m+1/2}\Big(\frac{d}{dt}\Big)^{2m-k+1}\Big((1-t^2)^{m-k+1/2}q_{k-1}(t)\Big) = q^*_{k-1}(t),$$

where q^*_{k-1} is a polynomial of degree $k-1$ such that $q_{k-1}(t) = 0$ if and only if $q^*_{k-1}(t) = 0$. In fact, if $q_{k-1}(t) = \sum_{l=0}^{k-1} a_l(1-t)^l$, then q^*_{k-1} is given by

$$q^*_{k-1}(t) = (-1)^{k-1}2^{2m-k+1}\sum_{l=0}^{k-1}(-m+l+1/2)_{2m-k+1}\,a_l\,s_l(t)$$

where s_l are polynomials of degree $k-1$ that are defined by

$$s_l(t) = (1-t)^l\sum_{j=0}^{k-l-1}\frac{(-k+l+1)_j(k-2m-1)_j}{(-m+l+1/2)_j j!}\Big(\frac{1-t}{2}\Big)^j.$$

Proof. We need to recall the definition of Jacobi polynomials. Let α and β be two arbitrary real numbers. Then the Jacobi polynomial $P_n^{(\alpha,\beta)}$ is given by Rodrigue's formula [13, p. 67, (4.3.1)])

$$P_n^{(\alpha,\beta)}(t) = \frac{(-1)^n}{n!2^n}(1-t)^{-\alpha}(1+t)^{-\beta}\Big(\frac{d}{dt}\Big)^n\left[(1-t)^{n+\alpha}(1+t)^{n+\beta}\right].$$

Let $q_{k-1}(t) = \sum_{l=0}^{k-1} a_l(1-t)^l$. It follows that

$$\begin{aligned}\phi(t) :=&(1-t^2)^{m+1/2}\left(\frac{d}{dt}\right)^{2m-k+1}\Big((1-t^2)^{m-k+1/2}q_{k-1}(t)\Big)\\ =&\sum_{l=0}^{k-1} a_l(1-t^2)^{m+1/2}\left(\frac{d}{dt}\right)^{2m-k+1}\Big((1-t)^{m-k+l+1/2}(1+t)^{m-k+1/2}\Big).\end{aligned}$$

Using Rodrigue's formula with $n = 2m-k+1$, $\alpha = -m+l-1/2$, and $\beta = -m-1/2$, we get

$$\phi(t) = \sum_{l=0}^{k-1} a_l(-1)^{k-1}2^{2m-k+1}(2m-k+1)!(1-t)^l P_{2m-k+1}^{(-m-1/2+l,-m-1/2)}(t).$$

It is known that the Jacobi polynomials with negative parameters satisfy the following relation ([13, p. 64, (4.22.3)])

$$\binom{n}{j-1}P_n^{(\alpha,\beta)}(t) = \binom{n+\alpha}{n-j+1}P_{j-1}^{(\alpha,\beta)}(t), \qquad \text{if} \quad n+\alpha+\beta+j=0,$$

which shows that $P_n^{(\alpha,\beta)}(t)$ is in fact a polynomial of degree $j-1$ with $j = -n-\alpha-\beta$. The condition $n+\alpha+\beta+j=0$ is satisfied in our case with n, α, β as defined above and $j = k-l$. Consequently, we conclude that

$$\begin{aligned}\phi(t) = \sum_{l=0}^{k-1} a_l(-1)^{k-1}2^{2m-k+1}(k-l-1)!&\frac{(-m+l+1/2)_{2m-k+1}}{(-m+l+1/2)_{k-l-1}}\\ &\times(1-t)^l P_{k-l-1}^{(-m-1/2+l,-m-1/2)}(t),\end{aligned}$$

after some simplification of the constants. This shows that ϕ is indeed a polynomial of degree $k-1$, which we called q_{k-1}^* in the statement. The explicit formula of q_{k-1}^* is derived from the explicit formula of $P_n^{(\alpha,\beta)}(t)$ ([13, p. 62, (4.21.2)]),

$$P_n^{(\alpha,\beta)}(t) = \binom{n+\alpha}{n}{}_2F_1\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-t}{2}\right),$$

where ${}_2F_1$ is the hypergeometric function defined by

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k k!}z^k, \qquad |z|<1.$$

Finally, let $s_l(t)$ be the polynomials in the statement. Writing $s_0(t), s_1(t), \ldots, s_{k-1}(t)$ in terms of $1, (1-t), \ldots, (1-t)^{k-1}$, the transition matrix is triangular with nonzero diagonal elements. Hence, the polynomials $s_0, s_1, \ldots, s_{k-1}$ are linearly independent. Consequently, $q_{k-1}^*(t) = 0$ if and only if $q_{k-1}(t) = 0$. ∎

LEMMA 2.4. *Let k and m be positive integers, $k \le 2m$. Let p_{2m-k} be a polynomial of degree $2m-k$ and q_{k-1} be a polynomial of degree $k-1$. If $\phi(t) = p_{2m-k}(t) + (1-t^2)^{m-k+1/2}q_{k-1}(t)$ vanishes on $2m+1$ distinct points in $[-1,1]$, then $\phi(t) \equiv 0$.*

Proof. Since ϕ has $2m+1$ zeros, Rolle's theorem implies that $\phi^{(2m-k+1)}(t)$ has at least k zeros inside $(-1,1)$. Using the previous lemma,

$$\phi^{(2m-k+1)}(t) = \left(\frac{d}{dt}\right)^{2m-k+1}\Big[(1-t^2)^{m-k+1/2}q_{k-1}(t)\Big] = (1-t^2)^{-m-1/2}q_{k-1}^*(t),$$

so that $q_{k-1}^*(t)$ has k zeros inside $(-1,1)$. Since q_{k-1}^* is a polynomial of degree $k-1$, we have $q_{k-1}^*(t) \equiv 0$. Consequently, $q_{k-1}(t) \equiv 0$ by the previous lemma. Therefore, $\phi(t) = p_{2m-k}(t)$, which must be zero since it is a polynomial of degree $2m-k$ and it vanishes on $2m+1$ points. □

COROLLARY 2.5. *Let k and m be positive integers and $k \le 2m$. Let p_{2m-k} be a polynomial of degree $2m-k$ and q_{k-1} be a polynomial of degree $k-1$. If*

$$\psi(\theta) = p_{2m-k}(\cos\theta)(\sin\theta)^k + (\sin\theta)^{2m-k+1} q_{k-1}(\cos\theta)$$

vanishes on $2m+1$ distinct points in $(0,\pi)$, then $\psi(\theta) \equiv 0$.

Proof. Using the fact that $\sin\theta$ is positive on $(0,\pi)$, the stated result follows from the previous proposition with $\phi(t) = \psi(\theta)/(\sin\theta)^k$ and $t = \cos\theta$. □

There is another way to state the result in Corollary 2.5. A system of functions $\{g_1, \ldots, g_n\}$ is called a Chebyshev system on the interval $(a,b) \subset \mathbb{R}$ if for any set of nonzero real numbers $c_1, \ldots, c_n$, the function $c_1 g_1 + \ldots + c_n g_n$ has at most $n-1$ zeros in (a,b).

COROLLARY 2.6. *Let k and m be integers and $k \le 2m$. The system of functions*

$$(\sin\theta)^k \{1, \cos\theta, \ldots, \cos(2m-k)\theta\} \cup (\sin\theta)^{2m-k+1}\{1, \cos\theta, \cdots, \cos(k-1)\theta\}$$

is a Chebyshev system on $(0,\pi)$. Equivalently, the system of functions

$$\{1, t, \ldots, t^{2m-k}\} \cup (1-t^2)^{2m-2k+1}\{1, t, \ldots, t^{k-1}\}$$

is a Chebyshev system on $(0,1)$.

Even in the case of $k=1$, which states that $\{1, \cos\theta, \ldots, \cos(2m-1)\theta, (\sin\theta)^{2m-1}\}$ is a Chebyshev system on $(0,\pi)$, this corollary is not obvious.

The following factorization theorem holds the key to our main result.

THEOREM 2.7. *Let m be an integer, $m \le s \le 2m+1$ and $n = 2m$ or $n = 2m-1$. Let $\theta_0, \theta_1, \ldots, \theta_{2\lambda}$ be distinct numbers in $(0,\pi)$, where $\lambda = s-m$. If $T \in \Pi_s(S^2)$ satisfies*

$$\widetilde{T}(\theta_j, \phi_i) = 0, \qquad 0 \le j \le 2\lambda, \quad 0 \le i \le 2m, \quad \phi_i \in \Theta_{\alpha,m},$$

where α is a number in $[0,2)$, then there is a spherical polynomial $T^* \in \Pi_{n-2\lambda-1}(S^2)$ such that

$$T(x,y,z) = \prod_{j=0}^{2\lambda}(z - \cos\theta_j) T^*(x,y,z).$$

In particular, $T^*(x,y,z) = 0$ if $s = 2m$ or $s = 2m-1$.

Proof. Using the formula (2.1), we write

$$\widetilde{T}(\theta,\phi) = a_0(\cos\theta) + \sum_{k=1}^{s} \left[a_k(\cos\theta)(\sin\theta)^k \cos k\phi + b_k(\cos\theta)(\sin\theta)^k \sin k\phi\right],$$

where a_k and b_k are polynomials of degree $s-k$. By Lemma 2.2, for $\phi_i \in \Theta_{\alpha,m}$,

$$\begin{aligned}\widetilde{T}(\theta_j, \phi_i) = a_0(\cos\theta_j) &+ \sum_{k=1}^{m} (\sin\theta_j)^k \left(a_k(\cos\theta_j)\cos k\phi_i + b_k(\cos\theta_j)\sin k\phi_i\right) \\ &+ \sum_{k=2m+1-s}^{m} \big(u_{2m-k+1}(\cos\theta_j)(\sin\theta_j)^{2m-k+1}\cos k\phi_i \\ &\qquad + v_{2m-k+1}(\cos\theta_j)(\sin\theta_j)^{2m-k+1}\sin k\phi_i\big)\end{aligned}$$

for $0 \le i \le 2m$ and $0 \le j \le 2\lambda$. For each fixed j, $\widetilde{T}(\theta_j, \phi_i) = 0$ shows that the trigonometric polynomial $\widetilde{T}(\theta_j, \cdot)$ of degree m vanishes on $2m+1$ points; the uniqueness of the trigonometric interpolation implies the following two cases.

Case 1: $0 \le k \le 2m - s$.

$$a_k(\cos\theta_i)(\sin\theta_j)^k = 0, \quad b_k(\cos\theta_j)(\sin\theta_j)^k = 0, \qquad 0 \le j \le 2\lambda,$$

where we assume $b_0(\cos\theta) = 0$. Since $\theta_j \in (0, \pi)$, this shows that $a_k(\cos\theta_j) = 0$ and $b_k(\sin\theta_j) = 0$. Consequently, there exist polynomials $a_k^*(t)$ and $b_k^*(t)$, both of degree $n - k - (2\lambda+1)$, such that $a_k(t) = \prod_{j=0}^{2\lambda}(t - \cos\theta_j)a_k^*(t)$ and $b_k(t) = \prod_{j=0}^{2\lambda}(t - \cos\theta_j)b_k^*(t)$.

Case 2: $2m - s + 1 \le k \le m$.

$$\begin{aligned} a_k(\cos\theta_j)(\sin\theta_j)^k + u_{2m-k+1}(\cos\theta_j)(\sin\theta_j)^{2m-k+1} &= 0, \\ b_k(\cos\theta_j)(\sin\theta_j)^k + u_{2m-k+1}(\cos\theta_j)(\sin\theta_j)^{2m-k+1} &= 0, \end{aligned} \qquad 0 \le j \le 2\lambda.$$

By Corollary 2.5, this shows that $a_k(t) = b_k(t) = 0$ and $u_{2m-k+1}(t) = v_{2m-k+1}(t) = 0$. The definition of u_{2m-k+1} and v_{2m-k+1} shows then,

$$\begin{aligned} a_{2m-k+1}(\cos\theta)\cos\alpha\pi + b_{2m-k+1}(\cos\theta)\sin\alpha\pi &= 0, \\ a_{2m-k+1}(\cos\theta)\sin\alpha\pi - b_{2m-k+1}(\cos\theta)\cos\alpha\pi &= 0, \end{aligned}$$

which implies that $a_{2m-k+1}(t) = b_{2m-k+1}(t) = 0$.

Together these two cases show that we have the following factorization,

$$\begin{aligned} \widetilde{T}(\theta, \phi) &= \prod_{j=0}^{2\lambda}(\cos\theta - \cos\theta_j) \\ &\times \Big(a_0^*(\cos\theta) + \sum_{j=1}^{2m-s} \big[a_k^*(\cos\theta)(\sin\theta)^k \cos k\phi + b_k^*(\cos\theta)(\sin\theta)^k \sin k\phi\big]\Big), \end{aligned}$$

which completes the proof. □

The fact that the interpolation points are equidistant on the circle is essential in this theorem. One important property of the factorization is that it allows us to repeat the argument to get a complete factorization of the polynomial.

THEOREM 2.8. *Let n and σ be positive integers. Let $\lambda_1, \ldots, \lambda_\sigma$ be nonnegative integers. Define $n_k = n_{k-1} - (2\lambda_k + 1)$ for $1 \le k \le \sigma$ with $n_0 = n$. Assume that $n_k \ge 0$ for $1 \le k \le \sigma - 1$. If $T \in \Pi_n(S^2)$ satisfies*

$$\widetilde{T}(\theta_{j,k}, \phi_{i,k}) = 0, \quad 0 \le j \le 2\lambda_k, \ 0 \le i \le 2(n_{k-1} - \lambda_k), \ 1 \le k \le \sigma,$$

where $\theta_{j,k}$, $0 \le j \le 2\lambda_k$ and $1 \le k \le \sigma$, are distinct numbers in $(0,\pi)$ and $\phi_{i,k} \in \Theta_{\alpha_k, n_{k-1}-\lambda_k}$ with $\alpha_k \in [0,2)$, then there exists a polynomial $T^* \in \Pi_{n_\sigma}(S^2)$ such that

$$T(x,y,z) = \prod_{k=1}^{\sigma}\prod_{j=0}^{2\lambda_k}(z - \cos\theta_{j,k})T^*(x,y,z).$$

In particular, $T(x,y,z) \equiv 0$ if $n_\sigma < 0$.

Proof. We apply the factorization in the theorem repeatedly with $s = n_{k-1}$, $m = n_{k-1} - \lambda_k$ and $\lambda = \lambda_k$ for $k = 1, 2, \ldots, \sigma$. □

We note that the interpolation points in the corollary are located on σ groups of latitudes $\{S^2(z_{j,k}) : 0 \le j \le 2\lambda_k\}$, $1 \le k \le \sigma$ and $z_{j,k} = \cos\theta_{j,k}$, and latitudes in different groups contain different number of nodes. More precisely, each of the latitudes in the k-th group, $S^2(z_{0,k}), S^2(z_{1,k}), \ldots, S^2(z_{2\lambda_k,k})$, contains $2(n_{k-1}-\lambda_k)+1$ equidistant points.

3. Interpolation on the unit sphere. Our main result on interpolation follows from the result on factorization. The following formula is used to show that the interpolation condition matches the dimension of the polynomial space:

$$\dim \Pi_s(S^2) = \dim \Pi_{s-2\lambda-1}(S^2) + (2\lambda+1)(2s-2\lambda+1). \tag{3.1}$$

THEOREM 3.1. *Let n and σ be positive integers, such that $n+1-\sigma$ is an even integer and $\sigma \le n+1$. Let $\lambda_1, \ldots, \lambda_\sigma$ be nonnegative integers such that*

$$\lambda_1 + \ldots + \lambda_\sigma = \frac{n+1-\sigma}{2}. \tag{3.2}$$

Define $n_k = n_{k-1} - (2\lambda_k + 1)$ for $1 \le k \le \sigma - 1$ with $n_0 = n$. Let

$$\widetilde{X} = \{(\theta_{j,k}, \phi_{i,k}) : 0 \le j \le 2\lambda_k,\ 0 \le i \le 2(n_{k-1} - \lambda_k),\ 1 \le k \le \sigma\},$$

where $\theta_{j,k}$, $0 \le j \le 2\lambda_k$ and $1 \le k \le \sigma$, are distinct numbers in $(0, \pi)$ and $\phi_{i,k} \in \Theta_{\alpha_k, n_{k-1}-\lambda_k}$ with $\alpha_k \in [0, 2)$. Then the set X solves Problem 1 in $\Pi_n(S^2)$.

Proof. Again it is sufficient to prove that the dimension of $\Pi_n(S^2)$ matches the interpolation conditions, and if $T \in \Pi_n(S^2)$ vanishes on X then $T(x,y,z) \equiv 0$. Under the condition (3.2), it follows that

$$n_\sigma := n_{\sigma-1} - (2\lambda_\sigma + 1) = n - (2\lambda_1 + 1) - \cdots - (2\lambda_\sigma + 1) = -1 < 0.$$

Hence, the factorization theorem in Theorem 2.8 shows that $T(x,y,z) \equiv 0$. Moreover, it follows from the equation (3.1) that

$$\dim \Pi_n(S^2) = (n+1)^2 = \sum_{k=1}^{\sigma} (2\lambda_k + 1)(2n_{k-1} - 2\lambda_k + 1),$$

so that the interpolation condition matches the dimension of $\Pi_n(S^2)$. □

For a fixed n this theorem contains a number of different interpolation processes. In fact, for each positive integer n, the number of sets X contained in Theorem 3.1 depends on the number of nonnegative integer solutions of the equation (3.2), where σ is also a variable which satisfies the condition that $n+1-\sigma$ is a nonnegative even integer. Every such solution of the equation (3.2) leads to a set of interpolation points that solves Problem 1. The number of solutions of such an equation grows exponentially as n goes to infinity. Moreover, the order of $\lambda_1, \ldots, \lambda_\sigma$ matters; that is, different permutations of a solution $\lambda_1, \ldots, \lambda_\sigma$ of the equation (3.2) give different sets of interpolation points.

Among the solutions of the equation (3.2), one extreme case is $\sigma = n+1$, for which the equation has only one solution $\lambda_1 = \ldots = \lambda_{n+1} = 0$. In this case, $n_k = n - k$, and the interpolation points are located on $n+1$ latitudes $S^2(z_0)$, $S^2(z_1), \ldots, S^2(z_n)$ and the latitude $S^2(z_k)$ contains $2k+1$ points. Hence, this is just a special case of Proposition 2.1.

The other extreme case is $\sigma = 1$. In this case, n needs to be even to keep $n+1-\sigma$ even. Assume $n = 2m$. There is only one solution of the equation which is $\lambda_1 = (n-\sigma+1)/2 = m$. We state this case as a corollary.

COROLLARY 3.2. *Let m be a positive integer. Let $\theta_0, \theta_1, \ldots, \theta_{2m}$ be $2m+1$ distinct numbers in $(0,\pi)$. Let X be defined by*

$$\widetilde{X} = \{(\theta_i, \phi_i) : 0 \le j \le 2m,\ 0 \le i \le 2m\},$$

where $\phi_i \in \Theta_{\alpha,m}$ with $\alpha \in [0, 2\pi)$. Then X solves Problem 1 in $\Pi_{2m}(S^2)$.

In this case, the interpolation points are located on $2m+1$ latitudes $S^2(z_i)$ with $z_i = \cos\theta_i$, each latitude has $2m+1$ equidistant points. In the special case that $\{\theta_0, \ldots, \theta_{2m}\}$ are symmetric (that is, $\theta_{2m-i} = \pi - \theta_i$), the corollary has been established in [14] using the fact that the uniqueness of the interpolation on S^2 of the symmetric set can be related to the interpolation on the unit disc B^2 of $\mathbb{R}^2$, which allows us to use the result in [1] for interpolation on B^2. The first nontrivial case of this corollary is $m = 1$, for which there are 3 latitudes, each with 3 points.

To illustrate the main result, we list all sets of interpolation points contained in Theorem 3.1 for small n. For $n = 1$, there is only one set, for which points are located on two latitudes with 3 points and 1 point, respectively, which is a special case of Proposition 2.1. The case $n = 2$ contains two cases, $\sigma = 1$ and $\sigma = 3$; the case $\sigma = 1$ is a special case of Corollary 3.2 with 3 latitudes and each with 3 points; the case $\sigma = 3$ is a special case of Proposition 2.1 of 3 latitudes with 5, 3, 1 points, respectively. In general, for each σ there can be multiple solutions of (3.2) and the order matters. The cases of $n = 3$ and $n = 4$ are given below.

EXAMPLE 3.3. $n = 3$. *That $n+1-\sigma$ is even implies that σ can be either 2 or 4. The set X contains 16 points.*

1. *$\sigma = 2 : \lambda_1 + \lambda_2 = 1$ has two solutions (order matters),*
   (a) *$\lambda_1 = 1, \lambda_2 = 0$: 3 latitudes each with 5 points and 1 latitude with one point;*
   (b) *$\lambda_1 = 0, \lambda_2 = 1$: 1 latitude with 7 points and 3 latitudes each with 3 points.*
2. *$\sigma = 4 : \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$ has only one solution, $\lambda_i = 0$. This is the special case of Proposition 2.1 of 4 latitudes with 7, 5, 3, 1 points, respectively.*

EXAMPLE 3.4. $n = 4$. *That $n+1-\sigma$ is even implies that σ can be either 1, 3, or 5. Here X contains 25 points.*

1. *$\sigma = 1 : \lambda_1 = 5$, this is a special case of Corollary 3.2; 5 latitudes each with 5 points.*
2. *$\sigma = 3 : \lambda_1 + \lambda_2 + \lambda_3 = 1$ has three solutions,*
   (a) *$\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0$: 3 latitudes each with 7 points, 1 latitude with 3 points and 1 latitude with one point;*
   (b) *$\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 0$: 1 latitude with 9 points, 3 latitudes each with 5 points and 1 latitude with one point;*
   (c) *$\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 1$: 1 latitude with 9 points, 1 latitude with 7 points and 3 latitudes each with 3 points;*
3. *$\sigma = 5 : \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 = 0$ has only one solution, $\lambda_i = 0$. This is the special case of Proposition 2.1 of 5 latitudes with 9, 7, 5, 3, 1 points, respectively.*

Thus, for $n = 3$, Theorem 3.1 contains 3 sets of 16 points that solve Problem 1, and for $n = 4$ it contains 5 sets of 25 points. The next two cases, $n = 5$ gives 8 sets of 36 points and $n = 6$ gives 13 sets of 49 points. In general, the number of different sets for each degree is a Fibonacci sequence (I thank a referee for pointing out this fact).

PROPOSITION 3.5. *Let γ_n denote the number of different sets in Theorem 3.1 that solves Problem 1 in $\Pi_n(S^2)$. Then $\{\gamma_n\}$ is a Fibonacci sequence: $\lambda_n = \lambda_{n-1} + \lambda_{n-2}$, $\lambda_1 = 1$ and $\lambda_2 = 2$.*

Proof. Denote by Ω_n the set of solutions of the equation $\lambda_1 + \ldots + \lambda_\sigma = (n + 1 - \sigma)/2$, where $\lambda_i \in \mathbb{N}_0$, $\sigma \in \mathbb{N}_0$ and $n + 1 - \sigma$ is even. Let $(\lambda_1, \ldots, \lambda_\sigma) \in \Omega_n$. If $\lambda_\sigma = 0$, then $\lambda_1 + \ldots + \lambda_{\sigma-1} = (n - 1 + 1 - (\sigma - 1))/2$ so that $(\lambda_1, \ldots, \lambda_{\sigma-1}) \in \Omega_{n-1}$ and this defines a one-to-one mapping from Ω_n to Ω_{n-1}. If $\lambda_\sigma > 0$, then $\lambda_1 + \ldots + \lambda_{\sigma-1} + (\lambda_\sigma - 1) = (n - 2 + 1 - \sigma)/2$ so that $(\lambda_1, \ldots, \lambda_{\sigma-1}, \lambda_\sigma - 1) \in \Omega_{n-2}$. Again this defines a one-to-one mapping from Ω_n to Ω_{n-2}. Since $\gamma_n = \#\Omega_n$, this shows that $\lambda_n = \lambda_{n-1} + \lambda_{n-2}$. ☐

Let us consider the interpolation process in Corollary 3.2 again. In this case, the interpolation points are on $2m + 1$ latitudes and each latitude has $2m + 1$ points. This seems to indicate that the interpolation polynomial should be a product type; that is, a product of two interpolation polynomials of one variable. However, since polynomials in $\Pi_n^2(S^2)$ have to have the form of (2.1), this is not the case. To find the formula for the interpolation polynomials, we need to find a formula for interpolation using the Chebyshev system in Corollary 2.6.

If we integrate an interpolation polynomial in Theorem 3.1 over the sphere, we get a cubature formula that is exact for polynomials of degree n; that is, the cubature formula is of degree n. In the case points are those in Corollary 3.2, the formula takes a particularly simple form and can be explicitly given.

PROPOSITION 3.6. *Let m be a positive integer. Let $\theta_0, \ldots, \theta_{2m}$ be distinct numbers in $(0, \pi)$ and let α be a number in $[0, 2)$. Then for all $f \in \Pi_{2m}(S^2)$,*

$$\int_{S^2} f(x, y, z) d\omega = \sum_{j=0}^{2m} \frac{\lambda_j}{2m+1} \sum_{i=0}^{2m} \widetilde{f}(\theta_j, \phi_i), \qquad \phi_i = \frac{(2i + \alpha)\pi}{2m+1},$$

where λ_j are given by

$$\lambda_j = \int_{-1}^{1} \prod_{i=0, i\neq j}^{2m} \frac{t - \cos\theta_i}{\cos\theta_j - \cos\theta_i} dt.$$

Proof. Let the interpolation polynomial T_{2m} be of the form (2.1). We use the quadrature formula

$$\frac{1}{2\pi} \int_0^{2\pi} \tau(t) dt = \frac{1}{2m+1} \sum_{j=0}^{2m} \tau(\theta_j^\alpha), \qquad \theta_j^\alpha = \frac{(2j + \alpha)\pi}{2m+1}, \tag{3.3}$$

which is known to hold for every trigonometric polynomial of degree $2m$. For $\alpha = 0$ this is the classical result in [15, Vol.2, p. 8], and for $\alpha \neq 0$ it follows from

$$\int_0^{2\pi} \tau(\theta + t) dt = \int_0^{2\pi} \tau(t) dt,$$

which holds for every θ and for every trigonometric polynomial τ. Using the formula (3.3) and the interpolation property of T_{2m}, it follows that

$$a_0(\cos\theta_j) = \frac{1}{2\pi} \int_0^{2\pi} \widetilde{T}_{2m}(\theta_j, \phi) d\phi = \frac{1}{2m+1} \sum_{i=0}^{2m} \widetilde{f}(\theta_j, \phi_i), \qquad \phi_i = \frac{(2i + \alpha)\pi}{2m+1}$$

for every θ_j, $0 \le j \le 2m$. Consequently, a_0 is uniquely determined by these interpolation conditions. It follows that

$$a_0(t) = \sum_{j=0}^{2m} \left(\frac{1}{2m+1} \sum_{i=0}^{2m} \widetilde{f}(\theta_j, \phi_i) \right) \ell_j(t), \qquad \ell_j(t) = \prod_{i=0, i\ne j}^{2m} \frac{t - \cos\theta_i}{\cos\theta_j - \cos\theta_i}.$$

Using the change of variable formula

$$\int_{S^2} f(x,y,z) d\omega = \int_0^\pi \int_0^{2\pi} \widetilde{f}(\theta,\phi) \sin\theta d\theta d\phi,$$

the integral of T_{2m} over the surface of the sphere is equal to

$$\int_{S^2} T_{2m}(x,y,z) d\omega = 2\pi \int_0^\pi a_0(\cos\theta) \sin\theta d\theta = 2\pi \int_{-1}^1 a_0(t) dt.$$

The stated formula follows from the formula of $a_0(t)$ derived above. □

In particular, if $\cos\theta_j$ are chosen so that λ_j are nonnegative, then the formula is a nonnegative cubature formula. For example, this holds if $\cos\theta_j$ are the zeros of the Legendre polynomial P_{2m+1} of degree $2m+1$ or zero of quasi Legendre orthogonal polynomial $P_{2m+1} + \alpha P_{2m}$ with mild conditions imposed on $\alpha \in \mathbb{R}$. The nodes of such a formula are located on circles, just as the usual product type formula. However, these cubature formulae are different from the usual product formulae. As it is well-known ([12]) that the the usual way of deriving the product formula is to treat it as the product of Gaussian quadrature for trigonometric polynomials and the Gaussian quadrature formula for the unit weight on $[-1,1]$; the result works for all linear combinations of $\cos j\theta(a_k \cos k\phi + b_k \sin k\phi)$ of degree $2m+1$, which contains the spherical polynomials of degree $2m+1$ as a subset. In our case, the cubature formula is derived by interpolation, and it holds for spherical polynomials of degree up to $2m$, which are trigonometric polynomials that are of the special form (2.1). Our formula uses twice as many nodes as the usual product formula, but we still have the freedom of choosing the latitudes (that is, θ_j) on which the interpolation points lie and, in the case of several groups of latitutes, the rotations (that is, α_k) of points on the latitudes for different groups. A proper choice of θ_j and α_k may lead to some cubature formula of higher degree. Furthermore, every set of interpolation points in Theorem 3.1 leads to a cubature formula of degree n, and one can ask the question of finding a cubature formula of the highest degree by choosing proper θ_j and α_k.

4. Trigonometric Interpolation. In order to construct a formula for the interpolation polynomial in Theorem 3.1, we need to understand the interpolation by trigonometric polynomials in Corollary 2.6. It turns out that this interpolation is closely related to a nontrivial Hermite-Birkhoff interpolation problem by trigonometric polynomials, which we discuss in this section.

Let $\mathcal{T}_n$ denote the space of trigonometric polynomials of degree n,

$$\mathcal{T}_n = \operatorname{span}\left\{ a_0 + \sum_{k=1}^n (a_k \cos\theta + b_k \sin\theta) : a_k, b_k \in \mathbb{R} \right\}.$$

A Birkhoff interpolation problem is usually described using the notion of *incidence matrices*, which are matrices whose entries are 0 and 1. Let $E = (e_{l,j})$ be such a matrix with s rows $l = 1,2,\ldots,s$ and n columns with $j = 0,1,\ldots,n-1$. Then

$$T^{(j)}(t_l) = f_{l,j}, \qquad \text{if} \quad e_{l,j} = 1, \quad T \in \mathcal{T}_n,$$

describes an interpolation process on the points $t_1, \ldots, t_s$. An incidence matrix E of $s \times n$ is said to satisfy the Pólya condition if

$$\sum_{k=1}^{j} \sum_{l=1}^{s} e_{l,k} \geq j+1, \qquad j = 0, 1, \ldots, n.$$

A sequence in an incidence matrix is a sequence of consecutive 1's in a row of E, say the l-th row with $e_{l,k} = 1$ for $k = i+1, \ldots, i+j$, $e_{l,i} = 0$ and $e_{l,i+j+1} = 0$, and it is an odd sequence if j is odd. A supported sequence is a sequence such that there are nonzero elements of E in both its upper and lower left sides, that is, there are $e_{i_1,j_1} = 1$ and $e_{i_2,j_2} = 1$ with $i_1 < l$, $i_2 > l$, $j_1 \leq i$ and $j_2 \leq i$. For example, the matrix

$$E = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

has, in its second row, two supported odd sequences of length 3 and 1, respectively. Since the trigonometric polynomials are periodic, the interpolation matrix should be considered periodic: Its last row should precede the first row; that is, the points in Θ are considered to be circular in the sense that the last point θ_s is in front of the first point θ_1. For example, if the matrix E in the above example is an incidence matrix for trigonometric interpolation, then its first row contains an odd supported sequence. Thus, for the interpolation by trigonometric polynomials, the supported sequence is a sequence that does not start from the first row, assuming that the matrix satisfies the Pólya condition. Although taking derivatives does not reduce the degree of a trigonometric polynomial, the analog of the Atkinson-Sharma theorem ([8] or [9, p. 23]) still holds giving the above understanding of the support sequence. The theorem states that a Birkhoff interpolation is poised if all its odd supported sequences begin in column 0 and E satisfies the Pólya condition.

We consider the following Hermite-Birkhoff interpolation problem:

Problem 2. Let m be a positive integer. Let $\Theta = \{\theta_i : 1 \leq i \leq 2m+1\}$ be a set of distinct points in $(0, \pi)$. Let k be an integer such that $1 \leq k \leq 2m$. Find conditions on Θ such that the interpolation problem

$$\begin{aligned} &T(\theta_i) = f_i, \qquad i = 1, 2, \ldots 2m+1, \\ &T^{(j)}(0) = g_{0,j}, \quad T^{(j)}(\pi) = g_{\pi,j}, \quad j = 0, 1, \ldots, k-2, k-1 \quad \text{and} \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad j = k+1, k+3, \ldots, 2m-k-1, \end{aligned}$$

has a unique solution in $\mathcal{T}_{2m}$ for any data f_i, $g_{0,j}$ and $g_{\pi,j}$.

Let Θ be the set as above. Assume that $0 < \theta_1 < \theta_2 < \ldots < \theta_{2m+1} < \pi$ and define $\theta_0 = 0$ and $\theta_{2m+2} = \pi$. Then the above problem is described by the incidence matrix E with $2m+3$ rows, whose elements $e_{l,j}$ are defined by

$$e_{0,j} = e_{2m+2,j} = 1, \qquad 0 \leq j \leq k-1 \ \text{ and } \ j = k+1, k+3, \ldots, 2m-k-1,$$

and $e_{l,0} = 1$ for $l = 1, 2, \ldots, 2m+1$; all other $e_{l,j}$ are 0. By the definition, there are exactly $2(2m)+1$ interpolation conditions, which is the same as the dimension of $\mathcal{T}_{2m}$. Clearly, the matrix E for this interpolation satisfies the Pólya condition. This matrix, however, has many odd supported sequences that do not begin in column 0; the analog of Atkinson-Sharma theorem does not apply. It turns out that Corollary 2.5 can be used to show that it is poised if Θ is a subset of $(0,\pi)$. The key ingredient is the following lemma.

LEMMA 4.1. *Let m be a positive integer. The trigonometric polynomial $T \in \mathcal{T}_{2m}$ satisfies the conditions*

$$T^{(j)}(0) = T^{(j)}(\pi) = 0, \quad j = 0, 1, \ldots, k-2, k-1 \quad \text{and} \\ j = k+1, k+3, \ldots, 2m-k-1,$$

for a fixed integer k, $0 \le k \le 2m$, if and only if T takes the form

$$T(\theta) = (\sin\theta)^k p(\cos\theta) + (\sin\theta)^{2m-k+1} q(\cos\theta), \tag{4.1}$$

where p is a polynomial of degree $2m-k$ and q is a polynomial of $k-1$.

Proof. First assume that T is of the special form (4.1). Let $T_1(\theta) = (\sin\theta)^k p(\cos\theta)$ and $T_2(\theta) = (\sin\theta)^{2m-k+1} q(\cos\theta)$. Since T_1 is an odd function if k is odd and an even function if k is even, and odd trigonometric polynomial vanishes at 0 and π, it follows from taking derivatives that

$$T_1^{(j)}(0) = T_1^{(j)}(\pi) = 0, \quad 0 \le j \le k-1 \text{ and } j = k+1, k+3, k+5, \ldots.$$

Similarly, since T_2 is odd if k is odd and even if k is even, it follows that

$$T_2^{(j)}(0) = T_2^{(j)}(\pi) = 0, \quad 0 \le j \le 2m-k \text{ and } j = 2m-k+2, 2m-k+4, \ldots.$$

Since $T = T_1 + T_2$ and $0 \le k \le m$, the stated result follows from the above two displayed equations.

On the other hand, the conditions $T^{(j)}(0) = T^{(j)}(\pi) = 0$ for $0 \le j \le k-1$ imply that

$$T(\theta) = (\sin\theta)^k S(\theta), \qquad S \in T_{2m-k}.$$

This can be considered as a simple consequence of the fact that the Hermite interpolation problem is unique for the trigonometric interpolation (consider the Hermite interpolation conditions on 0 and π, together with Lagrange interpolation on $2(2m-k)+1$ distinct points). Applying the Leibniz rule to T implies that

$$T^{(k+2l-1)}(\theta) = \sum_{j=0}^{k+2l-1} \binom{k+2l-1}{j} S^{(j)}(\theta) \frac{d^{k+2l-1-j}}{d\theta^{k+2l-1-j}} (\sin\theta)^k.$$

Since $(d^{k+2l-1-j}/d\theta^{k+2l-1-j})(\sin\theta)^k$ is even if j is odd and odd if j is even, it follows that these terms are nonzero at $\theta = 0$ or $\theta = \pi$ only if $j = 1, 3, 5, \ldots, 2l-1$, where $l = 1, 2, 3, \ldots, m-k$. Consequently, using induction on l shows that the conditions $T^{(j)}(0) = T^{(j)}(\pi) = 0$ for $j = k+1, k+3, \ldots, 2m-k-1$ are equivalent to the following conditions on S,

$$S^{(2l-1)}(0) = S^{(2l-1)}(\pi) = 0, \qquad l = 1, 2, 3, \ldots, m-k. \tag{4.2}$$

Since $S \in T_{2m-k}$, we can write $S(\theta) = a_0 + \sum_{j=1}^{2m-k} (a_j \cos j\theta + b_j \sin j\theta)$. Because the conditions (4.2) involve only odd derivatives, it applies only to the odd part of S. Hence, we only need to show that if $S_o(\theta) := \sum_{j=1}^{2m-k} b_j \sin j\theta$ satisfies (4.2), then $S_o(\theta) = (\sin\theta)^{2m-2k+1} q(\cos\theta)$. Since S_o automatically satisfies $S_o^{(2j)}(0) = S_o^{(2j)}(\pi) = 0$ for all j, it follows that it satisfies the Hermite interpolation conditions $S_o^{(j)}(0) = S_o^{(j)}(\pi) = 0$ for $0 \le j \le 2m-2k$. Consequently, it follows that $S_o(\theta) = (\sin\theta)^{2m-2k} R(\theta)$, $R \in \mathcal{T}_{k+2m}$. But S_o is odd, so must be R. Consequently, we can write $R(\theta) = \sin\theta q(\cos\theta)$, so that $S(\theta) = p(\cos\theta) + (\sin\theta)^{m-2k+1} q(\theta)$, which completes the proof. ☐

THEOREM 4.2. *Let m be a positive integer. Let $\theta_1 < \theta_2 < \ldots < \theta_{2m+1}$ be a set of distinct points in $(0,\pi)$. Then Problem 2 has a unique solution in $\mathcal{T}_{2m}$.*

Proof. Let T be a polynomial that satisfies $T^{(j)}(x_i) = 0$ for $e_{i,j} = 1$. The above lemma shows that T is of the form (4.2) and the problem reduces to showing that if

$$T(\theta) = (\sin\theta)^k p(\cos\theta) + (\sin\theta)^{2m-k+1} q(\cos\theta)$$

vanishes on $\theta_1, \ldots, \theta_{2m+1}$ then $T(\theta) \equiv 0$. This, however, is a consequence of Corollary 2.6. ☐

Acknowledgment. The author thanks referees for their careful review and valuable comments.

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