CONSTRUCTING FULLY SYMMETRIC CUBATURE FORMULAE FOR THE SPHERE

SANGWOO HEO AND YUAN XU

ABSTRACT. We construct symmetric cubature formulae of degrees in the 13-39 range for the surface measure on the unit sphere. We exploit a recently published correspondence between cubature formulae on the sphere and on the triangle. Specifically, a fully symmetric cubature formula for the surface measure on the unit sphere corresponds to a symmetric cubature formula for the triangle with weight function $(u_1 u_2 u_3)^{-1/2}$, where u_1, u_2, and u_3 are homogeneous coordinates.

1. INTRODUCTION

In this paper we construct symmetric cubature formulae on the surface of the sphere S^2 in $\mathbb{R}^3$ by using a correspondence between cubature formulae on the sphere and on the simplex established in [23]. Throughout this paper we denote by Π_n^d the space of polynomials of degree at most n in d variables ($d = 2$ or 3), and we denote by T the triangle with vertices at $(0,0)$, $(1,0)$ and $(0,1)$. Let $W(y_1^2, y_2^2, y_3^2)$ be a weight function defined on $\mathbb{R}^3$, normalized so that $\int_{S^2} W(y_1^2, y_2^2, y_3^2) d\omega = 1$. Associated with W, we define a weight function W_T on the triangle T by

$$(1.1) \quad W_T(u_1, u_2) = 2W(u_1, u_2, 1 - u_1 - u_2)/\sqrt{u_1 u_2 (1 - u_1 - u_2)}, \quad (u_1, u_2) \in T.$$

Then the correspondence between cubature formulae on S^2 and on T states that

Theorem 1.1. *Let W and W_T be defined as above. Suppose that there is a cubature formula of degree M on T given by*

$$(1.2) \qquad \int_T f(u_1, u_2) W_T(u_1, u_2) du_1 du_2 = \sum_{k=1}^{N} \lambda_k f(u_{k,1}, u_{k,2}), \qquad f \in \Pi_M^2,$$

whose N nodes lie on the triangle T. Then there is a cubature formula of degree $2M + 1$ on the unit sphere S^2,

$$(1.3) \qquad \int_{S^2} g(y_1, y_2, y_3) W(y_1^2, y_2^2, y_3^2) d\omega$$

$$= \sum_{k=1}^{N} \lambda_k \sum_{\varepsilon_i = \pm 1} g(\varepsilon_1 v_{k,1}, \varepsilon_2 v_{k,2}, \varepsilon_3 v_{k,3})/2^{a_k}, \quad g \in \Pi_{2M+1}^3,$$

where a_k is the number of nonzero elements among $v_{k,1}, v_{k,2}$ and $v_{k,3}$, and the nodes $(v_{k,1}, v_{k,2}, v_{k,3}) \in S^2$ are defined in terms of $(u_{k,1}, u_{k,2})$ by

$$(1.4) \qquad (v_{k,1}, v_{k,2}, v_{k,3}) = (\sqrt{u_{k,1}}, \sqrt{u_{k,2}}, \sqrt{1 - u_{k,1} - u_{k,2}}).$$

On the other hand, if there exists a cubature formula of degree $2M+1$ on S^2 in the form of (1.3), then there is a cubature formula of degree M on the simplex T in the form of (1.2) whose nodes $(u_{k,1}, u_{k,2}) \in T$ are defined by $(u_{k,1}, u_{k,2}) = (v_{k,1}^2, v_{k,2}^2)$.

The formula (1.3) is invariant under the change of signs, or invariant under the group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. The theorem establishes the equivalence between (1.2) and (1.3). In [23] this theorem is proved more generally for formulae on the sphere S^d and the simplex Σ^d for all d. For $d = 2$ we have used the more customary notation T for the simplex Σ^2. When $W(\mathbf{y}) = 1/4\pi$ is the reciprocal of the surface area of S^2, the corresponding weight function on T is the multiple of the weight function $(u_1 u_2 (1 - u_2 - u_3))^{-1/2}$, which we will denote by W_0; that is,

$$(1.5) \qquad W_0(u_1, u_2) = (u_1 u_2 (1 - u_1 - u_2))^{-1/2}/2\pi, \qquad (u_1, u_2) \in T.$$

In the following section we adopt a method of Lyness and Jespersen [15] to construct symmetric cubature formulae on T, which are formulae that are invariant under the symmetric group of the triangle, and use Theorem 1.1 to generate cubature formulae on the sphere. When the formula (1.2) on T is symmetric, the corresponding formula (1.3) in Theorem 1.1 is invariant under the octahedral group, which is the symmetric group of the unit cube $\{\pm 1, \pm 1, \pm 1\}$ in $\mathbb{R}^3$. In this case, the formula (1.3) is of the form

$$
\begin{aligned}
(1.6) \qquad & \int_{S^2} g(y_1, y_2, y_3) W(y_1^2, y_2^2, y_3^2) d\omega \\
& = \sum_{k=1}^{N} \mu_k \sum_{\sigma} \sum_{\varepsilon_i = \pm 1} g(\varepsilon_1 v_{k,\sigma_1}, \varepsilon_2 v_{k,\sigma_2}, \varepsilon_3 v_{k,\sigma_3}), \quad g \in \Pi_{2M+1}^3,
\end{aligned}
$$

where the second sum is taken over all permutations of $\sigma = (\sigma_1, \sigma_2, \sigma_3)$, and $\mu_k = \lambda_k/(b_k \cdot 2^{a_k})$ with $b_k = \sum_{\sigma} 1$ for the corresponding point. Formulae of this type have been constructed by Lebedev [10]–[13]. They are called *fully symmetric* in [20] and [7], and have been studied for S^d in [8], which contains another correspondence between fully symmetric formulae on S^d and cubature formulae on Σ^d, namely, a correspondence between the consistent rule structures on these two regions.

Numerical integration on the sphere has attracted a lot of attention; we refer to [1], [2], [4], [10]–[13], [17]–[20] and the references there. Most formulae have been constructed by making use of symmetry to reduce the number of moment equations that have to be solved (see, for example, Sobolev [19] and McLaren [17]). The fundamental result of Sobolev states that a cubature formula invariant under a finite group is exact for all polynomials in a subspace $\mathcal{P}$ if, and only if, it is exact for all polynomials in $\mathcal{P}$ that are invariant under the same group. The group that has been employed previously in this context is the octahedral group; Lebedev constructed in [10]–[13] cubature formulae of degree up to 59, many of which have the smallest number of nodes among all formulae that are known. Working with symmetric cubature formulae on T, we are able to find many formulae on S^2 of a structure that Lebedev did not consider (see the following section). There are also formulae that are invariant under the icosahedral group, which have, however, no corresponding formulae on the triangle, since they are not symmetric under

$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ in the first place. We refer to [4], [18], [20] and the references there for other papers that deal with cubature formulae on the sphere; see also [1], in which formulae are constructed making use of symmetry and a Taylor expansion formula.

2. Symmetric formulae on T
and fully symmetric formulae on S^2

In this section, we consider symmetric formulae with respect to the weight function W_0 on T, which correspond to cubature formulae with octahedral symmetry on S^2. In the first part of the section we present a method of constructing symmetric formulae given by Lyness and Jespersen in [15]. Our findings of cubature formulae are discussed in subsection 2.2, and we discuss the numerical computation in subsection 2.3.

2.1. Symmetric formulae on the triangle. Instead of T, Lyness and Jespersen used the equilateral triangle

$$\triangle = \{(x,y) : x \leq 1/2, \quad \sqrt{3}y - x \leq 1, \quad -\sqrt{3}y - x \leq 1\},$$

whose symmetric group $\mathcal{S}_3(\triangle)$ is generated by a rotation through an angle $2\pi/3$ and a reflection about the x-axis. The triangle T can be transformed into $\triangle$ by the affine transformation

$$\varphi : (x_1, y_1) \in T \mapsto (x, y) \in \triangle,$$
(2.1)
$$x = 3(x_1 + x_2)/2 - 1, \quad y = \sqrt{3}(x_2 - x_1)/2.$$

It is easy to see that invariance is preserved under φ; in particular, if a function f defined on $\triangle$ is invariant under $\mathcal{S}_3(\triangle)$, then the function $f \circ \varphi$ defined on T is invariant under $\mathcal{S}_3(T)$. The weight function W_0 on T becomes

$$W_\triangle^0(x,y) = 3^{-3/2}((1+x)^2 - 3y^2)^{-1/2}(1-2x)^{-1/2}/(2\pi).$$

A basis for the class of $\mathcal{S}_3(\triangle)$-invariant polynomials of degree at most n can be written down in terms of the polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$ as follows:

$$(2.2) \qquad r^{2i}(r^3 \cos 3\theta)^j, \qquad 0 \leq 2i + 3j \leq n.$$

Moreover, working with functions $g(r, \theta) = f(r\cos\theta, r\sin\theta)$ in polar coordinates, a basic invariant cubature formula takes the form

$$Q(r, \theta)g = \frac{1}{6} \sum_{j=1}^{3} \left\{ g\left(r, \theta + \frac{2\pi j}{3}\right) + g\left(r, -\theta + \frac{2\pi j}{3}\right) \right\},$$

which is just a sum over the $\mathcal{S}_3(\triangle)$-orbit of the point (r, θ). Because of the invariance of $Q(r, \theta)$, we assume that r can take negative values and $0 \leq \theta < \pi/3$. Three distinct types of orbits occur according to whether $r = 0$ (center of triangle); $r \neq 0$, $\cos 3\theta = 1$ (median of triangle); or $r \neq 0$, $\cos 3\theta \neq 1$. These three types are denoted as holistic type 0, holistic type 1 and holistic type 2, whose corresponding $Q(r, \theta)g$ requires 1, 3, or 6 function evaluations, respectively. Let n_i denote the number of orbits of type i in a symmetric cubature formula. The general symmetric cubature formula takes the form

$$(2.3) \qquad Q(g) = n_0 \lambda_0 g(0,0) + \sum_{i=1}^{n_1} \lambda_i Q(r_i, 0)g + \sum_{i=n_1+1}^{n_1+n_2} \lambda_i Q(r_i, \theta_i)g.$$

The number of nodes of this formula, denoted by $\mu(Q)$, is $\mu(Q) = n_0 + 3n_1 + 6n_2$. It is shown in [15] that the cubature formula $Q(g)$ is of degree M if its nodes and weights satisfy the following system of equations:

$$\lambda_0 + \sum_{i=1}^{n_1} \lambda_i + \sum_{i=n_1+1}^{n_1+n_2} \lambda_i = v_{0,0},$$

$$(2.4) \qquad \sum_{i=1}^{n_1} \lambda_i r_i^j + \sum_{i=n_1+1}^{n_1+n_2} \lambda_i r_i^j \cos 3k\theta_i = v_{j,3k}, \quad 2 \leq j \leq M, \ k = k_0,$$

$$\sum_{2i=n_1+1}^{n_1+n_2} \lambda_i r_i^j (\cos 3k_0\theta_i - \cos 3k\theta_i) = v_{j,3k}, \quad j = 6, \text{ or } 8 \leq j \leq M,$$

$$6 \leq 3k \leq j, \ j+k \text{ even},$$

where $k_0 = 0$ if j is even and $k_0 = 1$ if j is odd, and the numbers $v_{j,3k}$ are defined by

$$(2.5) \qquad v_{j,3k} = \int_\triangle r^{j+1} \cos 3k\theta \, W_\triangle^0(r\cos\theta, r\sin\theta) dr d\theta,$$

where the integral is over the region defined by $(r\cos\theta, r\sin\theta) \in \triangle$. For each M, the system contains

$$E(M) = [(M^2 + 6M + 12)/12]$$

equations, where $[x]$ denote the greatest integer less than or equal to x.

It is often useful to construct cubature formulae that have some nodes on the edges or at the vertices of the triangle. To describe such a formula, we use the cytolic types (orbits) introduced in [15]. The holistic type 1 ($r \neq 0$, $\cos 3\theta = 1$) orbits may be sub-partitioned into cytolic type 1: $r = -1$ (vertices), cytolic type 2: $r = 1/2$ (mid point of edges), and cytolic type 3: $-1 < r < 1/2$. The holistic type 2 ($r \neq 0$, $\cos 3\theta \neq 0$) orbits may be sub-partitioned into cytolic type 4: $r\cos\theta = 1/2$ (nodes on the edges but not at mid point of the edges nor at vertices), and cytolic type 5: $r\cos\theta \neq 1/2$ (interior points). From here on we treat only cytolic rules. The structure of a cytolic rule is denoted by $[m_0; m_1, m_2, m_3; m_4, m_5]$, where m_j denotes the number of components of cytolic type j contained in the rule. In particular, we have $n_0 = m_0$, $n_1 = m_1 + m_2 + m_3$, and $n_2 = m_4 + m_5$. We note that m_0, m_1 and m_2 can take only values either 0 or 1.

According to Theorem 1.1, a cytolic cubature formula of structure $[m_0; m_1, m_2, m_3; m_4, m_5]$ leads to a cubature formula on S^2 whose number of nodes is equal to

$$(2.6) \qquad N(S^2) = 8m_0 + 6m_1 + 12m_2 + 24m_3 + 24m_4 + 48m_5.$$

The nonlinear system of equations (2.4) remains in the same form for the cytolic type formulae; we only need to assign proper values of certain r_i and θ_i according to the given type. To form the nonlinear system of equations (2.4), we choose m_0 and m_i so that the number of parameters matches the number of equations. For the structure $[m_0; m_1, m_2, m_3; m_4, m_5]$, this means

$$(2.7) \qquad m_0 + m_1 + m_2 + 2m_3 + 2m_4 + 3m_5 = [(M^2 + 6M + 12)/12],$$

where M, as before, is the degree of the cubature formula. For each fixed M there may be a number of integer solutions to the above equation, leading to different types of cubature formulae. In this regard, the *consistency conditions* are very

useful. Following the argument in [15] for the holistic type, the conditions for the cytolic type are

$$2m_4 + 3m_5 \geq E(M-6),$$

(2.8)
$$m_1 + m_2 + 2(m_3 + m_4) + 3m_5 \geq E(M) - 1,$$

$$m_0 + m_1 + m_2 + 2(m_3 + m_4) + 3m_5 \geq E(M).$$

They are also included in the conditions found in [7] for the d-dimensional simplex. These conditions ensure that, within any subsystem of (2.4), the number of unknown parameters cannot be less than the number of equations. Another useful restriction is as follows.

Theorem 2.1. *A formula of degree M is of structure $[m_0; m_1, m_2, m_3; m_4, m_5]$ only if*

(2.9)
$$m_5 > \begin{cases} (M-9)/4, & \text{if } m_4 \neq 0 \text{ and } M \geq 9, \\ (M-6)/4, & \text{if } m_4 = 0 \text{ and } M \geq 6, \\ (M-3)/4, & \text{if } m_3 = 0 \text{ and } M \geq 3. \end{cases}$$

Proof. Let ℓ_i, $i = 1, 2, 3$, be the linear polynomials such that $\ell_i = 0$ give the equations of the sides, and we choose the sign so that the ℓ_i are nonnegative on $\triangle$. Let h_i, $i = 1, 2, 3$, be the linear polynomials such that $h_i = 0$ give the equations of the medians of $\triangle$. Furthermore, let $\mathbf{x}_i = (r_i \cos \theta_i, r_i \sin \theta_i)$ be points of type 3, and let g_i, $i = 1, 2, \ldots, m_5$, be the quadratic polynomials so that $g_i = 0$ gives the equation of the circle that has center at origin and radius r_i. If $m_4 \neq 0$, then the polynomial

$$\ell_1 \ell_2 \ell_3 h_1^2 h_2^2 h_3^2 g_1^2 \ldots g_{m_5}^2$$

will vanish on all nodes of the formula. Since the polynomial is positive on $\triangle$, its degree has to be bigger than the degree of the cubature formula, which leads to the desired inequality. If $m_4 = 0$, then the factors $\ell_1 \ell_2 \ell_3$ can be dropped from the polynomial, leading to the desired inequality in this case. If $m_3 = 0$, then $h_1^2 h_2^2 h_3^2$ can be dropped from the polynomial. $\qquad\square$

This theorem and its proof are extensions of the result in [15, p. 26], which deals with the cases of $M = 5, 6, 9$. There are other conditions that can be derived this way; for example, if both m_3 and m_4 are zero, then $m_5 > M/4$. For fixed M, it is possible to identify all possible integer solutions of (2.7) which also satisfy the restrictions (2.8) and (2.9); the number of the solutions, however, is still large even for moderate M. For more general conditions of this type, we refer to [7] and [16].

Particular choices of structure lead to a system (2.4) that may be split into subsystems, each having fewer independent variables: the smaller size of the subsystem makes them easier to solve. Such a split is possible since the third group of the equations in (2.4) does not contain r_i and λ_i for $i = 1, 2, \ldots, n_1$; and it occurs whenever m_4 and m_5 satisfy the equation

(2.10)
$$2m_4 + 3m_5 = E(M-6) = [(M^2 - 6M + 12)/12],$$

because the third group of equations contain $E(M-6)$ independent parameters. It is not hard to check that the integer solutions of the above equation exist for every $M \geq 7$, except $M = 10$; hence, the splitting occurs for each $M \neq 10$. One important class of formulae that admits the splitting corresponds to the cubature formulae constructed by Lebedev in [10]–[13] on S^2 with octahedral symmetry. Apart from

a few lower degree cases, Lebedev considers the formulae on S^2 that correspond to the structures

(2.11) $[1; 1, 0, 3m; m, m(m-1)]$ and $[1; 1, 1, 3m+1; m, m^2]$,

which are of degree $6m+2$ and $6m+5$, respectively; and he has constructed formulae for $m = 1, 2, 3, 4$.

2.2. Fully symmetric cubature formulae on S^2. We have attempted to find symmetric cubature formulae of degree up to 20 on the triangle. Our strategy for choosing the structure $[m_0; m_1, m_2, m_3; m_4, m_5]$ is as follows. We search for types whose corresponding formulae on S^2 have fewer nodes. This means finding m_0 and m_i, which satisfy (2.7), (2.8) and (2.9), so that N in (2.6) is minimal or close to minimal. To this end, we choose m_0, m_1 and m_2 with value one whenever possible, and then m_5 as small as possible. As a starting point, we choose m_4 and m_5 to satisfy (2.10), so that the system (2.4) is split into subsystems. There are some nonlinear systems for which we found no solution. For each $M \leq 20$, however, we found at least one cubature formula that has all of its nodes inside $\triangle$ and contains no negative weights; these correspond to cubature formulae on S^2 of degree up to 41. Note that, when such a formula for $\triangle$ has a node outside $\triangle$, Theorem 1.1 cannot be exploited to find a cubature formulae on S^2.

We report our findings as fully symmetric cubature formulae on S^2 and list the results in Table 2.1 below. Each formula is identified by its structure $[m_0; m_1, m_2, m_3; m_4, m_5]$, and we give its number of nodes. If a formula has all positive weight, we write P in the last column, otherwise, we write N.

The structures marked by [S] correspond to formulae in Stroud's book [20], structures marked by [L] correspond to formulae on S^2 found by Lebedev. The structures [0;1,0,2;1,0] of degree 13 and [1;1,0,2;2,0] of degree 17 have been constructed by Keast in [7], but the numerical values of the nodes and weights are not given there. All other formulae in the table appear to be new; in particular, these include formulae of degrees 21, 31, 33, 37 and 39, where no formulae of the same degree have been published previously, and formulae of degrees 25 and 27 of quality P. Three formulae, of degree 47, 53 and 59, may be found in [12] and [13]. Because the system of equations (2.4) is nonlinear, its solution need not be unique. In the cases of $[0; 0, 0, 3; 1, 2]$ of degree 21 and $[0; 0, 0, 4; 1, 10]$ of degree 39, we found two solutions in each case, and we mark these cases by (2) in the table.

Numerical computation that leads to symmetric cubature formulae for the unit weight function is carried out in [15] for $M \leq 11$ and in [3] for $M \leq 20$. The equations (2.4) in the cases of the unit weight function and the weight function W_0 are of the same form, except that the moments are different, which only changes the right hand sides of the equations. Since the equations are nonlinear, formulae of the same type may possess different quality for different weight functions. For example, for the structure $[1; 0, 0, 6; 1, 2]$ of degree 13, we found a formula for W_0 with some negative weights, while the formula for the unit weight function has all positive weights. The most interesting case, however, is perhaps the structure $[1; 0, 0, 8; 1, 7]$ of degree 19, in which we found no solution for the weight function W_0, while a solution exists for the unit weight function ([3]). This demonstrates once more the well-known phenomenon that the solutions of nonlinear systems may be sensitive to any changes in the value of any incidental constant.

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TABLE 2.1. Fully symmetric cubature formula on S^2

Degrees	Structure	# of Nodes	Quality
3	0;1,0,0;0,0	6	P [S]
	1;0,0,0;0,0	8	P [S]
5	1;1,0,0;0,0	14	P [S]
7	1;1,1,0;0,0	26	P [S]
	1;0,0,1;0,0	32	N [S]
9	1;1,0,0;1,0	38	P [L]
11	1;1,1,1;0,0	50	P [S]
13	1;1,1,1;1,0	74	N [L]
	0;1,0,2;1,0	78	P
15	1;1,0,2;1,0	86	P [L]
	0;1,1,2;1,0	90	P
17	1;1,0,3;1,0	110	P [L]
	1;1,0,2;2,0	110	N
19	1;1,1,3;0,1	146	P [L]
	1;1,1,2;1,1	146	P
	1;0,0,4;0,1	152	P
21	1;1,1,3;1,1	170	N
	1;1,1,2;2,1	170	N
	1;0,0,3;2,1	176	N
	0;0,0,3;1,2	192	P (2)
	1;0,0,2;0,3	200	P
23	1;1,1,4;1,1	194	P [L]
	0;1,0,4;2,1	198	P
	1;0,0,5;1,1	200	P
	1;0,0,4;2,1	200	N
25	1;1,0,5;2,1	230	N [L]
	1;0,0,5;1,2	248	P
	0;0,0,5;0,3	264	P
27	1;1,1,5;1,2	266	N [L]
	1;0,0,6;1,2	272	N
	1;1,0,5;0,3	278	N
	0;0,0,5;1,3	288	P
29	1;1,0,6;2,2	302	P [L]
	0;0,0,6;0,4	336	P
31	1;0,0,4;3,4	368	P
33	1;0,0,6;1,5	416	P
35	1;1,1,7;2,4	434	P [L]
	1;0,0,8;2,4	440	P
37	1;0,0,5;1,8	536	P
39	0;0,0,4;1,10	600	P (2)
41	1;1,0,9;3,6	590	P [L]

2.3. Remarks on numerical computation. The numerical computation was carried out on a DEC Alphastation 500 in double precision, using the DUNLSF Fortran subroutine in the IMSL Math/Library (Visual Numerics, Inc., 1994); however, moments $v_{j,3k}$ in (2.5) were computed exactly using Maple. The subroutine DUNLSF employs iterative techniques which require an initial estimate of the solution. For solving the nonlinear system (2.4), this means that we need to provide initial values for the weights λ_i and for the parameters r_i and θ_i that determine nodes. To determine the initial values, we have followed the strategy in [3] for solving the systems for the unit weight function. The node locations of the formulae for the weight function W_0 appear to be similar to those for the unit weight function: nodes are located closer to the edge of the triangle than the centroid, and are located closer to the median $\theta = \pi/3$ than $\theta = 0$. Our computation shows that whenever a formula of a given type has a solution, then even a rough initial estimate leads to the solution in reasonable computing time. For example, finding a formula of degree 19 needs less than 30 minutes.

For each formula of degree M, we compute the relative error and the absolute error of $\mathcal{I}(f) - \mathcal{I}_n(f)$ for all invariant polynomials f of degree $\leq M$, where $\mathcal{I}(f)$ stands for the integral of f with respect to W_0 on the triangle and $\mathcal{I}_n(f)$ stands for the numerical approximation (in principle but not in practice exact) to this quantity calculated using the cubature formula. We found that

$$\sup\{|\mathcal{I}(f) - \mathcal{I}_M(f)|/\mathcal{I}(f) : f \in \Pi_M^G\} \leq 0.5 \times 10^{-13}$$

for formulae of degree up to 19, but it was ten times larger for the degree 20. The numerical values of the parameters are given to 12 digits. The DUNLSF subroutine solves the nonlinear equations in the least square sense; that is, it finds the minimal solution of $\sum f_i^2(\mathbf{x})$, where $f_i = 0$ are nonlinear equations. In our computation, equations in (2.4) involve high powers of polynomials which, however, seem not to be sensitive to perturbations; for example, for $M = 20$, a perturbation in the 5-th decimal place of our solution did not change the order of the relative error 10^{-12}. For M large, the accuracy of the solution found by the DUNLSF subroutine is limited by the machine accuracy. Because the computer we used has limited precision of 15 digits, we stopped at $M = 21$.

3. Final comments

We comment on some perspectives that are not covered in the present paper.

Remark 3.1. Theorem 1.1 establishes the connection between cubature formulae on Σ^d and $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ symmetric cubature formulae on S^d. In [22] we also establish a connection between cubature formulae on the ball B^d and on S^d, and that connection has been used to construct cubature formulae on S^2 in [5]. Together, these results yield a correspondence between cubature formulae on Σ^d and $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ symmetric formulae on B^d. In particular, a cubature formula for the weight function W_0 on T corresponds to a formula for the weight function $1/\sqrt{1 - x_1^2 - x_2^2}$ on B^2. Thus, the results in Sections 2 and 3 also lead to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric formulae on B^2. On the other hand, those formulae constructed in [5] that are $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric lead to formulae on T. However, not all formulae on B^2 in [5] are fully symmetric. In fact, the correspondence between formulae on B^d and on S^d is not restricted to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric formulae.

Remark 3.2. The connection between cubature formulae on S^d, Σ^d and B^d may be exploited to treat a large class of weight functions. In particular, cubature formulae for the unit weight function on T correspond to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric formulae for $|x_1 x_2 x_3| d\omega$ on S^2 and for a weight function $|x_1 x_2|$ on B^2; and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ symmetric cubature formulae for the unit weight function on B^2 correspond to formulae for $(1/\sqrt{x_1 x_2})$ on T. For examples of formulae for the unit weight function on these domains, see the references in [18], [20].

Remark 3.3. The connection between formulae on the three domains may also be applied in higher dimensions. Although a number of formulae of lower degrees have been constructed for the unit weight function in the literature (see [2], [4], [18], [20]), it may be of interest to construct formulae for the weight function $(u_1 \cdots u_d (1 - u_1 - \ldots - u_d))^{-1/2}$ on Σ^d and use them to generate cubature formulae on S^d. To our knowledge, the calculation of symmetric cubature formulae for this weight function on Σ^d for $d > 2$ has not been undertaken previously, although the consistency conditions are available in [7] and [16]. For the unit weight function, some symmetric formulae of lower degrees on Σ^d have been constructed; see [2], [7], [18], [20] and the references given there.

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APPENDIX

We give the weights and nodes for the cubature formulae described in Section 2. The cubature formulae on S^2 are of the form (1.6) with $W(\mathbf{x}) = 1/4\pi$. Because of the symmetry, for each weight μ_k we need to specify only one node $(v_{k,1}, v_{k,2}, v_{k,3})$. For a formula of structure $[m_0; m_1, m_2, m_3; m_4, m_5]$, the nodes corresponding to m_0, m_1 and m_2 are

$$(\sqrt{1/3}, \sqrt{1/3}, \sqrt{1/3}), \quad (1, 0, 0), \quad (\sqrt{1/2}, \sqrt{1/2}, 0),$$

involving eight, six, and twelve function values, respectively; the weights corresponding to these are denoted by μ_0, μ_1, and μ_2, respectively. Note that some or all of μ_0, μ_1, μ_2 could be zero, which means that the corresponding node does not appear in the formula.

For each formula we specify the value of nonzero μ_i, $i = 0, 1, 2$ first; those that are not given are understood as zero. We then list in tabular form the other nodes $(v_{i,1}, v_{i,2}, v_{i,3})$ and the corresponding weights μ_i starting with $i = 3$ and following the ordering implied in the specified structure. That is, the weights corresponding to all nodes of type 3 are listed first, then all of those of type 4 followed by all those of type 5. Here are the first three tables.

Degree 13: $[0;1,0,2;1,0]$; $N = 78$ $\quad \mu_1 = 0.013866592105$

i	x_i	y_i	z_i	μ_i
3	0.286640146767	0.914152532416	0.286640146767	0.013050931863
4	0.659905001656	0.659905001656	0.359236381200	0.013206423223
5	0.539490098706	0.841991943785	0.0	0.011942663555

.

Degree 15: $[0;1,1,2;1,0]$; $N = 90$

$\mu_1 = 0.013191522874,$ $\mu_2 = 0.011024070845$

i	x_i	y_i	z_i	μ_i
3	0.337785899794	0.878522265967	0.337785899794	0.010538971114
4	0.658511676782	0.658511676782	0.364314072036	0.011656960715
5	0.399194381765	0.916866318264	0.0	0.010660818696

.

Degree 17: $[1;1,0,2;2,0]$; $N = 110$

$\mu_0 = 0.009103396603,$ $\mu_1 = -0.002664002664$

i	x_i	y_i	z_i	μ_i
3	0.357406744337	0.862856209461	0.357406744337	0.010777836655
4	0.678598344546	0.678598344546	0.281084637715	0.009161945784
5	0.542521185161	0.840042120165	0.0	0.009798544912
6	0.222866509741	0.974848972321	0.0	0.009559874447

Numerical values of the weights and abscissas of these and all other rules whose structures are given in Table 2.1 are listed in [6]. A machine-readable version of this technical memorandum is available on request from the authors.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OREGON, EUGENE, OREGON 97403-1222
E-mail address: yuan@math.uoregon.edu

DIVISION OF SCIENCE AND MATHEMATICS, UNIVERSITY OF MINNESOTA-MORRIS, MORRIS, MINNESOTA 56267
Current address: Department of Mathematics, University of Southern Indiana, Evansville, Indiana 47712
E-mail address: sheo@cda.mrs.umn.edu