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FUNK-HECKE FORMULA FOR ORTHOGONAL POLYNOMIALS ON SPHERES AND ON BALLS

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Abstract. Analogues of the Funk-Hecke formula for spherical harmonics are proved for Dunkl's h-harmonics associated to the reflection groups and for orthogonal polynomials related to h-harmonics on the unit ball. In particular, an analogue and its application is discussed for the weight function $(1-|\mathbf{x}|^2)^{\mu-1/2}$ on the unit ball in $\mathbb{R}^d$,

1. Introduction

The Funk-Hecke formula plays an important role in the theory of spherical harmonics. Let $S^d = \{\mathbf{x} : |\mathbf{x}| = 1\}$ be the unit sphere in $\mathbb{R}^{d+1}$, where $|\mathbf{x}|$ is the usual Euclidean norm. We denote by $d\omega$ the surface measure on S^d, and by ω_d the surface area $\omega_d = \int_{S^d} d\omega = 2\pi^{(d+1)/2}/\Gamma((d+1)/2)$. Let $C_n^{(\lambda)}$ denote the Gegenbauer polynomial of degree n, which is orthogonal with respect to $(1-t^2)^{\lambda-1/2}dt$ on $[-1,1]$ (cf. [13, p. 80]). Then the Funk-Hecke formula states that (cf. [8, p. 247] or [11, p. 29]).

Theorem A. *Let f be a continuous function on $[-1,1]$ and let Y_n be a spherical harmonic of degree n. Then for any $\mathbf{x} \in S^d$,*

$$\int_{S^d} f(\langle \mathbf{x}, \mathbf{y}\rangle) Y_n(\mathbf{y}) d\omega(\mathbf{y}) = \lambda_n Y_n(\mathbf{x}),$$

where $\langle \mathbf{x}, \mathbf{y}\rangle$ is the usual inner product in $\mathbb{R}^{d+1}$, and

$$\lambda_n = \frac{\omega_d}{C_n^{((d-1)/2)}(1)} \int_{-1}^1 f(t) C_n^{((d-1)/2)}(t)(1-t^2)^{(d-2)/2} dt \Big/ \int_{-1}^1 (1-t^2)^{(d-2)/2} dt.$$

The purpose of this paper is to prove analogues of the Funk-Hecke formula for Dunkl's h-harmonics associated with reflection groups and for certain orthogonal polynomials on the unit ball. The h-harmonics are homogeneous polynomials that are orthogonal with

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respect to a reflection invariant weight function on S^d. The simplest case is the group $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$, to which the associated weight function is h_κ^2 with

$$h_\kappa(\mathbf{y}) = C_\kappa |y_1|^{\kappa_1} \cdots |y_{d+1}|^{\kappa_{d+1}}, \qquad C_\kappa^2 = \frac{\Gamma(|\kappa|_1 + (d+1)/2)}{2\Gamma(\kappa_1 + 1/2) \cdots \Gamma(\kappa_{d+1} + 1/2)}, \tag{1.1}$$

where $\kappa_i \ge 0$ and $\mathbf{y} \in \mathbb{R}^{d+1}$ and C_κ is chosen so that the integral of h_κ^2 over S^d is one.

The relation between orthogonal polynomials on spheres and on balls has been discussed recently in [17], which implies, in particular, that orthogonal polynomials associated to the weight function

$$W_{\kappa,\mu}(\mathbf{x}) = C_{\kappa,\mu} |x_1|^{2\kappa_1} \cdots |x_d|^{2\kappa_d} (1 - |\mathbf{x}|^2)^{\mu - 1/2}, \tag{1.2}$$

defined on the unit ball $B^d = \{\mathbf{x} : |\mathbf{x}| \le 1\}$ in $\mathbb{R}^d$, where $\kappa_i \ge 0$ and $\mu \ge 0$, are related to the h-harmonics with respect to h_κ^2 with h_κ in (1.1); in particular, $C_{\kappa,\mu} = 2C_\gamma^2$ where C_γ is as in (1.1) with $\gamma = (\kappa, \mu)$. This relation allows us to derive an analogue of the Funk-Hecke formula for orthogonal polynomials with respect to $W_{\kappa,\mu}$. Of particular interest is the formula for the weight function $W_{0,\mu}$, the application of which to the construction of an orthogonal basis will also be discussed.

2. Funk-Hecke Formula for h-harmonics

Let G be a finite reflection group on $\mathbb{R}^{d+1}$ with the set $R_+ = \{\mathbf{v}_i : i = 1, 2, ..., m\}$ of positive roots; assume that $|\mathbf{v}_i| = |\mathbf{v}_j|$ whenever σ_i is conjugate to σ_j in G, where $\sigma_i = \sigma_{\mathbf{v}_i}$, $1 \le i \le m$, are reflections with respect to $\mathbf{v}_i$. For a nonzero vector $\mathbf{v} \in \mathbb{R}^{d+1}$ the reflection $\sigma_{\mathbf{v}}$ is defined by $x\sigma_{\mathbf{v}} := \mathbf{x} - 2(\langle \mathbf{x}, \mathbf{y} \rangle / |\mathbf{v}|^2)\mathbf{v}$, $\mathbf{x} \in \mathbb{R}^{d+1}$. Then G is a subgroup of the orthogonal group generated by the reflections $\{\sigma_i : 1 \le i \le m\}$. Let κ be a nonnegative function defined on R_+ such that $\kappa_i = \kappa_j$ whenever σ_i is conjugate to σ_j in G. The key ingredient of the theory of h-harmonics is a family of commuting first-order differential-difference operators, $\mathcal{D}_i$ (Dunkl's operators), defined by

$$\mathcal{D}_i f(\mathbf{x}) := \partial_i f(\mathbf{x}) + \sum_{j=1}^m k_j \frac{f(\mathbf{x}) - f(\mathbf{x}\sigma_j)}{\langle \mathbf{x}, \mathbf{v}_j \rangle} \langle \mathbf{v}_j, \mathbf{e}_i \rangle, \qquad 1 \le i \le d+1,$$

where ∂_i is the ordinary partial derivative with respect to x_i and $\mathbf{e}_1, \ldots, \mathbf{e}_{d+1}$ are the standard unit vectors of $\mathbb{R}^{d+1}$. The h-harmonics are homogeneous polynomials that satisfy the h-Laplacian equation $\Delta_h P = 0$, where $\Delta_h = \mathcal{D}_1^2 + \cdots + \mathcal{D}_{d+1}^2$. Let $\mathcal{P}_n^{d+1}$ denote the space of homogeneous polynomials of degree n in $d+1$ variables. Define an inner product associated with G and κ

$$(f, g)_{h_\kappa} = \int_{S^d} f(\mathbf{x}) g(\mathbf{x}) h_\kappa^2(\mathbf{x}) d\omega, \qquad h_\kappa(\mathbf{x}) := \mathrm{const} \prod_{i=1}^m |\langle \mathbf{x}, \mathbf{v}_i \rangle|^{\kappa_i}, \tag{2.1}$$

where const is chosen so that $(1,1)_{h_\kappa} = 1$. Then $P \in \mathcal{P}_n$ is an h-harmonic if and only if $(P, g)_{h_\kappa} = 0$ for all polynomials g of degree less than n. The space $\mathcal{H}_n^{d+1}(h_\kappa^2) :=$

$\mathcal{P}_n^{d+1} \cap \ker \Delta_h$ is the space of h-harmonics of degree n; its dimension is $\dim \mathcal{H}_n^{d+1}(h_\kappa^2) = \binom{n+d}{n} - \binom{n+d-2}{n-2}$. There is an intertwining operator V between the commutative algebras generated by the partial derivatives and by the Dunkl operators, which is the unique linear operator defined by

$$V\mathcal{P}_n \subset \mathcal{P}_n, \qquad V1 = 1, \qquad \mathcal{D}_i V = V\partial_i, \qquad 1 \le i \le d+1.$$

The intertwining operator allows us to draw analogues between the theory of ordinary harmonics and the theory of h-harmonics. For example, the reproducing kernel $P_n^h(\cdot,\cdot)$ of the space $\mathcal{H}_n^{d+1}(h_\kappa^2)$ is given in terms of V as ([6, 16])

$$P_n^h(\mathbf{x},\mathbf{y}) = \frac{n+|\kappa|_1+\mu+(d-1)/2}{|\kappa|_1+\mu+(d-1)/2} V[C_n^{(|\kappa|_1+\mu+(d-1)/2)}(\langle \mathbf{x},\cdot\rangle)](\mathbf{y}). \tag{2.2}$$

For $\kappa = 0$, $V = id$, the h-harmonics become the usual harmonic polynomials. For the general theory and many important properties of h-harmonics, we refer to [4-7] and the references there.

In the following we let $w^{(\lambda)}(t) = w_\lambda(1-t^2)^{\lambda-1/2}$, where $w_\lambda = \Gamma(\lambda+1)/(\pi^{1/2}\Gamma(\lambda+1/2))$ is chosen so that $\int_{-1}^1 w^{(\lambda)}(t)dt = 1$. The analogue of the Funk-Hecke formula for h-harmonics is as follows.

Theorem 2.1. *Let f be a continuous function on $[-1,1]$. Let $Y_n^h \in \mathcal{H}_n^{d+1}(h_\kappa^2)$. Then*

$$\int_{S^d} Vf(\langle \mathbf{x},\cdot\rangle)(\mathbf{y})Y_n^h(\mathbf{y})h_\kappa^2(\mathbf{y})d\omega = \lambda_n Y_n^h(\mathbf{x}), \qquad \mathbf{x} \in S^d, \tag{2.3}$$

where λ_n is a constant defined by

$$\lambda_n = \frac{1}{C_n^{(|\kappa|_1+(d-1)/2)}(1)} \int_{-1}^1 f(t) C_n^{(|\kappa|_1+(d-1)/2)}(t) w^{(|\kappa|_1+(d-1)/2)}(t)dt. \tag{2.4}$$

Proof. First we assume that f is a polynomial of degree m. Then we can write f in terms of the Gegenbauer polynomials as

$$\begin{aligned} f(t) &= \sum_{k=0}^m \lambda_k \frac{|\kappa|_1+k+(d-1)/2}{|\kappa|_1+(d-1)/2} C_k^{(|\kappa|_1+(d-1)/2)}(t) \\ &= \sum_{k=0}^m \lambda_k \tilde{C}_k^{(|\kappa|_1+(d-1)/2)}(1)\tilde{C}_k^{(|\kappa|_1+(d-1)/2)}(t), \end{aligned}$$

where $\tilde{C}_k^{(\lambda)}$ stands for orthonormal Gegenbauer polynomials with respect to $w^{(\lambda)}$ on $[-1,1]$, it is a multiple of $C_k^{(\lambda)}$. Using the orthonormality of $\tilde{C}_k^{(\lambda)}$, the coefficients λ_k are given by the formula

$$\lambda_k = \frac{1}{C_k^{(|\kappa|_1+(d-1)/2)}(1)} \int_{-1}^1 f(t) C_k^{(|\kappa|_1+(d-1)/2)}(t) w^{(|\kappa|_1+(d-1)/2)}(t)dt,$$

where we have used the fact that $[\tilde C_k^{(\lambda)}(1)]^2 = [(k+\lambda)/\lambda] C_k^{(\lambda)}(1)$. For $\mathbf{x}, \mathbf{y} \in S^d$, it then follows from (2.2) that

$$Vf(\langle \mathbf{x}, \cdot\rangle)(\mathbf{y}) = \sum_{k=0}^{m} \lambda_k \frac{|\kappa|_1 + k + (d-1)/2}{|\kappa|_1 + (d-1)/2} V C_k^{(|\kappa|_1 + (d-1)/2)}(\langle \mathbf{x}, \cdot\rangle)(\mathbf{y}) = \sum_{k=0}^{m} \lambda_k P_k^h(\mathbf{x}, \mathbf{y}).$$

Since $\mathcal{P}_n^h$ is the reproducing kernel of the space $\mathcal{H}_n^{d+1}(h_\kappa^2)$, it follows that for any $Y_n^h \in \mathcal{H}_n^{d+1}(h_\kappa^2)$ we have

$$\int_{S^d} Vf(\langle \mathbf{x}, \cdot\rangle)(\mathbf{y}) Y_n^h(\mathbf{y}) h_\kappa^2(\mathbf{y}) d\omega = \lambda_n Y_n^h, \qquad \mathbf{x} \in S^d,$$

where we note that if $m < n$, then $\lambda_n = 0$ so that both sides of the equation become zero. Together with the formula for λ_n, this gives the desired formula for f being a polynomial.

Now, if f is a continuous function on $[-1, 1]$, then we choose P_m to be a sequence of polynomials such that P_m converges to f uniformly on $[-1, 1]$. By [12, Theorem 1.2], the intertwining operator V is positive and $|V(g)(\mathbf{x})| \le \sup\{g(\mathbf{y}) : |\mathbf{y}| \le 1\}$ for $|\mathbf{x}| \le 1$. Let $g_\mathbf{x}(\mathbf{y}) = f(\langle \mathbf{x}, \mathbf{y}\rangle) - P_m(\langle \mathbf{x}, \mathbf{y}\rangle)$. It follows that, for m sufficiently large, $|g_\mathbf{x}(\mathbf{y})| \le \varepsilon$, $|\mathbf{y}| \le 1$, and $|V(g_\mathbf{x})(\mathbf{y})| \le \varepsilon$ for all $\mathbf{x} \in S^d$, from which it follows from the dominant convergence theorem that the desired result holds for f. $\square$

If $\kappa = 0$, then $V = id$ and the theorem reduces to Theorem A. At the time of this writing, the formula of the intertwining operator V is known only in the case of symmetric group of order 3 ([7]) and the group $\mathbb{Z}_2^{d+1}$ ([6, 15]). In the latter case, we consider the associated weight function h_κ given in (1.1), and we have that the intertwining operator is given by

$$Vf(\mathbf{x}) = \int_{[-1,1]^{d+1}} f(t_1 x_1, \ldots, t_{d+1} x_{d+1}) \prod_{i=1}^{d+1} w_{\kappa_i - 1/2}(1 + t_i)(1 - t_i^2)^{\kappa_i - 1} d\mathbf{t}. \tag{2.5}$$

This formula also holds if some of $\kappa_i = 0$, in such a case we take the formula under the limit relation

$$\lim_{\mu \to 0} w_{\mu - 1/2} \int_{-1}^{1} f(t)(1 - t^2)^{\mu - 1} dt = [f(1) + f(-1)]/2. \tag{2.6}$$

Moreover, in the case of $\mathbb{Z}_2^{d+1}$, an orthonormal basis for $\mathcal{H}_n^{d+1}(h_\kappa^2)$ can be given in terms of Jacobi polynomials (cf. [15]).

The condition that f is continuous on $[-1, 1]$ can be relaxed if V has an integral formula. For example, in the case of h_κ in (1.1), we can use (2.5) to show that f integrable is sufficient for (2.3) to hold.

As an application of the formula (2.3), let us consider the expansion of $Vf(\langle \mathbf{x}, \mathbf{y}\rangle)$ in h-harmonics. Since for each fixed $\mathbf{x} \in S^d$, $P_n^h(\mathbf{x}, \mathbf{y})$ is an element in $\mathcal{H}_n^{d+1}(h_\kappa^2)$, it follows from (2.3) that we have

$$\begin{aligned} P_n(h_\kappa^2, Vf(\langle \mathbf{x}, \cdot\rangle), \mathbf{y}) :=& \int_{S^d} Vf(\langle \mathbf{x}, \mathbf{u}\rangle) P_n^h(\mathbf{y}, \mathbf{u}) h_\kappa^2(\mathbf{u}) d\omega \\ =& \lambda_n P_n^h(\mathbf{x}, \mathbf{y}) = \lambda_n \frac{n + |\kappa|_1 + (d-1)/2}{|\kappa|_1 + (d-1)/2} V C_n^{(|\kappa|_1 + (d-1)/2)}(\langle \mathbf{x}, \mathbf{y}\rangle) \end{aligned}$$

by (2.2). The function $P_n(h_\kappa^2, g, \mathbf{y})$ is the projection of g on $\mathcal{H}_n^{d+1}(h_\kappa^2)$, which is the Fourier partial sum of g in terms of h-harmonics. Using (4.7.3) and (4.7.15) of [13, p. 80-81], we can verify that $(n+\mu)/(\mu \cdot C_n^{(\mu)}(1)) = a_n^2$, where $\tilde{C}_n^{(\mu)}(t) = a_n C_n^{(\mu)}(t)$. Hence, by the formula (2.4) of λ_n, we have that

$$P_n(h_\kappa^2, Vf(\langle \mathbf{x}, \cdot \rangle), \mathbf{y}) = \int_{-1}^{1} f(t) \tilde{C}_n^{(\mu)}(t) w^{(|\kappa|_1+(d-1)/2)}(t) dt \cdot V[\tilde{C}_n^{(\mu)}(\langle \mathbf{x}, \cdot \rangle)](\mathbf{y}).$$

In particular, if we sum the above equation for $n = 0, 1, \ldots, N$, then we conclude that

Corollary 2.2. *Let $S_N(h_\kappa^2, g) := \sum_{n=0}^N P_n(h_\kappa^2, g)$ be the N-th Fourier partial sum in terms of h-harmonics of degree at most N, and let $s_N(w^{(\mu)}, f)$ be the usual N-th partial sum of f in the Gegenbauer polynomials $C_n^{(\mu)}$. Then*

$$S_N(h_\kappa^2, Vf(\langle \mathbf{x}, \cdot \rangle), \mathbf{y}) = V[s_N(w^{(|\kappa|_1+(d-1)/2)}, f, \langle \mathbf{x}, \cdot \rangle)](\mathbf{y}). \tag{2.7}$$

The formula (2.7) shows that the expansion of $Vf(\langle \mathbf{x}, \cdot \rangle)$ is a function of the same type and is closely related to the expansion of f in Gegenbauer polynomials. For $\kappa = 0$, this shows that the expansion of $f(\langle \mathbf{x}, \cdot \rangle)$ in the ordinary harmonics is again a function of $\langle \mathbf{x}, \cdot \rangle$. The convergence of the Fourier expansion in h-harmonics is discussed in [16].

3. Funk-Hecke formula for orthogonal polynomials on B^d

We follow the study in [17] about the relation between orthogonal polynomials on B^d and those on S^d. The discussion in [17] holds for rather general weight functions, we shall restrict to the weight function $W_{\kappa,\mu}$ in (1.2).

Let Π_n^d denote the space of polynomials of degree at most n in d variables. Let $\mathcal{V}_n^d(W_{\kappa,\mu})$ denotes the subspace of Π_n^d that consists of polynomials orthogonal to Π_{n-1}^d with respect to $W_{\kappa,\mu}$ on B^d. Let $\{P_k^n\}$ denote a basis of $\mathcal{V}_n^d(W_{\kappa,\mu})$, where the superscript n means that P_k^n is a polynomial of degree n and the subscript k satisfies $1 \le k \le r_n^d$, where $r_n^d = \dim \mathcal{P}_n^d = \binom{n+d-1}{n}$. The result in [17] shows that the orthogonal polynomials in $\mathcal{V}_n^d(W_{\kappa,\mu})$ are related to h-harmonics with respect to the weight function h_κ^2 defined in (1.1) on S^d, where $\kappa_{d+1} = \mu$. More precisely, it is proved that the polynomials

$$Y_k^n(\mathbf{y}) = r^n P_k^n(\mathbf{x}), \qquad \mathbf{y} = r(\mathbf{x}, x_{d+1}), \quad r = |\mathbf{y}|, \tag{3.1}$$

are homogeneous polynomials of degree n in $\mathbf{y}$ and they are orthogonal with respect to h_κ^2. Moreover, if $\{P_k^n\}$ is an orthonormal basis of $\mathcal{V}_n^d(W_{\kappa,\mu})$, then $\{Y_k^n\}$ is an orthonormal basis for the subspace of all h-harmonics in $\mathcal{H}_n^{d+1}(h_\kappa^2)$ that are even in the last coordinate x_{d+1}. In fact, if we define $\tilde{Y}_k^n(\mathbf{y}) = r^n x_{d+1} Q_k^{n-1}(\mathbf{x})$, where Q_k^{n-1} are orthogonal polynomials with respect to $W_{\kappa,\mu+1}$, then Y_k^n and $\tilde{Y}_k^n$ together form a basis for $\mathcal{H}_n^{d+1}(h_\kappa^2)$.

This connection between orthogonal polynomials allows us to write down the Funk-Hecke formula for orthogonal polynomials with respect to $W_{\kappa,\mu}$. Let us define an operator V_B that corresponds to the intertwining operator (2.5) by

$$V_B f(\mathbf{x}) = \int_{[-1,1]^d} f(t_1 x_1, \ldots, t_d x_d) \prod_{i=1}^d c_{\kappa_i}(1+t_i)(1-t_i^2)^{\kappa_i - 1} d\mathbf{t}. \tag{3.2}$$

We note that the operator V_B is the same as V in (2.5) for the weight function h_κ^2 with $\kappa_{d+1} = 0$. The corresponding Funk-Hecke formula states that

Theorem 3.1. *Let f be a continuous function on $[-1,1]$. Let $P_n \in \mathcal{V}_n^d(W_{\kappa,\mu})$. Then*

$$\int_{B^d} V_B f(\langle \mathbf{x}, \cdot\rangle)(\mathbf{y}) P_n(\mathbf{y}) W_{\kappa,\mu}(\mathbf{y}) d\mathbf{y} = \lambda_n P_n(\mathbf{x}), \qquad |\mathbf{x}| = 1, \tag{3.3}$$

where λ_n is the same as in Theorem 2.1 with $\kappa_{d+1} = \mu$.

Proof. From the definition of $V_B f$ and the definition of Vf in (2.5) with $\kappa_{d+1} = \mu$, it follows that

$$V_B f(\langle \mathbf{x}, \cdot\rangle)(\mathbf{y}) = Vf(\langle (\mathbf{x}, x_{d+1}), \cdot\rangle)(\mathbf{y}, 0) = Vf(\langle (\mathbf{x}, 0), \cdot\rangle)(\mathbf{y}, y_{d+1}),$$

where we take $\mathbf{x} \in B^d$ and $x_{d+1} = \sqrt{1 - |\mathbf{x}|^2}$ so that $(\mathbf{x}, x_{d+1}) \in S^d$. In the following we also write $\mathbf{y} \in B^d$ and $(\mathbf{y}, y_{d+1}) \in S^d$. Then, using the relation between P_k^n and Y_k^n and the integration formula

$$\int_{S^d} f(\mathbf{y}) d\omega = \frac{1}{2} \int_{B^d} [f(\mathbf{x}, \sqrt{1-|\mathbf{x}|^2}) + f(\mathbf{x}, -\sqrt{1-|\mathbf{x}|^2})] d\mathbf{x} \Big/ \sqrt{1-|\mathbf{x}|^2},$$

which can be easily verified, see, for example [1, p. 216], it follows that

$$\begin{aligned}
&\int_{B^d} V_B f(\langle \mathbf{x}, \cdot\rangle)(\mathbf{y}) P_k^n(\mathbf{y}) W_{\kappa,\mu}(\mathbf{y}) d\mathbf{y} \\
&= \int_{S^d} Vf(\langle (\mathbf{x}, 0), \cdot\rangle)(\mathbf{y}, y_{d+1}) Y_k^n(\mathbf{y}, y_{d+1}) h_\kappa^2(\mathbf{y}, y_{d+1}) d\omega \\
&= \lambda_n Y_k^n(\mathbf{x}, 0) = \lambda_n P_k^n(\mathbf{x}),
\end{aligned}$$

where the second equal sign follows from the Funk-Hecke formula of Theorem 2.1, which requires that $|(\mathbf{x}, 0)| = |\mathbf{x}| = 1$. Since P_k^n is a basis of $\mathcal{V}_n^d(W_\mu)$, the proof is complete. $\square$

The most interesting case of the formula (3.3) is when $\kappa = 0$, that is, in the case of the weight function $W_\mu(\mathbf{x}) = W_{0,\mu}(\mathbf{x}) = C_\mu (1 - |\mathbf{x}|^2)^{\mu - 1/2}$. Indeed, using the limit relation (2.6) several times, we have that $V_B = id$ for W_μ. The result is the following corollary.

Corollary 3.2. *Let f be a continuous function on $[-1,1]$. Let $P_n \in \mathcal{V}_n^d(W_\mu)$. Then*

$$\int_{B^d} f(\langle \mathbf{x}, \mathbf{y}\rangle) P_n(\mathbf{y}) W_\mu(\mathbf{y}) d\mathbf{y} = \lambda_n P_n(\mathbf{x}), \qquad |\mathbf{x}| = 1, \tag{3.4}$$

where λ_n is the same as in Theorem 2.1 with $|\kappa|_1 = \mu$.

We remark that the formulae (3.3) and (3.4) do not hold for $\mathbf{x} \in B^d$ in general. Indeed, the reproducing kernel of $\mathcal{V}_n^d(W_\mu)$ is given by ([18])

$$\begin{aligned}
\mathbf{P}_n(W_\mu; \mathbf{x}, \mathbf{y}) &= \sum_k P_k^n(\mathbf{x}) P_k^n(\mathbf{y}) = w_{\mu - 1/2} \frac{n + \mu + (d-1)/2}{\mu + (d-1)/2} \\
&\times \int_{-1}^1 C_n^{(\mu + (d-1)/2)}(\langle \mathbf{x}, \mathbf{y}\rangle + \sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}\, t)(1-t^2)^{\mu-1}(t) dt,
\end{aligned} \tag{3.5}$$

where $\{P_k^n\}$ forms an orthonormal basis of $\mathcal{V}_n^d(W_\mu)$. This formula shows that only for $|\mathbf{x}| = 1$ we have $\mathbf{P}_n(W_\mu; \mathbf{x}, \mathbf{y}) = \text{const}\, C_n^{(\mu+(d-1)/2)}(\langle \mathbf{x}, \mathbf{y}\rangle)$, which allows us to expand $f(\langle \mathbf{x}, \mathbf{y}\rangle)$ in terms of $\mathbf{P}_n(W_\mu; \mathbf{x}, \mathbf{y})$ and makes the formula (3.4) possible. For a function g defined on B^d, its Fourier n-th partial sum $S_n(W_\mu, g)$ in terms of orthogonal polynomials with respect to W_μ is defined by

$$S_n(W_\mu, g; \mathbf{x}) = \int_{B^d} g(\mathbf{y}) \sum_{k=0}^{n} \mathbf{P}_k(W_\mu; \mathbf{x}, \mathbf{y}) W_\mu(\mathbf{y}) d\mathbf{y}, \qquad \mathbf{x} \in B^d.$$

The formula (3.5) of the reproducing kernel also shows that $C_n^{(\mu+(d-1)/2)}(\langle \mathbf{x}, \eta\rangle)$, $|\eta| = 1$, is an element of $\mathcal{V}_n^d(W_\mu)$, that is, an orthogonal polynomial of degree n with respect to W_μ. As a consequence of this fact and Corollary 3.2, we can follow the proof of Corollary 2.2 to conclude that

Corollary 3.3. *For $|\mathbf{y}| = 1$ and $n \geq 0$, we have*

$$S_n(h_\kappa^2, f(\langle \mathbf{x}, \cdot\rangle), \mathbf{y}) = s_n(w^{(|\kappa|_1+(d-1)/2)}, f, \langle \mathbf{x}, \mathbf{y}\rangle).$$

A function of the form $f(\langle \mathbf{x}, \cdot\rangle)$ is called a ridge function. The corollary states that the Fourier expansion of a ridge function in terms of orthogonal polynomials with respect to W_μ is again a ridge function; in particular, it is one variable in nature. Let us mention that we showed in [18] that the Fourier expansion of a radial function, that is, a function of the form $f(|\mathbf{x}|)$, is again a radial function.

Another consequence of Corollary 3.2 and the fact that $C_n^{(\mu+(d-1)/2)}(\langle \mathbf{x}, \eta\rangle)$, $|\eta| = 1$, is an orthogonal polynomial with respect to W_μ is the following equation.

Corollary 3.4. *Let η and ξ satisfy $|\eta| = |\xi| = 1$. Then*

$$\int_{B^d} C_n^{(\mu+(d-1)/2)}(\langle \mathbf{x}, \xi\rangle) C_n^{(\mu+(d-1)/2)}(\langle \mathbf{x}, \eta\rangle) W_\mu(\mathbf{x}) d\mathbf{x} = \lambda_n C_n^{(\mu+(d-1)/2)}(\langle \eta, \xi\rangle), \tag{3.6}$$

where $\lambda_n = (\mu + (d-1)/2)/(n + \mu + (d-1)/2)$.

The formula (3.6) follows from (3.3) by obvious choices of f and P_n; the value of λ_n can be evaluated using (4.7.15) in [13, p. 81].

Since for every $|\eta| = 1$ the polynomial $C_n^{(\mu+(d-1)/2)}(\langle \mathbf{x}, \eta\rangle)$ is an orthogonal polynomial with respect to W_μ, one may ask the question of how to choose a set of η_i such that the polynomials $C_n^{(\mu+(d-1)/2)}(\langle \mathbf{x}, \eta_i\rangle)$ form a basis for $\mathcal{V}_n^d(W_\mu)$; furthermore, one may ask if it is possible to choose η_i so that the basis is an orthonormal one.

The second question may be answered with the help of Corollary 3.4, which shows that the polynomials $C_n^{(\mu+(d-1)/2)}(\langle \mathbf{x}, \eta_i\rangle)$, $1 \leq i \leq r_n^d$, are mutually orthogonal if, and only if, $C_n^{(\mu+(d-1)/2)}(\langle \eta_i, \eta_j\rangle) = 0$ for every pair of $i \neq j$, in other words, $\langle \eta_i, \eta_j\rangle$ are zeros of the Gegenbauer polynomial $C_n^{(\mu+(d-1)/2)}(t)$. Since this polynomial has only n zeros, it is unlikely that such a system of points η_i exist for $d > 2$. We should mention that the idea of sets of points on spheres with a specified set of possible inner product values was studied in [3]. Let us examine the case $d = 2$ more carefully. Let $t_1, t_2, \ldots, t_n$ be zeros of the Gegenbauer polynomial $C_n^{(\mu+1/2)}(t)$. It is well-known that all t_i are inside $(-1, 1)$, we write $t_i = \cos\theta_i$, $1 \le i \le n$. For $d = 2$, we can write η_i as $\eta_i = (\cos\phi_i, \sin\phi_i)$, $1 \le i \le r_n^2 = (n+1)$. Then

$$\langle \eta_i, \eta_j \rangle = \cos\phi_i \cos\phi_j + \sin\phi_i \sin\phi_j = \cos(\phi_i - \phi_j).$$

Hence, $\langle \eta_i, \eta_j \rangle = t_k$ for some k becomes $\phi_i - \phi_j = \theta_k \mod \pi$. However, from this equation and $\phi_{i+1} - \phi_j = \theta_l \mod \pi$ for some l, it then follows that $\phi_{i+1} - \phi_i = \theta_k - \theta_l \mod \pi$. Hence, in order that $\langle \eta_i, \eta_{i+1} \rangle$ is equal to one of t_k, we would have to have $\theta_k - \theta_l = \theta_m$, some m, for all k and l. This property on the zeros of the Gegenbauer polynomials is very restricted, and is not possessed by $C_n^{(\mu+1/2)}$ for $\mu > 0$ and $\mu \ne 1/2$. The case $\mu = 1/2$, however, provides a nice exception. In this case, the Gegenbauer polynomial is the Chebyshev polynomial of the second kind, $C_n^{(1)}(t) = U_n(t)$ and

$$U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad t = \cos\theta.$$

The zeros of U_n is given by $t_k = \cos k\pi/(n+1)$ for $k = 1, \ldots, n$, and $\phi_k = k\pi/(n+1)$ satisfies $\phi_k - \phi_l = \phi_{k-l}$. Hence, we can take $\eta_i = (\cos\phi_i, \sin\phi_i)$ and conclude that $U_n(\langle, \mathbf{x}, \eta_i \rangle)$ are mutually orthogonal with respect to the weight function $W_{1/2}(\mathbf{x}) = 1/\pi$, a multiple of the Lebesgue measure on B^2. We state this result as the following theorem, which was discovered in [10] and more recently in [2].

Theorem 3.5. *An orthonormal basis for the space $\mathcal{V}_n^2$ with respect to the Lebesgue measure on the unit disk B^2 is given by the polynomials*

$$U_k^n(x_1, x_2) = \frac{1}{\sqrt{\pi}} U_n(x_1 \cos\frac{k\pi}{n+1} + x_2 \sin\frac{k\pi}{n+1}), \qquad 0 \le k \le n. \tag{3.7}$$

In the statement of this theorem we have used the usual Lebesgue measure, not the normalized $W_{1/2}$, which leads to the constant factor $1/\sqrt{\pi}$ in the polynomials, where we have used the fact that $U_n(1) = n+1$ in (3.6) to determine the constant. From this basis of orthonormal polynomials and the formula (3.5) of the reproducing kernel, we also get the following interesting formula for the Chebyshev polynomials of the second kind,

$$\begin{aligned} &\sum_{k=0}^n U_n(x_1 \cos\frac{k\pi}{n+1} + x_2 \sin\frac{k\pi}{n+1}) U_n(y_1 \cos\frac{k\pi}{n+1} + y_2 \sin\frac{k\pi}{n+1}) \\ &\qquad = \frac{n+1}{\pi} \int_{-1}^1 U_n(x_1 y_1 + x_2 y_2 + \sqrt{1 - x_1^2 - x_2^2}\sqrt{1 - y_1^2 - y_2^2}\, t)(1 - t^2)^{-1/2} dt, \end{aligned}$$

$(x_1, x_2) \in B^2$ and $(y_1, y_2) \in B^2$. Moreover, if $\{P_k^n\}_{k=0}^n$ is another orthonormal basis of $\mathcal{V}_n^2(W_{1/2})$, for example, the orthonormal basis given in [9],

$$P_k^n(x_1, x_2) = b_k^n C_{n-k}^{(k+1)}(x_1)(1 - x_1^2)^{k/2} C_k^{(1/2)}(x_2/\sqrt{1 - x_1^2}), \qquad 0 \le k \le n, \tag{3.8}$$

where b_k^n are normalization constants, then it follows from the formula (3.5) that

$$\sum_{k=0}^{n} P_k^n(x_1,x_2)P_k^n(\cos\frac{j\pi}{n+1},\sin\frac{j\pi}{n+1}) = (n+1)U_n(x_1\cos\frac{j\pi}{n+1}+x_2\sin\frac{j\pi}{n+1}),$$

which shows how the orthonormal basis $\{U_k^n\}$ can be written in term of the basis of P_k^n. Since the transformation between the two orthonormal bases is an orthogonal matrix (cf. [14]), we then conclude the following curious result.

Corollary 3.6. *If $\{P_k^n\}$ is an orthonormal basis for $\mathcal{V}_n^2(W_{1/2})$, $W_{1/2}(\mathbf{x}) = 1/\pi$ being the normalized Lebesgue measure on B^2, then the matrix $\frac{1}{n+1}\Big[P_k^n(\cos\frac{j\pi}{n+1},\sin\frac{j\pi}{n+1})\Big]_{k,j=0}^{n}$ is an orthogonal matrix. In particular, we have*

$$\sum_{k=0}^{n} P_k^n(\cos\frac{i\pi}{n+1},\sin\frac{i\pi}{n+1})P_k^n(\cos\frac{j\pi}{n+1},\sin\frac{j\pi}{n+1}) = (n+1)^2\delta_{i,j}.$$

We note that the statement of this corollary is far from obvious for the basis $\{P_k^n\}$ given in (3.8).

We now turn our attention to the question of how to choose η_i such that the polynomials $C_n^{(\mu+(d-1)/2)}(\langle\mathbf{x},\eta_i\rangle)$ form a basis for $\mathcal{V}_n^d(W_\mu)$. We start with

Lemma 3.7. *The set $\{C_n^{(\mu+(d-1)/2)}(\langle\mathbf{x},\eta_i\rangle)\}$, where $|\eta_i| = 1$, $1 \le i \le r_n^d$, is a basis for $\mathcal{V}_n^d(W_\mu)$ if the matrix $A_n^{(\mu)} := [C_n^{(\mu+(d-1)/2)}(\langle\eta_i,\eta_j\rangle)]$ is nonsingular.*

Proof. If $A_n^{(\mu)}$ is nonsingular, then the expression $\sum_k c_k C_n^{(\mu+(d-1)/2)}(\langle\mathbf{x},\eta_i\rangle) = 0$, upon setting $\mathbf{x} = \eta_j$, implies that $c_k = 0$ for all k. □

The condition that $A_n^{(\mu)}$ is nonsingular is closely related to the interpolation at the set of points η_i. As in the case for spherical harmonics (cf. [11, p. 32]), let us call $\{\eta_1,\ldots,\eta_{r_n^d}\}$ a set of *regular* points for W_μ, if there is a unique polynomial P in $\mathcal{V}_n^d(W_\mu)$ such that $P(\eta_j) = y_j$, $1 \le j \le r_n^d$, for any prescribed data $\{y_j\}$. It is easy to see that this interpolation property is equivalent to the statement that any $P \in \mathcal{V}_n^d(W_\mu)$ vanishing on all η_j necessarily vanishes everywhere.

Lemma 3.8. *The matrix $A_n^{(\mu)}$ is nonnegative definite. It is positive definite if and only if the set $\{\eta_1,\ldots,\eta_{r_n^d}\}$ is regular.*

Proof. Let $\{P_k^n\}$ be an orthonormal basis of $\mathcal{V}_n^d(W_\mu)$. Example of such a basis can be given explicitly in terms of the Gegenbauer polynomials (analogy to (3.8) in which $d = 2$ and $\mu = 1/2$). From the formula of the reproducing kernel (3.5) we have that

$$\sum_{k=1}^{r_n^d} P_k^n(\eta_i)P_k^n(\eta_j) = \frac{n+\mu+(d-1)/2}{\mu+(d-1)/2} C_n^{(\mu+(d-1)/2)}(\langle\eta_i,\eta_j\rangle).$$

Hence, for any nonzero vector $\mathbf{c} \in \mathbb{R}^{r_n^d}$, it follows that

$$\mathbf{c}^T A_n^{(\mu)} \mathbf{c} = \frac{n + \mu + (d-1)/2}{\mu + (d-1)/2} \sum_{k=1}^{r_n^d} \Big[\sum_i c_i P_k^n(\eta_i) \Big]^2 \ge 0.$$

This proves that $A_n^{(\mu)}$ is nonnegative definite. If $\mathbf{c}^T A_n^{(\mu)} \mathbf{c} = 0$, then the above equation shows that $\sum_i c_i P_k^n(\eta_i) = 0$ for $1 \le k \le r_n^d$. If $\{\eta_i\}$ is regular, then the matrix $(P_k^n(\eta_i))_{i,k}$ is nonsingular, which shows that all c_i are zero and that $A_n^{(\mu)}$ is positive definite. We note that we also have

$$\frac{n + \mu + (d-1)/2}{\mu + (d-1)/2} \det A_n^{(\mu)} = [\det(P_k^n(\eta_j))_{k,j}]^2,$$

from which the direction that $A_n^{(\mu)}$ positive definite implies $\{\eta_j\}$ regular follows readily. $\square$

There are many choices of η_j so that $\det A_n^{(\mu)}$ is nonzero and the set of polynomials $\{C_n^{(\mu+(d-1)/2)}(\langle \mathbf{x}, \eta_i \rangle)\}$ forms a basis of $\mathcal{V}_n^d(W_\mu)$. In fact, the set of points that is not regular has measure zero. Nevertheless, there is no easy method to check that a given set is regular or to identify a regular set. In this respect, the following result for $d = 2$ is of interest.

Theorem 3.9. *The set $\{(\cos k\pi/(n+1), \sin k\pi/(n+1)), 0 \le k \le n\}$ is regular for W_μ, $\mu > 0$, on B^2. Consequently, the set $\{C_n^{(\mu+1/2)}(x \cos k\pi/(n+1) + y \sin k\pi/(n+1))\}$ forms a basis for $\mathcal{V}_n^2(W_\mu)$.*

Proof. By Theorem 3.5, this set is regular for $W_{1/2}$. In fact, $A_n^{(1/2)} = I$, the identity matrix. In order to show that $A_n^{(\mu)}$ is positive definite, we use the connection formula of the Gegenbauer polynomials (cf. [13, p. 99]), which states that

$$C_n^{(\mu+(d-1)/2)}(t) = \sum_{k=0}^{[n/2]} a_{k,n}^{(\mu)} C_{n-2k}^{(1/2)}(t),$$

where $a_{k,n}^{(\mu)}$ are positive constants. Using this formula, it follows that for any nonzero vector $\mathbf{c} \in \mathbb{R}^{r_n^d}$,

$$\mathbf{c}^T A_n^{(\mu)} \mathbf{c} = \sum_{k=0}^{[n/2]} a_{k,n}^{(\mu)} \mathbf{c}^T A_{n-2k}^{(1/2)} \mathbf{c} \ge a_{0,n}^{(\mu)} \mathbf{c}^T A_n^{(1/2)} \mathbf{c} = a_{0,n}^{(\mu)} \mathbf{c}^T \mathbf{c} > 0,$$

using the fact that $A_n^{(1/2)}$ is the identity, which shows that $A_n^{(\mu)}$ is positive definite. $\square$

Finally, we make the following observation about orthogonal and orthonormal bases. Using the Gram-Schmidt process, we can generate an orthonormal basis from any given basis for $\mathcal{V}_n^d(W_\mu)$. For a basis of the form $\{C_n^{(\mu+(d-1)/2)}(\langle \mathbf{x}, \eta_i\rangle)\}$, however, the Gram-Schmidt process is implicitly in the matrix $A_n^{(\mu)}$. Indeed, since $A_n^{(\mu)}$ is positive definite, it has a positive definite square root; we denote by $C = (c_{ij})$ the matrix $C = [A_n^{(\mu)}]^{-1/2}$. Then we claim that the polynomials P_k^n defined by

$$P_k^n(\mathbf{x}) = \sum_{j=1}^{r_n^d} c_{j,k} C_n^{(\mu+(d-1)/2)}(\langle \mathbf{x}, \eta_i\rangle)$$

form an orthonormal basis of $\mathcal{V}_n^d(W_\mu)$. Indeed, by (3.6) in Corollary 3.4 we have

$$\int_{B^d} P_k^n(\mathbf{x}) P_l^n(\mathbf{x}) W(\mathbf{x}) d\mathbf{x} = \sum_{i=1}^{r_n^d}\sum_{j=1}^{r_n^d} c_{i,k} c_{j,l} C_n^{(\mu+(d-1)/2)}(\langle \eta_i, \eta_j\rangle),$$

which is equal to $\delta_{k,l}$, since the right hand side is the (i,j)-th element of the matrix $C^T A_n^{(\mu)} C = I$.

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