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ORTHOGONAL POLYNOMIALS ON THE BALL AND THE SIMPLEX FOR WEIGHT FUNCTIONS WITH REFLECTION SYMMETRIES

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Abstract. Generalized classical orthogonal polynomials on the unit ball B^d and the standard simplex T^d are orthogonal with respect to weight functions that are reflection invariant on B^d and, after a composition, on T^d, respectively. They are also eigenfunctions of a second order differential-difference operator that is closely related to Dunkl's h-Laplacian for the reflection groups. Under a proper limit, the generalized classical orthogonal polynomials on B^d converge to the generalized Hermite polynomials on $\mathbb{R}^d$, and those on T^d converge to the generalized Laguerre polynomials on $\mathbb{R}^d_+$. The latter two are related to the Calogero-Sutherland models associated to the Weyl groups of type $\mathcal{A}$ and type $\mathcal{B}$.

1. Introduction

Recently the Calogero-Sutherland models associated to the Weyl groups of type A and type B are studied by several authors (cf. [2, 3 ,4, 7, 12, 13, 15, 17] and the references there). The model of type A is the Calogero model with harmonic term, which consists of d particles on the circle with inverse square interaction in an harmonic potential, whose Hamiltonian is, after changing variable $x_k = \exp(i\theta_k)$ and conjugation with the ground state,

$$\mathcal{H}_A := \sum_{i=1}^{d}\Big(\frac{\partial^2}{\partial y_i^2} - 2y_j\frac{\partial}{\partial y_i}\Big) + 2\kappa \sum_{1\le i<j\le d}\frac{1}{y_i-y_j}\Big[\Big(\frac{\partial}{\partial y_i}-\frac{\partial}{\partial y_j}\Big) - \frac{1-(ij)}{y_i-y_j}\Big], \tag{1.1}$$

where (ij) denotes the transposition $x_i \leftrightarrow x_j$. The eigenfunctions of $\mathcal{H}_A$ are polynomials orthogonal on $\mathbb{R}^d$ with respect to the weight function

$$W_\kappa^H(\mathbf{y}) = h_\kappa^2(\mathbf{y})\exp(-|\mathbf{y}|^2), \quad \text{where} \quad h_\kappa(\mathbf{y}) = \prod_{1\le i<j\le d}|y_i-y_j|^\kappa, \tag{1.2}$$

where $|\mathbf{y}|$ denotes the standard Euclidean norm of $\mathbf{y}$ ([3,4]). The weight function h_κ is invariant under the symmetric group $\mathcal{S}_d$, a Weyl group of type A. The polynomials are

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also called non-symmetric Hermite polynomials ([3]) of type A, and they can be written in terms of the non-symmetric Jack polynomials defined in [22] for general root systems. The model of type B is associated to the Hamiltonian

$$\mathcal{H}_B := \sum_{i=1}^{d}\Big(y_i\frac{\partial^2}{\partial y_i^2}+(\kappa_1+1-y_i)\frac{\partial}{\partial y_i}\Big) + \kappa_0 \sum_{1\le i<j\le d}\frac{1}{y_i-y_j}\Big[2\Big(y_i\frac{\partial}{\partial y_i}-y_j\frac{\partial}{\partial y_j}\Big)-\frac{y_i+y_j}{y_i-y_j}(1-(ij))\Big]. \tag{1.3}$$

The polynomial part of the eigenfunctions of $\mathcal{H}_B$ are the generalized Laguerre polynomials (cf. [3, 4]); they are orthogonal on $\mathbb{R}_+^d$ with respect to the weight function

$$W_\kappa^L(\mathbf{y}) = h_\kappa^*(\mathbf{y})\exp(-|\mathbf{y}|_1), \quad \text{where} \quad h_\kappa^*(\mathbf{y}) = \prod_{1\le i<j\le d}|y_i-y_j|^{\kappa_0}\prod_{i=1}^{d}|y_i|^{\kappa_1} \tag{1.4}$$

and $|\mathbf{y}|_1 = y_1+\cdots+y_d$. The associated weight function $h_\kappa(\mathbf{y}) := h_\kappa^*(y_1^2,\ldots,y_d^2)$ is invariant under the hyperoctahedral group $\mathcal{B}_d$, a Weyl group of type B.

The function h_κ is an example of weight functions invariant under reflection groups which are associated to h-harmonics studied by Dunkl in [8-10]. For a nonzero vector $\mathbf{v}\in\mathbb{R}^d$ the reflection $\sigma_\mathbf{v}$ is defined by $\mathbf{x}\sigma_\mathbf{v} := \mathbf{x}-2(\langle\mathbf{x},\mathbf{v}\rangle/|\mathbf{v}|^2)\mathbf{v}$, $\mathbf{x}\in\mathbb{R}^d$, where $\langle\mathbf{x},\mathbf{y}\rangle$ denote the usual Euclidean inner product. Let G be a finite reflection group on $\mathbb{R}^d$, which is a subgroup of the orthogonal group generated by the reflections. Let R_+ be a fixed positive root system associated with G, normalized so that $\langle\alpha,\alpha\rangle = 2$ for all $\alpha\in R_+$. Let κ be a multiplicity function defined on R_+, that is, $\kappa : R_+\mapsto\mathbb{R}$ is a G-invariant function. We shall assume that $\kappa(\alpha)\ge 0$ for all $\alpha\in R_+$. Then the function h_κ defined by

$$h_\kappa(\mathbf{x}) = \prod_{\alpha\in R_+}|\langle\mathbf{x},\alpha\rangle|^{\kappa(\alpha)}, \qquad \mathbf{x}\in\mathbb{R}^d, \tag{1.5}$$

is a positive homogeneous G-invariant function of degree $\gamma_\kappa := \sum_{\alpha\in R_+}\kappa(\alpha)$. In the case where G is $\mathcal{S}_d$ or $\mathcal{B}_d$, h_κ is given as in (1.2) and in the line below (1.4).

In this paper we study polynomials on the unit ball B^d and the simplex T^d, where

$$B^d = \{\mathbf{x}\in\mathbb{R}^d : |\mathbf{x}|\le 1\} \quad \text{and} \quad T^d = \{\mathbf{x}\in\mathbb{R}^d : x_1\ge 0,\ldots,x_d\ge 0, 1-|\mathbf{x}|_1\ge 0\},$$

that are orthogonal with respect to reflection invariant weight functions. For h_κ in (1.5) associated to a reflection group, we define a weight function on B^d by

$$W_{\kappa,\mu}^B(\mathbf{x}) = h_\kappa^2(\mathbf{x})(1-|\mathbf{x}|^2)^{\mu-1/2}, \qquad \mu > -1/2. \tag{1.6}$$

We call polynomials that are orthogonal with respect to $W_{\kappa,\mu}^B$ *generalized classical orthogonal polynomials on B^d*. When $\kappa = 0$ we have $h_\kappa(\mathbf{x}) = 1$ and the weight function $W_\mu^B(\mathbf{x}) := W_{0,\mu}^B(\mathbf{x}) = (1-|\mathbf{x}|^2)^{\mu-1/2}$ is associated to the classical orthogonal polynomials on the ball (cf. [1 or 14]). If G is a semi-direct product of the abelian group $\mathbb{Z}_2^d$ and another reflection group, then the weight function $h_\kappa(\mathbf{x})$ associated to G is even in each of its variables. For such an h_κ, we define a weight function on T^d by

$$W_{\kappa,\mu}^T(\mathbf{x}) = h_\kappa^2(\sqrt{x_1}, \ldots, \sqrt{x_d})(1 - |\mathbf{x}|_1)^{\mu-1/2}/\sqrt{x_1 \cdots x_d}, \qquad \mu > -1/2. \tag{1.7}$$

We call polynomials that are orthogonal to $W_{\kappa,\mu}^T$ *generalized classical orthogonal polynomials on T^d*. The classical orthogonal polynomials on the simplex are associated to the weight function $W_{\kappa,\mu}^T$ for $h_\kappa(\mathbf{x}) = |x_1|^{\kappa_1} \cdots |x_d|^{\kappa_d}$ (cf. [1 or 14]).

Using relations between orthogonal structures on S^d, B^d and T^d studied recently in ([30, 31]), we shall show that the generalized classical orthogonal polynomials are closely related to the h-harmonics, which are homogeneous polynomials orthogonal with respect to $h_\kappa^2 d\omega$ on the surface of the sphere S^{d-1}. The key ingredient in Dunkl's theory of h-harmonics is a family of first-order differential-difference operators, $\mathcal{D}_i$, called Dunkl's operators, which generates a commutative algebra ([9]); these operators are defined by

$$\mathcal{D}_i := \partial_i + \sum_{\alpha \in R_+} k(\alpha) \frac{id - \sigma_\alpha}{\langle \mathbf{x}, \alpha \rangle} \langle \alpha, \mathbf{e}_i \rangle, \qquad 1 \le i \le d, \tag{1.8}$$

where ∂_i is the ordinary partial derivative with respect to x_i, $\mathbf{e}_1, \ldots, \mathbf{e}_d$ are the standard unit vectors of $\mathbb{R}^d$, id stands for the identity operator and σ_α also denotes the operator defined by the reflection σ_α, $\sigma_\alpha f(\mathbf{x}) = f(\mathbf{x}\sigma_\alpha)$. The h-Laplacian, which plays the role similar to that of the ordinary Laplacian, is defined by $\Delta_h = \mathcal{D}_1^2 + \ldots + \mathcal{D}_d^2$. We keep the notation Δ for the ordinary Laplacian, $\Delta = \partial_1^2 + \cdots + \partial_d^2$. Then Δ_h is equal to ([8])

$$\Delta_h = \Delta + \sum_{\alpha \in R_+} 2k(\alpha) \Big[\frac{\langle \nabla, \alpha \rangle}{\langle \mathbf{x}, \alpha \rangle} - \frac{id - \sigma_\alpha}{\langle \mathbf{x}, \alpha \rangle^2} \Big]. \tag{1.9}$$

The h-harmonics are defined as the homogeneous polynomials satisfying the equation $\Delta_h P = 0$. We will show that the generalized classical orthogonal polynomials on B^d and on T^d are eigenfunctions of a second order differential-difference operator, which is essentially the spherical part of an h-Laplacian Δ_h. For the classical orthogonal polynomials, the operator becomes the differential operator in [1, 14] and this connection does not seem to have been observed before. We will also show that under a proper limit of $\mu \to \infty$, the generalized classical orthogonal polynomials on B^d converge to the generalized Hermite orthogonal polynomials, which are orthogonal with respect to the weight function $\prod_{i<j} |y_i^2 - y_j^2|^\kappa e^{-|\mathbf{y}|^2}$, and those polynomials on T^d converge to the generalized Laguerre polynomials. In particular, for the symmetric group and the hyperoctahedral group the limits lead to the situation of type A and B in (1.2) and (1.4), respectively.

The paper is organized as follows. We study orthogonal polynomials on B^d in Section 2 and their limit to the Hermite polynomials in Section 3. We discuss the structure on T^d in Section 4 and their limit relation to the Laguerre type polynomials in Section 5.

2. Orthogonal polynomials on the ball

2.1 Preliminary. For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$ and $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we write $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$. The number $|\alpha|_1 = \alpha_1 + \cdots + \alpha_d$ is called the total degree of $\mathbf{x}^\alpha$. We denote by Π^d the set of polynomials in d variables on $\mathbb{R}^d$ and by Π_n^d the subset of polynomials of total degree at most n. We also denote by $\mathcal{P}_n^d$ the space of homogeneous polynomials of degree n on $\mathbb{R}^d$ and we let $r_n^d = \dim \mathcal{P}_n^d$. It is well-known that

$$\dim \Pi_n^d = \binom{n+d}{n} \quad \text{and} \quad r_n^d = \dim \mathcal{P}_n^d = \binom{n+d-1}{d}.$$

Let Ω denote either B^d or T^d and let W be a weight function defined on Ω. Given an order among the monomials $\{\mathbf{x}^\alpha\}$, we can use the Gram-Schmidt process to generate a sequence of orthogonal polynomials with respect to the inner product of $L^2(W d\mathbf{x})$. We denote by $\{P_\alpha^n\}$, $|\alpha|_1 = n$ and $n \ge 0$, a sequence of orthonormal polynomials with respect to W on Ω, where the superscript n means that $P_\alpha^n \in \Pi_n^d$. For each fixed n the set $\{P_\alpha^n\}$ is a basis of the space $\mathcal{V}_n^d(W)$, the set of polynomials of degree n that are orthogonal to all polynomials of lower degree, whose dimension is r_n^d. For the reflection invariant weight function h_κ in (1.5) associated to a reflection group G, we denote by $\mathcal{H}_n^d(h_\kappa^2)$ the space of h-harmonics of degree n; that is, $\mathcal{H}_n^d(h_\kappa) = \mathcal{P}_n^d \cap \ker \Delta_h$. It is known that ([8])

$$\dim \mathcal{H}_n^d(h_\kappa^2) = \dim \mathcal{P}_n^d - \dim \mathcal{P}_{n-2}^d = r_n^d - r_{n-2}^d.$$

We start with recalling the relation between orthogonal polynomials on B^d and those on S^d. This relation is established in [30] for a general class of weight functions. Restricted to the weight function $W_{\kappa,\mu}^B$, it states that orthogonal polynomials with respect to $W_{\kappa,\mu}^B$ are related to the h-harmonics associated to $h_\kappa^2(\mathbf{x})|\mathbf{x}|^{2\mu}$, which is invariant under the direct product $G \times \mathbb{Z}_2$. To be more precise, we write for $\mathbf{y} \in \mathbb{R}^{d+1}$,

$$\mathbf{y} = (y_1, \ldots, y_d, y_{d+1}) = (\mathbf{y}', y_{d+1}) = r(\mathbf{x}, x_{d+1}), \tag{2.1}$$

where $r = |\mathbf{y}|$ and $(\mathbf{x}, x_{d+1}) \in S^d$. The notation implies that $|\mathbf{x}|^2 = 1 - |x_{d+1}|^2 \le 1$; that is, $\mathbf{x} \in B^d$. We denote by $\{P_\alpha^n\}$ and $\{Q_\alpha^n\}$ systems of orthonormal polynomials that form a basis for $\mathcal{V}_n^d(W_{\kappa,\mu}^B)$ and $\mathcal{V}_n^d(W_{\kappa,\mu+1}^B)$, respectively. We define

$$Y_{\alpha,n}(\mathbf{y}) = r^n P_\alpha^n(\mathbf{x}) \qquad \text{and} \qquad Y_{\beta,n}^*(\mathbf{y}) = r^n x_{d+1} Q_\beta^{n-1}(\mathbf{x}), \tag{2.2}$$

where $|\alpha|_1 = n$, $|\beta|_1 = n-1$, and we define $Y_{\beta,0}^*(\mathbf{y}) = 0$. Since $h_\kappa(-\mathbf{x}) = h_\kappa(\mathbf{x})$ by (1.5), it follows from the main result in [30] that, as functions of $\mathbf{y}$, $Y_{\alpha,n}$ and $Y_{\beta,n}^*$ are in fact homogeneous polynomials that are orthonormal in $L^2(h_{\kappa,\mu}^2 d\omega, S^d)$, where $d\omega$ denotes the surface measure and we define

$$h_{\kappa,\mu}^2(\mathbf{x}, x_{d+1}) := h_\kappa^2(\mathbf{x})|x_{d+1}|^{2\mu} = \prod_{\alpha \in R_+} |\langle \mathbf{x}, \alpha\rangle|^{2\kappa(\alpha)} |x_{d+1}|^{2\mu}. \tag{2.3}$$

Since h_κ is homogeneous, (2.3) defines $h_{\kappa,\mu}$ as a function in $\mathbb{R}^{d+1}$. On the other hand, since $h_{\kappa,\mu}$ is even in y_{d+1}, the space $\mathcal{H}_n^{d+1}(h_{\kappa,\mu}^2)$ must have a basis that consists of homogeneous polynomials that are either even in y_{d+1} or odd in y_{d+1}. Using the notation (2.1) and $x_{d+1}^2 = 1 - |\mathbf{x}|^2$, we can write this basis in terms of algebraic polynomials in variables $\mathbf{x}$ in the form of (2.2). The following is the precise statement proved in [30].

Theorem 2.1. *The relation (2.2) defines an one-to-one correspondence between an orthonormal basis of the space $\mathcal{H}_n^{d+1}(\tilde h_{\kappa,\mu})$ and an orthonormal basis of the space $\mathcal{V}_n^d(\tilde W_{\kappa,\mu}^B)\oplus x_{d+1}\mathcal{V}_{n-1}^d(\tilde W_{\kappa,\mu+1}^B)$, where the tilde means that the weight function is normalized to have unit integral on the corresponding domain.*

This theorem allows us to derive properties of orthogonal polynomials on the ball from those on the sphere. It plays the fundamental role in our study in this section. The simplest case is the abelian group $G=\mathbb{Z}_2^d$, which has the associated weight function

$$h_\kappa(\mathbf{x};\mathbb{Z}_2^d)=\prod_{i=1}^d |x_i|^{\kappa_i}, \qquad \kappa_i\ge 0, \quad \mathbf{x}\in\mathbb{R}^d, \tag{2.4}$$

where we include $\mathbb{Z}_2^d$ in the notation to distinguish it from the other cases. The corresponding weight function on B^d is

$$W_{\kappa,\mu}^B(\mathbf{x};\mathbb{Z}_2^d)=|x_1|^{2\kappa_1}\cdots|x_d|^{2\kappa_d}(1-|\mathbf{x}|^2)^{\mu-1/2}. \tag{2.5}$$

In particular, when $\kappa=0$, the function $W_\mu^B(\mathbf{x}):=W_{0,\mu}^B(\mathbf{x};\mathbb{Z}_2^d)$ is the classical weight function associated to the classical orthogonal polynomials on B^d. Other examples will be given later in the section.

2.2 Differential-difference equation for orthogonal polynomials on B^d. The h-harmonic Y satisfies the equation $\Delta_h Y=0$. We can write down the operator Δ_h explicitly as in (1.9). For $h_{\kappa,\mu}$ associated to $G\times\mathbb{Z}_2$, we shall denote its associated h-Laplacian by $\Delta_h^{G\times\mathbb{Z}_2}$ to distinguish it from the usual Δ_h. Let R_+ denote the fixed set of positive roots of G as in (1.5). For $\alpha\in R_+$, we denote $\tilde\alpha=(\alpha,0)$, which is a vector in $\mathbb{R}^{d+1}$. Then for $\mathbf{y}\in\mathbb{R}^{d+1}$, we have

$$\begin{aligned}\Delta_h^{G\times\mathbb{Z}_2} f(\mathbf{y}) = \Delta f(\mathbf{y}) &+ \sum_{\alpha\in R_+} 2k(\alpha)\Big[\frac{\langle\nabla f(\mathbf{y}),\tilde\alpha\rangle}{\langle \mathbf{y},\tilde\alpha\rangle}-\frac{f(\mathbf{y})-f(\mathbf{y}\sigma_\alpha)}{\langle\mathbf{y},\tilde\alpha\rangle^2}\Big]\\ &+2\mu\Big[\frac{1}{y_{d+1}}\frac{\partial f(\mathbf{x})}{\partial y_{d+1}}-\frac{f(\mathbf{y})-f(y_1,\ldots,y_d,-y_{d+1})}{y_{d+1}^2}\Big]\end{aligned} \tag{2.6}$$

for $h_{\kappa,\mu}$, where $\nabla=(\partial_1,\ldots,\partial_{d+1})^T$ is the gradient and the reflection σ_α acts on the first d variables of $\mathbf{y}$. Using Theorem 2.1, we can derive a second order differential-difference equation for orthogonal polynomials with respect to $W_{\kappa,\mu}$. In order to do so, we need to make a change of variables. Since, by (2.2), the polynomials $P_\alpha^n(W_{\kappa,\mu}^B)$ associated to $W_{\kappa,\mu}^B$ correspond to h-harmonics that are even in y_{d+1}, we only need to deal with the upper half space $\{\mathbf{y}\in\mathbb{R}^{d+1}: y_{d+1}\ge 0\}$. In order to write the operator for $P_\alpha^n(W_{\kappa,\mu}^B)$ in terms of $\mathbf{x}\in B^d$, we choose the following mapping

$$\mathbf{y}\mapsto(r,\mathbf{x}): \quad y_1=rx_1,\ldots,y_d=rx_d, \quad y_{d+1}=r\sqrt{1-x_1^2-\cdots-x_d^2}, \tag{2.7}$$

which is one-to-one from $\{\mathbf{y}\in\mathbb{R}^{d+1}: y_{d+1}\ge 0\}$ to itself. We shall rewrite the h-Laplacian Δ_h in terms of the new coordinates $(r,\mathbf{x})$, whose restriction to S^d leads to the operator for $P_\alpha^n(W_{\kappa,\mu}^B)$. We start with the ordinary Laplacian.

Lemma 2.2. *In terms of the coordinates* $(r, \mathbf{x})$ *in* $\{\mathbf{y} \in \mathbb{R}^{d+1} : y_{d+1} \ge 0\}$

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{d}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_0, \tag{2.8}$$

where the spherical part Δ_0 *is given by*

$$\Delta_0 = \Delta_{\mathbf{x}} - \langle \mathbf{x}, \nabla_{\mathbf{x}}\rangle^2 - (d-1)\langle \mathbf{x}, \nabla_{\mathbf{x}}\rangle, \tag{2.9}$$

$\Delta_{\mathbf{x}}$ *and* $\nabla_{\mathbf{x}}$ *denote the Laplacian and the gradient with respect to* $\mathbf{x}$*, respectively.*

Proof. Writing r and x_i in terms of $\mathbf{y}$ under the change of variables $\mathbf{y} \mapsto (r, \mathbf{x})$ in (2.7) and finding the partial derivatives $\partial x_i/\partial y_i$, we use the chain rule to conclude that

$$\frac{\partial}{\partial y_i} = x_i \frac{\partial}{\partial r} + \frac{1}{r}\Big(\frac{\partial}{\partial x_i} - x_i \langle \mathbf{x}, \nabla_{\mathbf{x}}\rangle\Big), \qquad 1 \le i \le d+1, \tag{2.10}$$

where for $i = d+1$ we use the convention that $x_{d+1} = \sqrt{1 - |\mathbf{x}|_2^2}$ and we define that $\partial/\partial x_{d+1} = 0$. Repeat the above operation and simplify, we obtain the second order derivatives after tedious computation,

$$\begin{aligned} \frac{\partial^2}{\partial y_i^2} = x_i^2 \frac{\partial^2}{\partial r^2} + \frac{1-x_i^2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Big[\frac{\partial^2}{\partial x_i^2} - (1-x_i^2)\langle \mathbf{x}, \nabla_{\mathbf{x}}\rangle - x_i\langle \mathbf{x}, \nabla_{\mathbf{x}}\rangle \frac{\partial}{\partial x_i}\Big] \\ + \Big(x_i \frac{\partial}{\partial x_i} - x_i^2\langle \mathbf{x}, \nabla_{\mathbf{x}}\rangle\Big)\Big(\frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\Big) - \Big(x_i \frac{\partial}{\partial x_i} - x_i^2\langle \mathbf{x}, \nabla_{\mathbf{x}}\rangle\Big)\langle \mathbf{x}, \nabla_{\mathbf{x}}\rangle. \end{aligned}$$

Adding these equations for $1 \le i \le d+1$, we see that the last two terms become zero since $x_1^2 + \cdots + x_{d+1}^2 = 1$ and we end up with the equation

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{d}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Big[\Delta_{\mathbf{x}} - d\langle \mathbf{x}, \nabla_{\mathbf{x}}\rangle - \sum_{i=1}^{d} x_i \langle \mathbf{x}, \nabla_{\mathbf{x}}\rangle \frac{\partial}{\partial x_i}\Big].$$

The term with summation sign in the above equation can be verified to be the same as $\langle \mathbf{x}, \nabla_{\mathbf{x}}\rangle^2 - \langle \mathbf{x}, \nabla_{\mathbf{x}}\rangle$, from which the desired formula follows. □

We note that (2.8) is the same as the standard equation that defines the spherical Laplacian (cf. [20, p. 78]), in which Δ_0 is usually written in terms of polar coordinates. For our purpose, we write Δ_0 in terms of $\mathbf{x}$. Let us also mention that Δ_0 in (2.9) can also be written in the following form,

$$\Delta_0 = \sum_{1 \le i < j \le d+1} \Big(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}\Big)^2, \tag{2.11}$$

where we again take $x_{d+1}^2 = 1 - x_1^2 - \cdots - x_d^2$ and assume that $\partial/\partial x_{d+1} = 0$. This formula can be easily verified by expanding the right hand side. We will not use it in this paper. Recall the formula of $\Delta_h^{G\times\mathbb{Z}_2}$ in (2.6). We have

Lemma 2.3. *Acting on functions on $\mathbb{R}^{d+1}$ that are even in $\mathbf{y}_{d+1}$, the operator $\Delta_h^{G\times\mathbb{Z}_2}$ takes the form*

$$\Delta_h^{G\times\mathbb{Z}_2} = \frac{\partial^2}{\partial r^2} + \frac{d+2\gamma_\kappa+2\mu}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{h,0}^{G\times\mathbb{Z}_2} \tag{2.12}$$

in terms of the coordinates $(r,\mathbf{x})$ in $\{\mathbf{y}\in\mathbb{R}^{d+1} : y_{d+1}\ge 0\}$, where the spherical part $\Delta_{h,0}^{G\times\mathbb{Z}_2}$, acting on functions of variables $\mathbf{x}$, is given by

$$\Delta_{h,0}^{G\times\mathbb{Z}_2} = \Delta_h - \langle \mathbf{x},\nabla\rangle^2 - (2\gamma_\kappa+2\mu+d-1)\langle\mathbf{x},\nabla\rangle, \tag{2.13}$$

in which Δ_h denote the h-Laplacian for h_κ associated to G, $\gamma_\kappa = \sum_{\alpha\in R_+}\kappa(\alpha)$ is the homogeneous order of h_κ, and $\nabla = \nabla_{\mathbf{x}}$.

Proof. Since we assume that the operator acts on functions that are even in y_{d+1}, it follows from (2.10) that the last term in the formula of $\Delta_h^{G\times\mathbb{Z}_2}$ in (2.6) becomes

$$2\mu\frac{1}{y_{d+1}}\frac{\partial}{\partial y_{d+1}} = 2\mu\Big[\frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\langle\mathbf{x},\nabla_{\mathbf{x}}\rangle\Big],$$

where we have used the fact that $y_{d+1} = rx_{d+1}$ and $x_{d+1}^2 = 1 - x_1^2 - \ldots - x_d^2$. Moreover, under the change of variables (2.7) we can use (2.10) to verify that

$$\frac{\langle\nabla,\tilde\alpha\rangle}{\langle\mathbf{y},\tilde\alpha\rangle} = \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\langle\nabla_{\mathbf{x}},\alpha\rangle}{\langle\mathbf{x},\alpha\rangle} - \frac{1}{r^2}\langle\mathbf{x},\nabla_{\mathbf{x}}\rangle.$$

Since σ_α acts on the first d variables of y_{d+1}, the difference part in the sum of the equation (2.6) becomes $r^{-2}(id-\sigma_\alpha)/\langle\mathbf{x},\alpha\rangle$. Upon taking summation over $\alpha\in R_+$, we conclude from (2.6) that $\Delta_h^{G\times\mathbb{Z}_2}$ takes the form of (2.12) with $\Delta_{0,h}^{G\times\mathbb{Z}_2}$ given by

$$\Delta_{h,0}^{G\times\mathbb{Z}_2} = \Delta_0 - 2(\gamma_\kappa+\mu)\langle\mathbf{x},\nabla_{\mathbf{x}}\rangle + 2\sum_{\alpha\in R_+}k(\alpha)\Big[\frac{\langle\nabla_{\mathbf{x}},\alpha\rangle}{\langle\mathbf{x},\alpha\rangle} - \frac{id-\sigma_\alpha}{\langle\mathbf{x},\alpha\rangle^2}\Big].$$

The desired formula of $\Delta_{0,h}^{G\times\mathbb{Z}_2}$ follows from the formula of Δ_0 in (2.9) and from the formula of the h-Laplacian in (1.9). ☐

Recall that $\mathcal{V}_n^d(W_{\kappa,\mu}^B)$ stands for the space of orthogonal polynomials of degree n with respect to the weight function $W_{\kappa,\mu}^B$ on B^d.

Theorem 2.4. *Polynomials in $\mathcal{V}_n^d(W_{\kappa,\mu}^B)$ satisfy the differential-difference equation*

$$\Big[\Delta_h - \langle\mathbf{x},\nabla\rangle^2 - (2\gamma_\kappa+2\mu+d-1)\langle\mathbf{x},\nabla\rangle\Big]P = -n(n+d+2\gamma_\kappa+2\mu-1)P. \tag{2.14}$$

Proof. Let P be a polynomial in $\mathcal{V}_n^d(W_{\kappa,\mu}^B)$. By Theorem 2.1, the homogeneous polynomial $Y(\mathbf{y}) = r^nP(\mathbf{x})$, where $\mathbf{y} = r(\mathbf{x},x_{d+1})$ as in (2.1), satisfies the equation $\Delta_h^{G\times\mathbb{Z}_2}Y = 0$. Since Y is homogeneous of degree n, using (2.12) we conclude that

$$0 = \Delta_h^{G\times\mathbb{Z}_2}Y(\mathbf{y}) = r^{n-2}[n(n+d+2\gamma_\kappa+2\mu-1)P(\mathbf{x}) + \Delta_{h,0}^{G\times\mathbb{Z}_2}P(\mathbf{x})]$$

from which the desired result follows from the formula of $\Delta_{h,0}^{G\times\mathbb{Z}_2}$ in (2.13). $\square$

We note that the left hand side of the equation (2.14) is $\Delta_{h,0}^{G\times\mathbb{Z}_2}P$. Writing explicitly out, the equation (2.14) is as follows,

$$\Delta_h P - \sum_{i=1}^{d}\sum_{j=1}^{d} x_i x_j \frac{\partial^2 P}{\partial x_i x_j} - (2\gamma_\kappa + 2\mu + d)\sum_{i=1}^{d} x_i \frac{\partial P}{\partial x_i} = -n(n+d+2\gamma_\kappa+2\mu-1)P. \tag{2.14a}$$

This extends the differential equation satisfied by the classical orthogonal polynomials. Indeed, elementary computation shows that (2.14a) can also be written in the form,

$$\Delta_h P - \sum_{j=1}^{d} \frac{\partial}{\partial x_j} x_j \Big[(2\gamma_\kappa + 2\mu - 1)P + \sum_{i=1}^{d} x_i \frac{\partial P}{\partial x_i}\Big] = -(n+d)(n+2\gamma_\kappa+2\mu-1)P. \tag{2.14b}$$

In particular, if $h_\kappa(\mathbf{x}) = 1$ or $\kappa = 0$, then $\Delta_h = \Delta$ and the difference part in (2.14) disappears. In this case, the weight function $W_{\kappa,\mu}^B$ becomes the classical weight function $W_\mu(\mathbf{x}) = (1-|\mathbf{x}|^2)^{\mu-1/2}$, and the equation (2.14b) is exactly the second order differential equation satisfied by the classical orthogonal polynomials [14, Vol. 2, p. 275]. The fact that this classical differential equation is derived in this manner appears to be new. We formulate these results as follows.

Corollary 2.4. *Under the correspondence $Y(\mathbf{y}) = r^n P(\mathbf{x})$, where $\mathbf{y} = r(\mathbf{x}, x_{d+1}) \in \mathbb{R}^{d+1}$, the equation (2.14) is the same as $\Delta_h^{G\times\mathbb{Z}_2} Y = 0$. In particular, the partial differential equation satisfied by the classical orthogonal polynomials on B^d is equivalent to $\Delta_h Y = 0$ for $h_\kappa(\mathbf{y}) = |y_{d+1}|^\mu$.*

Another two interesting cases are related to the type A or type B of the Calgero-Sutherland models; that is, G is either the symmetric group $\mathcal{S}_d$ or the hyperoctahedral group $\mathcal{B}_d$. We list the corresponding operator and the weight function below.

Case 2.1. The symmetric group $\mathcal{S}_d$:

$$W_{\kappa,\mu}^B(\mathbf{x}) = \prod_{1\le i<j\le d} |x_i - x_j|^{2\kappa}(1-|\mathbf{x}|^2)^{\mu-1/2}, \tag{2.15}$$

which corresponds to h_κ in (1.2), and the corresponding h-Laplacian is given by

$$\Delta_h = \Delta + 2\kappa \sum_{1\le i<j\le d} \frac{1}{x_i - x_j}\Big[(\partial_i - \partial_j) - \frac{1-(ij)}{x_i - x_j}\Big]. \tag{2.16}$$

We only give Δ_h here, since the differential-difference operator (2.14) can be easily written down knowing Δ_h. In this case the operator (2.14) is related to the operator $\mathcal{H}_A$ in (1.1), see the discussion in Section 3.

Case 2.2. The hyperoctahedral group $\mathcal{B}_d$:

$$W_{\kappa,\mu}^B(\mathbf{x}) = \prod_{i=1}^{d} |x_i|^{2\kappa_1} \prod_{1\le i\le j} |x_i^2 - x_j^2|^{2\kappa_0}(1-|\mathbf{x}|^2)^{\mu-1/2}. \tag{2.17}$$

Since the group $\mathcal{B}_d$ is generated by the reflections in $x_i = 0$, $1 \le i \le d$ and $x_i \pm x_j = 0$, $1 \le i, j \le d$; there are two conjugacy classes of reflections, hence two parameters for h_κ. The corresponding h-Laplacian is given by

$$\Delta_h = \Delta + \kappa_1 \sum_{i=1}^d \Big(\frac{2}{x_i}\partial_i - \frac{1-\sigma_i}{x_i^2}\Big) \tag{2.18}$$
$$+2\kappa_0 \sum_{1\le i<j\le d} \Big[\frac{\partial_i - \partial_j}{x_i - x_j} - \frac{1-\sigma_{ij}}{(x_i-x_j)^2}\Big] + 2\kappa_0 \sum_{1\le i<j\le d} \Big[\frac{\partial_i + \partial_j}{x_i + x_j} - \frac{1-\tau_{ij}}{(x_i+x_j)^2}\Big],$$

where σ_i, σ_{ij} and τ_{ij} denote the reflections in the hyperoctahedral group $\mathcal{B}_d$ defined by $\mathbf{x}\sigma_i = \mathbf{x} - 2x_i\mathbf{e}_i$, $\mathbf{x}\sigma_{ij} = (ij)\mathbf{x}$ and $\tau_{ij} = \sigma_{ij}\sigma_i\sigma_j$, and we also use the same notation to denote the operators defined by the action of these reflections. In this case the differential-difference operator (2.14) is related to the operator $\mathcal{H}_B$ in (1.3), see the discussion in Section 5.

2.3 Orthonormal bases on B^d. We can write down an orthonormal basis for $\mathcal{V}^d(W^B_{\kappa,\mu})$ in terms of a basis of h-harmonics. To do so, it is convenient to introduce the following class of orthogonal polynomials. Let $w^{(\mu,\lambda)}$ denote the weight function

$$w^{(\mu,\lambda)}(x) = w_{\mu,\lambda}(1-x^2)^{\mu-\frac12}|x|^{2\lambda}, \quad \text{where} \quad w_{\mu,\lambda} = \frac{\Gamma(\lambda+\mu+1)}{\Gamma(\mu+\frac12)\Gamma(\lambda+\frac12)}.$$

and $\lambda, \mu > -1/2$. The normalization constant $w_{\lambda,\mu}$ is chosen so that the integral of $w^{(\lambda,\mu)}$ over the integral $[-1,1]$ is 1. A family of orthogonal polynomials associated to $w^{(\mu,\lambda)}$ on $[-1,1]$ can be defined by the following formula

$$C_n^{(\mu,\lambda)}(x) = c_\lambda \int_{-1}^1 C_n^{(\lambda+\mu)}(xt)(1+t)(1-t^2)^{\lambda-1}dt, \tag{2.19}$$

where $c_\lambda = 1/\int_{-1}^1 (1-t^2)^{\lambda-1}dt$ for $\lambda > 0$ and $C_n^{(\lambda)}$ denote the standard Gegenbauer polynomials (see, for example, [26, p. 80], where $P_n^{(\lambda)}$ is used in stead of $C_n^{(\lambda)}$). From (2.19) we can also define $C_n^{(\mu,\lambda)}$ by a generating function ([28, (2.12)]). When $\lambda \to 0$, $C_n^{(\mu,\lambda)}$ reduces to the Gegenbauer polynomials $C_n^{(\mu)}$. The polynomials $C_n^{(\mu,\lambda)}$ are orthogonal but not normalized with respect to $w^{(\mu,\lambda)}$. We denote the corresponding orthonormal polynomials by $\tilde C_n^{(\mu,\lambda)}$ (In [28] the notation $D_n^{(\mu,\lambda)}$ is used). These polynomials can be written in terms of Jacobi polynomials; we have

$$\tilde C_{2n}^{(\mu,\lambda)}(x) = d_n(\lambda,\mu) P_n^{(\mu-\frac12,\lambda-\frac12)}(2x^2-1), \tag{2.20a}$$

$$\tilde C_{2n+1}^{(\mu,\lambda)}(x) = d_n(\lambda+1,\mu)\sqrt{\frac{\lambda+\mu+1}{\lambda+1/2}}\, x P_n^{(\mu-\frac12,\lambda+\frac12)}(2x^2-1), \tag{2.20b}$$

where

$$d_n(\lambda,\mu) = \sqrt{\frac{\Gamma(\lambda+\frac12)\Gamma(\mu+\frac12)}{\Gamma(\lambda+\mu+1)}}\sqrt{\frac{(2n+\lambda+\mu)\Gamma(n+\lambda+\mu)\Gamma(n+1)}{\Gamma(n+\lambda+\frac12)\Gamma(n+\mu+\frac12)}}.$$

To define an orthonormal basis for $\mathcal{V}_n^d(W_{\kappa,\mu}^B)$, we will be concerned with the normalization constants. For convenience, we shall use the normalized weight functions that have unit integral, which we denote by adding a tilde to make the distinction. Thus, we use $\tilde{h}_\kappa$ to denote the normalized weight function of h_κ; it is normalized so that the integral of h_κ^2 on the unit sphere S^{d-1} is one. Evidently, $\tilde{h}_\kappa$ is equal to a constant multiple of h_κ, and the normalization constant can be computed by use of Selberg's integral, as shown in the work of Macdonald, Heckman, Opdam and several others, see [18, 19, 21] and the references there. We also denote the normalized weight function on B^d by $\tilde{W}_{\kappa,\mu}^B$, normalized to have unit integral on B^d. Using the standard polar coordinates, we can verify that

$$\tilde{W}_{\kappa,\mu}^B(\mathbf{x}) = w_{\kappa,\mu}^B \tilde{h}_\kappa^2(\mathbf{x})(1-|\mathbf{x}|^2)^{\mu-1/2} \quad \text{where} \quad w_{\kappa,\mu}^B = \frac{2\Gamma(\gamma_\kappa+\mu+(d+1)/2)}{\Gamma(\gamma_\kappa+d/2)\Gamma(\mu+1/2)}. \tag{2.21}$$

We are now ready to write down an orthonormal basis on ball. To emphasize the dependence on the weight function, we denote elements of the basis by $P_{j,\beta}^n(W_{\kappa,\mu}^B)$; they are defined by

$$P_{j,\beta}^n(W_{\kappa,\mu}^B;\mathbf{x}) = c_{j,n}^B \tilde{C}_{2j}^{(\mu,n-2j+\gamma_\kappa+\frac{d-1}{2})}(|\mathbf{x}|)S_{n-2j,\beta}^h(\mathbf{x}), \tag{2.22}$$

where $\{S_{n-2j,\beta}^h\}$ is an orthonormal basis of h-harmonics of degree $n-2j$ in $L^2(\tilde{h}_\kappa^2 d\omega, S^{d-1})$ ($\beta = (\beta_1,\ldots,\beta_d) \in \mathbb{N}_0^d$ with $|\beta|_1 = n-2j$ and $\beta_d = 0$ or 1), and the constant $c_{k,n}^B$ is given by

$$[c_{j,n}^B]^2 = \frac{\Gamma(\gamma_\kappa+d/2)\Gamma(\gamma_\kappa+\mu+n-2j+(d+1)/2)}{\Gamma(\gamma_\kappa+n-2j+d/2)\Gamma(\gamma_\kappa+\mu+(d+1)/2)}. \tag{2.23}$$

Since the weight function $w^{(\mu,\lambda)}$ is symmetric with respect to the origin, the polynomials $\tilde{C}_{2n}^{(\mu,\lambda)}$ are even polynomials. Hence, it follows that $P_{j,\beta}^n(W_{\kappa,\mu}^B)$ are indeed polynomials of degree n. Furthermore, we have

Proposition 2.5. *An orthonormal basis of $\mathcal{V}_n^d(\tilde{W}_{\kappa,\mu}^B)$ is given by*

$$\left\{c_{j,n}^B \tilde{C}_{2j}^{(\mu,n-2j+\gamma_\kappa+\frac{d-1}{2})}(|\mathbf{x}|)S_{n-2j}^h(\mathbf{x}) : 0 \le 2j \le n,\, S_{n-2j}^h \in \mathcal{H}_{n-2j}^d(h_\kappa^2)\right\}.$$

Proof. Using the polar coordinates $\mathbf{x} = r\mathbf{x}'$, $\mathbf{x}' \in S^{d-1}$ and $r = |\mathbf{x}|$, and the standard change of variables

$$\int_{B^d} g(\mathbf{x})d\mathbf{x} = \int_0^1 r^{d-1}\int_{S^{d-1}} g(r\mathbf{x}')d\omega(\mathbf{x}')dr,$$

it follows from the fact that $\tilde{C}_n^{(\mu,\lambda)}$ and S_k^h are orthonormal that

$$\begin{aligned}&\int_{B^d} P_{k,\alpha}^n(\mathbf{x})P_{j,\beta}^m(\mathbf{x})\tilde{W}_{\kappa,\mu}^B(\mathbf{x})d\mathbf{x}\\ =&[c_{j,n}^B]^2 w_{\kappa,\mu}^B\int_0^1 r^{2\gamma_\kappa+d+2(n-2j)-1}(1-r^2)^{\mu-1/2}dr\,\delta_{k,j}\delta_{n,m}\delta_{\beta,\alpha} = \delta_{k,j}\delta_{n,m}\delta_{\beta,\alpha}\end{aligned}$$

for all choices of indices, where we have also used the fact that S_n^h are homogeneous. $\square$

We should point out that explicit formulae of orthonormal bases of $\mathcal{H}_n^d(h_\kappa^2)$ are known only in the case of the abelian group $\mathbb{Z}_2^d$ (see, for example, [27]). In this case, the weight function $W_{\kappa,\mu}$ is given in (2.5) and the orthonormal basis in Proposition 2.5 can be explicitly given, using the explicit formulae for the h-harmonics. In the following, however, we choose to give another orthonormal basis for $W_{\kappa,\mu}$, making use of the product nature of the weight function. In order to do so, we use the following convention. For $\kappa = (\kappa_1, \dots, \kappa_d)$ and $1 \le j \le d$, we introduce the notation $\kappa^j = (\kappa_j, \dots, \kappa_d)$. Similarly, we define α^j. For $\mathbf{x} \in \mathbb{R}^d$, we introduce the notation that $\mathbf{x}_j = (x_1, \dots, x_j)$ and we write $\mathbf{x}_0 = 0$. Then an orthonormal basis for the space $\mathcal{V}_n^d(W_{\kappa,\mu}^B)$ is given by

$$P_\alpha(W_{\kappa,\mu}^B, \mathbf{x}) = A_\alpha^n \prod_{j=1}^{d-1} \tilde{C}_{\alpha_j}^{(|\alpha^{j+1}|_1 + |\kappa^{j+1}|_1 + \mu + \frac{d-j}{2}, \kappa_j)}\Big(\frac{x_j}{\sqrt{1-|\mathbf{x}_{j-1}|^2}}\Big) \times \Big(\frac{1-|\mathbf{x}_j|^2}{1-|\mathbf{x}_{j-1}|^2}\Big)^{|\alpha^{j+1}|_1/2} \tilde{C}_{\alpha_d}^{(\mu,\kappa_d)}\Big(\frac{x_d}{\sqrt{1-|\mathbf{x}_{d-1}|^2}}\Big) \tag{2.24}$$

for all α such that $|\alpha|_1 = n$, where the constant A_α^n is defined by

$$[A_\alpha^n]^2 = \frac{\Gamma(\mu + \kappa_d + 1/2)}{\Gamma(\gamma_\kappa + \mu + (d+1)/2)} \prod_{j=1}^{d-1} \frac{\Gamma(|\alpha^{j+1}|_1 + |\kappa^j|_1 + \mu + (d-j+2)/2)}{\Gamma(|\alpha^{j+1}|_1 + |\kappa^{j+1}|_1 + \mu + (d-j+1)/2)}. \tag{2.25}$$

These formulae can be verified using the product nature of the integral, or they can be derived from the compact formulae in [28] for the h-harmonics associated to $\mathbb{Z}_2^d$, making use of the correspondence in Theorem 2.1. In using the formulae in [28], we need to make the substitution $\mu = \kappa_{d+1}$, $\alpha_j = k_{j-1} - k_j$ and $\cos\theta_{d-j} = x_j/\sqrt{1-|\mathbf{x}_{j-1}|^2}$. We should also point out that the notation α^j in [28] is defined in the reversed order $\alpha^j = (\alpha_1, \dots, \alpha_j)$.

There is another interesting orthogonal basis for $W_{\kappa,\mu}^B$, which is related to the non-symmetric Jack polynomials. See the discussion in Section 2.5.

2.4 Intertwining operator and reproducing kernel.

The reproducing kernel $P_n(h_\kappa^2; \mathbf{x}, \mathbf{y})$ of the space $\mathcal{H}_n^d(h_\kappa^2)$ of h-harmonics is defined by the relation

$$\int_{S^{d-1}} P_n(h_\kappa^2; \mathbf{x}, \mathbf{y}) f(\mathbf{y}) h_\kappa^2(\mathbf{y}) d\omega = f(\mathbf{y}), \qquad \forall f \in \mathcal{H}_n^d(h_\kappa^2). \tag{2.26}$$

Evidently, it satisfies

$$P_n(h_\kappa^2; \mathbf{x}, \mathbf{y}) = \sum_\alpha S_{n,\alpha}^h(\mathbf{x}) S_{n,\alpha}^h(\mathbf{y}), \qquad \mathbf{x}, \mathbf{y} \in S^{d-1}, \tag{2.27}$$

where $\{S_{n,\alpha}^h\}$ forms an orthonormal basis of $\mathcal{H}_n(h_\kappa^2)$ and the kernel is independent of the particular choice of the bases. For the ordinary harmonics it is the so called zonal polynomial, which is known to equal to a constant multiple of $C_n^{((d-1)/2)}(\langle \mathbf{x}, \mathbf{y}\rangle)$. Using the intertwining operator V associated to h_κ, it is possible to write down an analog formula for h-harmonics. We define the intertwining operator first, which intertwines between the commutative algebra generated by Dunkl's operators $\mathcal{D}_i$ and the algebra generated by the ordinary partial differential operators ∂_i.

Let G be a reflection group and κ be the multiplicity function defined on R_+. The intertwining operator V_κ is a linear operator on polynomials defined uniquely by

$$V_\kappa \mathcal{P}_n^d \subset \mathcal{P}_n^d, \qquad V_\kappa 1 = 1, \qquad \mathcal{D}_i V_\kappa = V_\kappa \partial_i, \qquad 1 \le i \le d, \tag{2.28}$$

and it is one-to-one on each $\mathcal{P}_n^d$, $n \in \mathbb{N}_0$. Moreover, V_κ communicates with the group action. It is conjectured in [10] and proved in [24] that V_κ is a positive operator for every reflection group. An explicit formula of this operator is known only in the cases of the abelian group $\mathbb{Z}_2^d$ ([10, 28]) and the symmetric group $\mathcal{S}_3$ ([11]). For the weight function (2.4) in the $\mathbb{Z}_2^d$ case, the intertwining operator takes the form

$$V_\kappa^{\mathbb{Z}_2^d} f(\mathbf{x}) = \int_{[-1,1]^d} f(t_1x_1, \ldots, t_dx_d) \prod_{i=1}^d c_{\kappa_i}(1+t_i)(1-t_i^2)^{\kappa_i - 1} d\mathbf{t}, \tag{2.29}$$

where c_λ is as in (2.19). We will also need the following lemma in the next section, which follows easily from the definition of V_κ.

Lemma 2.6. *The intertwining operator V_κ satisfy the property that $V_\kappa[f(\cdot)](a\mathbf{x}) = V_\kappa[f(a(\cdot))](\mathbf{x})$, where a is a nonzero scalar and f is a polynomial.*

Using the intertwining operator, the reproducing kernel $P_n(h_\kappa^2)$ in (2.27) can be written in terms of the Gegenbauer polynomials ([29]),

$$P_n(h_\kappa^2; \mathbf{x}, \mathbf{y}) = \frac{n + \gamma_\kappa + (d-1)/2}{\gamma_\kappa + (d-1)/2} V_\kappa[C_n^{(\gamma_\kappa + (d-1)/2)}(\langle \mathbf{x}, \cdot \rangle)](\mathbf{y}), \tag{2.30}$$

where $\mathbf{x}, \mathbf{y} \in S^d$. This formula has been used to study summability of h-harmonic expansions in [29]. Making use of the connection between orthogonal polynomials on B^d and h-harmonics, we can then derive a compact formula for the reproducing kernel of $\mathcal{V}_n(W_{\kappa,\mu}^B)$. This kernel is defined by

$$P_n(W_{\kappa,\mu}^B; \mathbf{x}, \mathbf{y}) = \sum_{|\alpha|_1 = n} P_\alpha^n(W_{\kappa,\mu}^B; \mathbf{x}) P_\alpha^n(W_{\kappa,\mu}^B; \mathbf{y}), \tag{2.31}$$

where $\{P_\alpha^n(W_{\kappa,\mu}^B)\}$ is an orthonormal basis of $\mathcal{V}_n(W_{\kappa,\mu}^B)$. It is easy to see that the kernel is independent of a particular choice of the basis (see, for example, [27]). We have

Theorem 2.7. *Let V_κ be the intertwining operator associated with h_κ. Then*

$$P_n(W_{\kappa,\mu}^B; \mathbf{x}, \mathbf{y}) = \frac{n + \gamma_\kappa + \mu + (d-1)/2}{\gamma_\kappa + \mu + (d-1)/2} c_\mu \times \tag{2.32}$$
$$\int_{-1}^1 V_\kappa[C_n^{(\gamma_\kappa + \mu + (d-1)/2)}(\langle \mathbf{x}, \cdot \rangle + t\sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2})](\mathbf{y})(1-t^2)^{\mu-1} dt.$$

Proof. The weight function $h_{\kappa,\mu}$ in (2.3) is invariant under the group $G \times \mathbb{Z}_2$. Its intertwining operator, denoted by $V_{\kappa,\mu}$, satisfies the relation $V_{\kappa,\mu} = V_\kappa \times V_\mu^{\mathbb{Z}_2}$, where V_κ denotes the intertwining operator associated to h_κ, and $V_\mu^{\mathbb{Z}_2}$ is the operator associated to $h(\mathbf{y}) = |y_{d+1}|^\mu$. Hence, it follows from (2.29) that

$$V_{\kappa,\mu} f(\mathbf{x}, x_{d+1}) = c_\mu \int_{-1}^{1} V_\kappa[f(\cdot, x_{d+1}t)](\mathbf{x})(1+t)(1-t^2)^{\mu-1} dt.$$

Using the fact that $P_n(W_{\kappa,\mu}^B)$ correspond to h-harmonics in $\mathcal{H}_n^{d+1}(h_{\kappa,\mu}^2)$ that are even in $\mathbf{y}_{d+1}$, see (2.2), the desired formula follows from the above formula and (2.30). $\square$

In the case of classical orthogonal polynomials associated to W_μ^B, we have $h_\kappa(\mathbf{x}) = 1$ and $V_\kappa = id$. In this case, the formula (2.31) first appears in [33] and is used to study the convergence of the Fourier orthogonal expansion.

2.5 Non-symmetric orthogonal polynomials on B^d. In the next section, we will show that the orthogonal polynomials $P_{j,\beta}^n(W_{\kappa,\mu}^B)$ converge, under a proper limit, to the generalized Hermite polynomials. In particular, it follows that in the case that G is the Weyl group of type A, $P_{j,\beta}^n(W_{\kappa,\mu}^B)$ converge to the generalized Hermite polynomials of type A. This suggests that we study the analogy of the non-symmetric Hermite polynomials, following [3, 4], in this case. These polynomials can be given in terms of the non-symmetric Jack polynomials $E_\eta(\mathbf{x}, 1/\kappa)$ which were defined in [22] and further studied in [16, 25]. To define the polynomials E_η, which are labeled by $\eta \in \mathbb{N}_0^d$, we need to introduce a partial order $\prec$ on d-tuples. A partition (with no more than d parts) is a composition $\eta \in \mathbb{N}_0^d$ such that $\eta_1 \ge \eta_2 \ge \ldots \ge \eta_d$. For $\eta \in \mathbb{N}_0^d$, we let η^+ be the unique partition such that $\eta^+ = w\eta$ for some $w \in S_d$. For $\eta, \nu \in \mathbb{N}_0^d$ we say that $\nu \le \eta$ if $\sum_{i=1}^j \nu_i \le\ge \sum_{i=1}^j \eta_i$ for $1 \le j \le d$. Also $\nu < \eta$ means $\nu \le \eta$ and $\nu \ne \eta$. Further we define $\nu \prec \eta$ if $|\eta| = |\nu|$ and either $\nu^+ < \eta^+$ or $\nu^+ = \eta^+$ and $\nu < \eta$. Then the non-symmetric Jack polynomials are uniquely defined as being the simultaneous eigenfunctions of the (mutually commuting) Cherednik operators ([5])

$$\xi_i = \frac{1}{\kappa} x_i \partial_i + x_i \sum_{j<i} \frac{1-(ij)}{x_i - x_j} + \sum_{j>i} \frac{x_j(1-(ij))}{x_i - x_j} + 1 - i \tag{2.33}$$

which have an expansion of the form

$$E_\eta(\mathbf{x}, 1/\kappa) = \mathbf{x}^\eta + \sum_{\nu \prec \eta,\, |\nu|_1 = |\eta|_1} a_{\eta,\nu} \mathbf{x}^\nu. \tag{2.34}$$

The non-symmetric Hermite polynomials, denoted by E_η^H, are defined as the unique eigenfunctions of (1.1) of the form $E_\eta + \sum_{|\nu|_1 < |\eta|_1} c_{\eta,\nu} E_\nu$. Moreover, as shown in [3, 23],

$$E_\eta^H(\mathbf{x}; 1/\kappa) = \exp(-\Delta_h/2) E_\eta(\mathbf{x}; 1/\kappa). \tag{2.35}$$

Similarly, we can define the non-symmetric orthogonal (see below) polynomials on B^d as the unique eigenfunctions of (2.14), with Δ_h given by (2.16), of the form

$$E_\eta^B(\mathbf{x}; 1/\kappa, \mu) = E_\eta(\mathbf{x}; 1/\kappa) + \sum_{|\nu|_1 < |\eta|_1} c_{\eta,\nu} E_\nu(\mathbf{x}; 1/\kappa). \tag{2.36}$$

The definition is justified below in Theorem 2.9. First, we need the projection operator $\operatorname{proj}_{B^d} : \mathcal{P}_n^d \mapsto \mathcal{V}_n^d(W_{\kappa,\mu}^B)$, which projects a homogeneous polynomial P to $\mathcal{V}_n^d(W_{\kappa,\mu}^B)$. We have

Lemma 2.8. *The operator* $\operatorname{proj}_{B^d} P : \mathcal{P}_n^d \mapsto \mathcal{V}_n^d(W_{\kappa,\mu}^B)$ *is given by*

$$\operatorname{proj}_{B^d} P(\mathbf{x}) = \sum_{p=0}^{[n/2]} \frac{1}{4^p p!(-n-\gamma_\kappa-\mu-d/2+3/2)_p} \Delta_h^p P(\mathbf{x}). \tag{2.37}$$

Proof. Let $P \in \mathcal{P}_n^d$. We define a polynomial $Y(\mathbf{y}) = r^n P(\mathbf{x})$ in variables $\mathbf{y} = r(\mathbf{x}, x_{d+1})$ as in (2.1). Then Y is a homogeneous polynomial of degree n in $\mathbf{y}$. We use the connection between $W_{\kappa,\mu}^B$ and $h_{\kappa,\mu}$. A formula for the h-harmonic projection operator $\operatorname{proj}^G : \mathcal{P}_n^d \mapsto \mathcal{V}_n^d(h_\kappa^2)$ was found in [8, p. 38]. For the weight function $h_{\kappa,\mu}$ in (2.3) associated to the group $G \times \mathbb{Z}_2$, the formula for the projection operator is given by

$$\operatorname{proj}^{G\times\mathbb{Z}_2} Y(\mathbf{y}) := \sum_{j=0}^{[n/2]} \frac{|\mathbf{y}|^{2j}}{4^j j!(-n-\gamma_\kappa-\mu-d/2+3/2)_j} \left(\Delta_h^{G\times\mathbb{Z}_2}\right)^j Y.$$

Since P is homogeneous of degree n, it follows from (2.13) and the Euler's equation that $\Delta_{h,0}^{G\times\mathbb{Z}_2} P = \Delta_h P - n(n+d+2\gamma_\kappa+2\mu-1)P$. Hence, applying $\Delta_h^{G\times\mathbb{Z}_2}$ on the polynomial $Y(\mathbf{y}) = r^n P(\mathbf{x})$, it follows from (2.12) and (2.13) that $\Delta_h^{G\times\mathbb{Z}_2} Y = r^{n-2}\Delta_h P$, which implies that $(\Delta_h^{G\times\mathbb{Z}_2})^j Y = r^{n-2j}\Delta_h^j P$. Therefore, using the fact that $r = |\mathbf{y}|$, we conclude that $\operatorname{proj}^{G\times\mathbb{Z}_2} Y(\mathbf{y}) = r^n T_h P(\mathbf{x})$, where $T_h P(\mathbf{x})$ denote the right hand side of (2.37). Since $\Delta^{G\times\mathbb{Z}_2} \operatorname{proj}^{G\times\mathbb{Z}_2} Y = 0$, it then follows from (2.12) and (2.13) that $T_h P$ satisfies the equation (2.14); that is, $T_h P \in \mathcal{V}_n^d(W_{\kappa,\mu}^B)$. Hence, $T_h P = \operatorname{proj}_{B^d} P$. $\square$

Theorem 2.9. *The non-symmetric orthogonal polynomials* $E_\eta^B(\mathbf{x}; 1/\kappa)$ *on* B^d *satisfies*

$$E_\eta^B(\mathbf{x}; 1/\kappa, \mu) = \operatorname{proj}_{B^d} E_\eta(\mathbf{x}; 1/\kappa), \qquad |\eta|_1 = n. \tag{2.38}$$

Proof. We write the equation (2.14) satisfied by $E_\eta^B(\mathbf{x}; 1/\kappa, \mu)$ as

$$(C - \Delta_h) E_\eta^B(\mathbf{x}; 1/\kappa, \mu) = |\eta|_1(|\eta|_1 + d + 2\gamma_\kappa + 2\mu - 1) E_\eta^B(\mathbf{x}; 1/\kappa, \mu), \tag{2.39}$$

where the operator C, which is only used locally in this proof, is given by

$$C := \langle \mathbf{x}, \nabla \rangle^2 + (2\gamma_\kappa + 2\mu + d - 1)\langle \mathbf{x}, \nabla \rangle.$$

Let P be a homogeneous polynomial of degree n. By Euler's formula, $\langle \mathbf{x}, \nabla \rangle P = nP$. Since $\Delta_h^j P$ is a homogeneous polynomial of degree $n - 2j$, it follows from the definition of C that the following commutator relation holds,

$$(C\Delta_h^j - \Delta_h^j C)P = -4j(n - j + \gamma_\kappa + \mu + (d-1)/2)\Delta_h^j P.$$

Consequently, it from the definition of $\operatorname{proj}_{B^d}$ that

$$[C, \operatorname{proj}_{B^d}] := C \operatorname{proj}_{B^d} - \operatorname{proj}_{B^d} C = \Delta_h \operatorname{proj}_{B^d}.$$

Now, using the fact that $\operatorname{proj}_{B^d} CP = n(n+d+2\gamma_\kappa+2\mu-1)\operatorname{proj}_{B^d} P$ for $P \in \mathcal{P}_n^d$, we see that $\operatorname{proj}_{B^d} E_\eta(\mathbf{x}; 1/\kappa)$ satisfies the same equation as (2.39). $\square$

The use of the commutator relation in the proof follows [23, Theorem 3.4]. A heuristic proof can be given following [3].

3. Generalized Hermite polynomials

We study the generalized Hermite polynomials, which are orthogonal polynomials with respect to the weight function W_κ^H as defined in (1.2) but with h_κ being the reflection invariant weight function in (1.5). Orthogonal polynomials of this type have been studied in [23]. In the case when G is a Weyl group of type A or type B, they are associated to the models (1.1) and (1.3), as we shall see later in the section, and they have been studied by several authors, as we mentioned in Section 1. We shall study them as limit of the generalized classical orthogonal polynomials on B^d.

3.1. Orthogonal bases. We start with the the generalized Hermite polynomials of one variable. For each $n \ge 0$, these polynomials are orthogonal with respect to the weight function $w_H^{(\mu)}(\mathbf{x}) = |x|^{2\mu} e^{-x^2}$, where $\mu > -1/2$, on $\mathbb{R}$. These polynomials can be given in terms of the usual Laguerre polynomials L_n^α (cf. [26, p.100]) as follows,

$$H_{2k}^\mu(x) = (-1)^k 2^{2k} k! L_k^{\mu-1/2}(x^2), \tag{3.1a}$$

$$H_{2k+1}^\mu(x) = (-1)^k 2^{2k+1} k! x L_k^{\mu+1/2}(x^2) \tag{3.1b}$$

([6]; see also [23] and the references there for some of their properties). When $\mu = 0$, they are the ordinary Hermite polynomials, usually denoted by H_n (cf. [26, p. 105]). For later use, we shall denote by b_n^μ the norm of H_n^μ with respect to the normalized weight function $\tilde{w}_H^{(\mu)}(x) = w_H^{(\mu)}(x)/\Gamma(\mu+1/2)$, which has unit integral on $\mathbb{R}$. That is

$$b_n^\mu := \int_{-\infty}^{\infty} [H_n^\mu(x)]^2 \tilde{w}_H^{(\mu)}(x)dx = 2^{2n}\Gamma\big([\tfrac{n}{2}]+1\big)\Gamma\big([\tfrac{n+1}{2}]+\mu+\tfrac{1}{2}\big)/\Gamma(\mu+1/2), \tag{3.2}$$

the second equality can be verified using (3.1). For later use, we introduce the notation $\tilde{H}_n^\mu := (b_n^\mu)^{-1/2} H_n^\mu$; the polynomials $\{\tilde{H}_n^\mu\}$ form an orthonormal basis in $L^2(\tilde{w}_H^{(\mu)}dx, \mathbb{R})$. Using a well-known integral ([26, p. 107, (5.6.5)])

$$L_n^\alpha(x^2) = \frac{(-1)^n \pi^{-1/2}\Gamma(n+\alpha+1)}{\Gamma(\alpha+1/2)(2n)!} \int_{-1}^1 H_{2n}(xt)(1-t^2)^{\alpha-1/2}dt$$

and the formula (3.1) we can verify the following relation between H_n^μ and H_n

$$\frac{H_n^\mu(x)}{b_n^\mu} = c_\mu \int_{-1}^1 \frac{H_n(xt)}{2^n n!}(1+t)(1-t^2)^{\mu-1}dt. \tag{3.3}$$

We note that the normalization constant for the Hermite polynomial H_n is $b_n^0 = 2^n n!$.

We now use H_n^μ to define an orthonormal basis for the weight function W_κ^H on $\mathbb{R}^d$. As in Section 2.3, we will use the normalized weight function

$$\tilde{W}_\kappa^H(\mathbf{x}) = w_\kappa^H \tilde{h}_\kappa^2(\mathbf{x}) e^{-|\mathbf{x}|^2} \quad \text{where} \quad w_\kappa^H = 1/\Gamma(\gamma_\kappa + d/2). \tag{3.4}$$

The elements of the orthonormal basis in $L^2(\tilde W_\kappa^H d\mathbf{x}, \mathbb{R}^d)$ are denoted by $P_{j,\beta}^n(W_\kappa^H)$ and defined by

$$P_{j,\beta}^n(W_\kappa^H;\mathbf{x}) = c_{j,n}^H \tilde H_{2j}^{n-2j+\gamma_\kappa+\frac{d-1}{2}}(|\mathbf{x}|) S_{n-2j,\beta}^h(\mathbf{x}), \tag{3.5}$$

where $S_{n-2j,\beta}^h$ is an h-harmonics of degree $n-2j$ normalized in $L^2(\tilde h_\kappa^2 d\omega, S^{d-1})$ and the normalization constant $c_{j,n}^H$ is given by

$$[c_{j,n}^H]^2 = \Gamma(\gamma_\kappa + d/2)/\Gamma(\gamma_\kappa + n - 2j + d/2). \tag{3.6}$$

Since the weight function $w_H^{(\mu)}$ is even in $\mathbb{R}^d$, the polynomials H_{2j}^μ are even functions. Hence, $P_{j,\beta}^n(W_\kappa^H;\mathbf{x})$ defined in (3.4) are indeed polynomials of degree n on $\mathbb{R}^d$. We have

Proposition 3.1. *An orthonormal basis of $\mathcal{V}_n^d(\tilde W_\kappa^H)$ is given by*

$$\left\{ c_{j,n}^H \tilde H_{2j}^{n-2j+\gamma_\kappa+\frac{d-1}{2}}(|\mathbf{x}|) S_{n-2j}^h(\mathbf{x}) : 0 \le 2j \le n, S_{n-2j}^h \in \mathcal{H}_{n-2j}^d(h_\kappa^2) \right\}.$$

The proof follows from changing variables to the usual polar coordinates, as in Proposition 2.5. In the case of the abelian group $\mathbb{Z}_2^d$, the weight function W_κ^H takes the form

$$W_\kappa^H(\mathbf{x};\mathbb{Z}_2^d) = |x_1|^{2\kappa_1}\cdots|x_d|^{2\kappa_d} e^{-|\mathbf{x}|^2} = w^{(\kappa_1)}\ldots w^{(\kappa_d)}. \tag{3.7}$$

In this case, an orthonormal basis can be given by a product of H_n^μ, which consists of

$$\tilde H_\alpha^\kappa(\mathbf{x}) := \tilde H_{\alpha_1}^{\kappa_1}(x_1)\cdots \tilde H_{\alpha_d}^{\kappa_d}(x_d). \tag{3.8}$$

We also use H_α^κ to denote the product $H_{\alpha_1}^{\kappa_1}(x_1)\cdots H_{\alpha_d}^{\kappa_d}(x_d)$. In particular, when $\kappa = 0$, they are the usual multiple Hermite polynomials. Another orthogonal basis associated with W_κ^H are the non-symmetric Hermite polynomials for the symmetric group defined in [3], see (2.35).

3.2 Limit relations. It is well-known that the Hermite polynomials H_n and the Gegenbauer polynomials are related by the following simple limit relation ([26, p. 107])

$$H_n(x)/n! = \lim_{\mu\to\infty} \mu^{-n/2} C_n^{(\mu)}(x/\sqrt{\mu}). \tag{3.9}$$

Together with (3.3) and (2.19), it implies an analogous relation between H_n^λ and $C_n^{(\mu,\lambda)}$,

$$(2^n/b_n^\lambda) H_n^\lambda(x) = \lim_{\mu\to\infty} \mu^{-n/2} C_n^{(\mu,\lambda)}(x/\sqrt{\mu}). \tag{3.10}$$

The following is an extension of this limit relation to higher dimensions.

Theorem 3.2. *For $P^n_{j,\beta}(W^B_{\kappa,\mu})$ in (2.18) and $P^n_{j,\beta}(W^H_\kappa)$ in (3.4), we have*

$$\lim_{\mu\to\infty} P^n_{j,\beta}(W^B_{\kappa,\mu};\mathbf{x}/\sqrt{\mu}) = P^n_{j,\beta}(W^H_\kappa;\mathbf{x}) \tag{3.11}$$

Proof. We need to derive a limit relation for $\tilde{C}^{(\mu,\lambda)}_n$ from (3.10). In order to determine the constants involved, we use the formula

$$\tilde{C}^{(\mu,\lambda)}_n(1)\tilde{C}^{(\mu,\lambda)}_n(x) = \frac{n+\lambda+\mu}{\lambda+\mu}C^{(\mu,\lambda)}_n(x)$$

([28, p. 182]), where the value of $\tilde{C}^{(\lambda,\mu)}_n(1)$ can be derived from (2.20),

$$[\tilde{C}^{(\mu,\lambda)}_n(1)]^2 = \frac{(n+\lambda+\mu)\Gamma(\mu+1/2)}{\Gamma(\lambda+1/2)\Gamma(\lambda+\mu+1)}\frac{\Gamma([(n+1)/2]+\lambda+\mu)\Gamma([n/2]+\lambda+1/2)}{\Gamma([(n+1)/2]+\mu+1/2)\Gamma([n/2]+1)},$$

which is equal to $(b^\lambda_n)^{-1}2^{2n}\mu^n(1+\mathcal{O}(\mu^{-1}))$ asymptotically as $\mu\to\infty$. Hence, using (3.10) we can conclude that

$$\tilde{H}^\lambda_n(x) = \lim_{\mu\to\infty}\tilde{C}^{(\mu,\lambda)}_n(x/\sqrt{\mu}). \tag{3.12}$$

Since h-harmonics are homogeneous polynomials, we have $S^h_{j,\beta}(\mathbf{x}/\sqrt{\mu}) = \mu^{-j/2}S^h_{j,\beta}(\mathbf{x})$. From (2.23) and (3.6) we can also verify that $\mu^{-(n-2j)/2}c^B_{j,n}\to c^H_{j,n}$ as $\mu\to\infty$, from which the desired result follows from the explicit formulae (2.22) and (3.5). □

In the case of $G=\mathbb{Z}^d_2$, a similar conclusion holds for the product type orthonormal basis (2.24) associated to the weight function (2.5) and the basis (3.8) associated to the weight function (3.7). Upon using (3.12), the verification amounts to show that the normalization constant A^n_α of $P_\alpha(W^B_{\kappa,\mu})$ goes to 1 as $\mu\to\infty$, which can be verified readily from (2.25). We formulate the result as follows for the record.

Theorem 3.3. *The multiple generalized Hermite polynomials (3.8) are the limit of the orthogonal polynomials (2.25) associated to the weight function (2.5) on the ball,*

$$\lim_{\mu\to\infty}P_\alpha(W^B_{\kappa,\mu};\mathbf{x}/\sqrt{\mu}) = \tilde{H}^\kappa_\alpha(\mathbf{x}).$$

In particular, when $\kappa=0$, the theorem states that the classical orthogonal polynomials on B^d associated to W^B_μ converge to the multiple Hermite polynomials. This also holds for polynomials in a bi-orthogonal basis on B^d derived from a generating function, see [14, Vol. 2, p. 289]. The limit relation also holds for the non-symmetric orthogonal polynomials on B^d. Recall the definition of E^H_η in (2.35) and E^B_η in (2.36).

Theorem 3.4. *For non-symmetric orthogonal polynomials $E_\eta^B(\mathbf{x}, 1/\kappa, \mu)$,*

$$\lim_{\mu\to\infty} \mu^{-|\eta|_1/2} E_\eta^B(\mathbf{x}/\sqrt{\mu}, 1/\kappa, \mu) = E_\eta^H(\mathbf{x}, 1/\kappa).$$

Proof. We use the expression of E_η^B in Theorem 2.9. Changing variables $x_i \to x_i/\sqrt{\mu}$, we see that $\partial/\partial x_i$ becomes $\sqrt{\mu}\partial/\partial x_i$. It follows that Δ_h becomes $\mu\Delta_h$. Hence, using the fact that E_η is homogeneous of degree $|\eta|_1$, we derive from (2.36) and (2.38) that,

$$E_\eta^B(\mathbf{x}/\sqrt{\mu}; 1/\kappa, \mu) = \sum_{p=0}^{[n/2]} \frac{\mu^{p+|\eta|_1/2}}{4^p p!(-n-\gamma_\kappa-\mu-d/2+3/2)_p} \Delta_h^p E_\eta(\mathbf{x}; 1/\kappa).$$

Since $(-n-\gamma_\kappa-\mu-d/2+3/2)_p = (-1)^p\mu^p(1+\mathcal{O}(\mu^{-1}))$ as $\mu\to\infty$, taking the limit $\mu\to\infty$, it follows that

$$\lim_{\mu\to\infty} \mu^{-|\eta|_1/2} E_\eta^B(\mathbf{x}/\sqrt{\mu}; 1/\kappa, \mu) = \sum_{p=0}^{[n/2]} \frac{\mu^p}{4^p p!} \Delta_h^p E_\eta(\mathbf{x}; 1/\kappa).$$

Since $\Delta_h^p E_\eta(\mathbf{x}; 1/\kappa) = 0$ for $p > [n/2]$, it follows that the right hand side of the above equation is equal to $\exp(\Delta_h/4)E_\eta(\mathbf{x}; 1/\kappa)$. □

As a consequence of the limit relation, we can derive a differential-difference equation for the generalized Hermite polynomials from the equation satisfied by $P_{j,\beta}^n(W_{\kappa,\mu}^B)$.

Theorem 3.5. *Orthogonal polynomials of degree n with respect to the weight function W_κ^H satisfying the following differential-difference equation,*

$$(\Delta_h - 2\langle \mathbf{x}, \nabla\rangle)P = -2nP. \tag{3.13}$$

Proof. Since $\{P_{j,\beta}^n(W_\kappa^H)\}$ form a basis for $\mathcal{V}_n^d(W_\kappa^H)$, it suffices to prove that these polynomials satisfy (3.13). We use the limit relation (3.11). The polynomials $P_{j,\beta}^n(W_{\kappa,\mu}^B)$ satisfy the equation (2.14). Changing variables $x_i \to x_i/\sqrt{\mu}$ as in the previous proof, it follows that Δ_h becomes $\mu\Delta_h$ and (2.14) becomes

$$\Big[\mu\Delta_h - \sum_{i=1}^d\sum_{j=1}^d x_i x_j \partial_i\partial_j - (2\gamma_\kappa + 2\mu + d)\langle \mathbf{x}, \nabla\rangle\Big] P(\mathbf{x}/\sqrt{\mu})$$
$$= -\, n(n+d+2\gamma_\kappa+2\mu-1)P(\mathbf{x}/\sqrt{\mu}).$$

Taking $P = P_{j,\beta}^n(W_{\kappa,\mu}^B)$ in the above equation, dividing the equation by μ and letting $\mu\to\infty$, we conclude that $P_{j,\beta}^n(W_\kappa^H)$ satisfy the equation (3.13). □

Equation (3.13) has been derived in [23], where a completely different method is used. In the case of $G = \mathcal{S}_d$, the symmetric group, the left hand side of the equation is exactly the operator $\mathcal{H}_A$ in (1.1). In the case of $G = \mathcal{B}_d$, the hyperoctahedral group, the left hand side of (3.13) is closely related to $\mathcal{H}_B$ in (1.3) as we will show in the following section. For the case $G = O(d)$, the equation with $\Delta = \Delta_h$ is the classical differential equation satisfied by the multiple Hermite polynomials (cf. [14, p. 288]).

As another consequence of the limit relation (3.11), we derive the following Mehler-type formula for the generalized Hermite polynomials.

Theorem 3.6. *Let $\{P_\alpha^n(W_\kappa^H)\}$, $|\alpha|_1 = n$ and $n \in \mathbb{N}_0$, denote an orthonormal basis with respect to W_κ^H on $\mathbb{R}^d$. Then for $0 < z < 1$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$*

$$\sum_{n=0}^{\infty} \sum_{|\alpha|_1 = n} P_\alpha^n(W_\kappa^H; \mathbf{x}) P_\alpha^n(W_\kappa^H; \mathbf{y}) z^n \tag{3.14}$$
$$= \frac{1}{(1-z^2)^{\gamma_\kappa + d/2}} \exp\Big\{ - \frac{z^2(|\mathbf{x}|^2 + |\mathbf{y}|^2)}{1-z^2} \Big\} V_\kappa \Big[\exp\Big\{ \frac{2z\langle \mathbf{x}, \cdot\rangle}{1-z^2} \Big\} \Big](\mathbf{y}).$$

Proof. The inner sum over α of (3.14) is the reproducing kernel of $\mathcal{V}_n^d(W_\kappa^H)$, which is independent of the particular bases. Hence, we can work with the basis $P_{j,\beta}^n(W_\kappa^H)$ in (3.5). We use the compact formula of the reproducing kernel $P_n(W_{\kappa,\mu}^B)$ in (2.32). Throughout this proof, we write $\lambda = \gamma_\kappa + \mu + (d-1)/2$. Then we can write (2.32) as

$$P_n(W_{\kappa,\mu}^B; \mathbf{x}, \mathbf{y}) = \frac{n+\lambda}{\lambda} c_\mu \int_{-1}^{1} C_n^{(\lambda)}(z) V_\kappa[K_\mu(\mathbf{x}, \mathbf{y}, (\cdot), z)](\mathbf{y}) (1-z^2)^{\lambda - 1/2} dz$$

where we have set $z = \langle \mathbf{x}, \cdot\rangle + t\sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}$ and K_μ is defined by

$$K_\mu(\mathbf{x}, \mathbf{y}, \mathbf{u}, z) = \Big[1 - \frac{(z - \langle \mathbf{x}, \mathbf{u}\rangle)^2}{(1-|\mathbf{x}|^2)(1-|\mathbf{y}|^2)}\Big]^{\mu-1} \frac{1}{\sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2}(1-z^2)^{\lambda-1/2}},$$

if z satisfies $(z - \langle \mathbf{x}, \mathbf{u}\rangle)^2 \le (1-|\mathbf{x}|^2)(1-|\mathbf{y}|^2)$, and $K_\mu(\mathbf{x}, \mathbf{y}, \mathbf{u}, z) = 0$ otherwise. In particular, if we expand $V_\kappa[K_\mu(\mathbf{x}, \mathbf{y}, (\cdot), z)]$ as a Gegenbauer series of $C_n^{(\lambda)}(z)$, then the above formula gives the coefficients of the expansion. Taking into consideration of the normalization constants, we have

$$\frac{c_{\lambda+1/2}}{c_\mu} \sum_{n=0}^{\infty} \frac{\Gamma(2\lambda)\Gamma(n+1)}{\Gamma(n+2\lambda)} P_n(W_{\kappa,\mu}^B; \mathbf{x}, \mathbf{y}) C_n^{(\lambda)}(z) = V_\kappa[K_\mu(\mathbf{x}, \mathbf{y}, (\cdot), z)]. \tag{3.15}$$

Let $\mathbf{x}$ and $\mathbf{y}$ be replaced by $\mathbf{x}/\sqrt{\mu}$ and $\mathbf{y}/\sqrt{\mu}$, respectively, in (3.15). Using the fact that $c_{\lambda+1/2}/c_\mu \to 1$ as $\mu \to \infty$ and that $\mu^{-n} C_n^{(\mu)}(z) \to 2^n z^n/n!$, we can use (3.11) to conclude that the left hand side of the (3.15) converges to the left hand side of (3.14). Moreover, from the definition of K_μ, we can write

$$\begin{aligned} K_\mu\Big(\frac{\mathbf{x}}{\sqrt{\mu}}, \frac{\mathbf{y}}{\sqrt{\mu}}, \frac{\mathbf{u}}{\sqrt{\mu}}, z\Big) =& (1-z^2)^{-\gamma_\kappa - d/2} (1-|\mathbf{x}|^2/\mu)^{-\mu+1} (1-|\mathbf{y}|^2/\mu)^{-\mu+1} \\ & \times \Big(1 - \frac{|\mathbf{x}|^2 + \mathbf{y}|^2 - 2z\langle \mathbf{x}, \mathbf{u}\rangle}{\mu(1-z^2)} + O(\mu^{-2})\Big)^{\mu-1}, \end{aligned}$$

which implies, upon taking limit $\mu \to \infty$, that

$$\lim_{\mu\to\infty} K_\mu\Big(\frac{\mathbf{x}}{\sqrt{\mu}}, \frac{\mathbf{y}}{\sqrt{\mu}}, \frac{\mathbf{u}}{\sqrt{\mu}}, z\Big) = \frac{1}{(1-z^2)^{\gamma_\kappa+d/2}} \exp\Big\{ - \frac{z^2(|\mathbf{x}|^2 + |\mathbf{y}|^2)}{1-z^2}\Big\} \exp\Big\{\frac{2z\langle \mathbf{x}, \mathbf{u}\rangle}{1-z^2}\Big\}.$$

Since V_κ is a bounded linear operator in the space of continuous functions (That V_κ is positive implies $|V_\kappa f(\mathbf{u}) - V_\kappa f(0)| \le V(|f(\mathbf{u}) - f(0)|) \le cV(\varepsilon) = c\varepsilon$ whenever $|f(\mathbf{u}) - f(0)| < \varepsilon$), we can take limit inside V_κ. Hence, using

$$V_\kappa[K_\mu(\mathbf{x}/\sqrt{\mu}, \mathbf{y}/\sqrt{\mu}, (\cdot), z)](\mathbf{y}/\sqrt{\mu}) = V_\kappa[K_\mu(\mathbf{x}/\sqrt{\mu}, \mathbf{y}/\sqrt{\mu}, (\cdot)/\sqrt{\mu}, z)](\mathbf{y}),$$

which follows from Lemma 2.6, we see that the right hand side of (3.15) converges to the right hand side of (3.14), which concludes the proof. $\square$

This formula has been proved in [23] using a different approach. For the generalized Hermite polynomials of type A and type B, it is already established in [4]. When $\kappa = 0$, then $V_\kappa = id$ and the formula reduces to

$$\sum_{n=0}^{\infty} \sum_{|\alpha|_1 = n} \tilde{H}_\alpha^\kappa(\mathbf{x}) \tilde{H}_\alpha^\kappa(\mathbf{y}) z^n = \prod_{i=1}^{d} \frac{1}{(1-z^2)^{1/2}} \exp\left\{ \frac{2x_i y_i z - z^2(x_i^2 + y_i^2)}{1 - z^2} \right\},$$

which is the classical Mehler formula for the Hermite polynomials ([14, Vol. 2, p.194]). Using the reproducing kernel $P_n(W_{\kappa,\mu}^B; \mathbf{x}, \mathbf{y})$, we can also obtain a generating function for $P_n(W_\kappa^H; \mathbf{x}, \mathbf{y})$, but the formula involves the reproducing kernel of h-harmonics; see [3, 23] for the relevant results. However, in the case of the generalized multiple Hermite polynomials H_α^κ in (3.8), the generating function has a compact form. Let us define $b_\alpha^\kappa = b_{\alpha_1}^{\kappa_1} \cdots b_{\alpha_d}^{\kappa_d}$, where b_n^μ is defined in (3.2).

Theorem 3.7. *For all $\mathbf{x} \in B^d$ and $\mathbf{y} \in \mathbb{R}^d$,*

$$\sum_{|\alpha|_1 = n} \frac{H_\alpha^\kappa(\mathbf{y})}{b_\alpha^\kappa} \mathbf{x}^\alpha = \frac{1}{\sqrt{\pi} 2^n n!} \int_{-\infty}^{\infty} \int_{[-1,1]^d} H_n(t_1 x_1 y_1 + \cdots + t_d x_d y_d + s\sqrt{1 - |\mathbf{x}|^2}) \times \prod_{i=1}^{d} c_{\kappa_i} (1 + t_i)(1 - t_i^2)^{\kappa_i - 1} d\mathbf{t}\, e^{-s^2} ds.$$

Proof. We need the following limit relation which can be verified using (2.20) and (3.2)

$$\lim_{\mu \to \infty} \mu^{-n/2} \tilde{C}_n^{(\mu,\lambda)}(x) = 2^n [b_n^\lambda]^{-1/2} x^n.$$

From the explicit formula (2.24) and the fact that $A_\alpha^n \to 1$ as $\mu \to \infty$, it follows that

$$\lim_{\mu \to \infty} \mu^{-n/2} P_\alpha^n(W_{\kappa,\mu}^B; \mathbf{x}) = 2^n [b_\alpha^\kappa]^{-1/2} \mathbf{x}^\alpha,$$

which implies, together with Theorem 3.3, the following limit relation

$$\lim_{\mu \to \infty} \mu^{-n/2} \sum_{|\alpha|_1 = n} P_\alpha^n(W_{\kappa,\mu}^B; \mathbf{x}) P_\alpha^n(W_{\kappa,\mu}^B; \mathbf{y}/\sqrt{\mu}) = 2^n \sum_{|\alpha|_1 = n} H_\alpha^\kappa(\mathbf{y}) \mathbf{x}^\alpha / b_\alpha^\kappa.$$

The sum on the left hand side is $P_n(W_{\kappa,\mu}^B, \mathbf{x}, \mathbf{y}/\sqrt{\mu})$ in (2.32). On the other hand, if we multiply the right hand side of (2.32) by $\mu^{-n/2}$, substitute $\mathbf{y}$ by $\mathbf{y}/\sqrt{\mu}$ and then change variable $t \to s/\sqrt{\mu}$, its limit as $\mu \to \infty$, using (3.9), is

$$\frac{1}{n!\sqrt{\pi}} \int_{-\infty}^{\infty} V_\kappa\big[H_n(\langle \mathbf{x}, \cdot \rangle + s\sqrt{1 - |\mathbf{x}|^2}\,)\big](\mathbf{y})\, e^{-s^2} ds.$$

The desired formula follows from the compact formula of V_κ in (2.29). $\square$

Corollary 3.7. *For* $|\mathbf{x}| = 1$ *and* $\mathbf{y} \in \mathbb{R}^d$,

$$\sum_{|\alpha|_1=n} \frac{H_\alpha^\kappa(\mathbf{y})}{b_\alpha^\kappa} \mathbf{x}^\alpha = \frac{1}{2^n n!} \int_{[-1,1]^d} H_n(t_1 x_1 y_1 + \cdots + t_d x_d y_d) \prod_{i=1}^d c_{\kappa_i} (1+t_i)(1-t_i^2)^{\kappa_i - 1} d\mathbf{t}.$$

In particular, in the case of $\kappa = 0$, the above formula reduces to the following well known formula for the classical Hermite polynomials (see, for example, [14, p. 288]),

$$\sum_{|\alpha|_1=n} \frac{H_\alpha(\mathbf{y})}{\alpha!} \mathbf{x}^\alpha = \frac{1}{n!} H_n(x_1 y_1 + \cdots + x_d y_d), \qquad |\mathbf{x}| = 1, \tag{3.16}$$

where we write $\alpha! = \alpha_1! \cdots \alpha_d!$. We note that even in this classical case, the formula in Theorem 3.7 appears to be new; in comparison to (3.16), it takes the form

$$\sum_{|\alpha|_1=n} \frac{H_\alpha(\mathbf{y})}{\alpha!} \mathbf{x}^\alpha = \frac{1}{n! \sqrt{\pi}} \int_{-\infty}^{\infty} H_n(x_1 y_1 + \cdots + x_d y_d + s\sqrt{1 - |\mathbf{x}|^2}) e^{-s^2} ds \tag{3.17}$$

for all $\mathbf{x} \in B^d$ and $\mathbf{y} \in \mathbb{R}^d$. The above formula (3.17) gives the extension of the formula (3.16) from $\mathbf{x}$ on the boundary of B^d to the entire B^d.

4. Orthogonal polynomials on the simplex

In this section we consider orthogonal polynomials on the the standard simplex T^d in $\mathbb{R}^d$, which is defined by

$$T^d = \{\mathbf{x} \in \mathbb{R}^d : x_1 \geq 0, \ldots, x_d \geq 0, 1 - |\mathbf{x}|_1 \geq 0\}.$$

In [31] we showed that orthogonal polynomials on the simplex T^d are related to those on the unit ball B^d. We first recall this relation.

Let $W^B(\mathbf{x}) := W(x_1^2, \ldots, x_d^2)$ be a weight function defined on B^d, and assume that W^B has unit integral on B^d. Associated to W^B we define a weight function W^T on the simplex T^d by

$$W^T(\mathbf{u}) = W(u_1, \ldots, u_d)/\sqrt{u_1 \cdots u_d}, \quad \mathbf{u} \in T^d. \tag{4.1}$$

Then W^T also has unit integral on T^d. A polynomial P is invariant under the group $\mathbb{Z}_2^d$, if P is even in each of its variables; such a polynomial must be of even degree. We denote by $\mathcal{V}_{2n}^d(W^B, \mathbb{Z}_2^d)$ the space of polynomials in $\mathcal{V}_{2n}^d(W^B)$ that are invariant under the group $\mathbb{Z}_2^d$. Let P_α^{2n} be a polynomial in $\mathcal{V}_{2n}^d(W^B, \mathbb{Z}_2^d)$. Since it is even in each of its variables, we can write it in the form of

$$P_\alpha^{2n}(\mathbf{x}) = R_\alpha^n(x_1^2, \ldots, x_d^2), \qquad |\alpha|_1 = n \tag{4.2}$$

where R_α^n is a polynomial of degree n. It turns out that R_α^n is a polynomial in $\mathcal{V}_n^d(W^T)$. In fact, the relation defines a one-to-one correspondence.

Theorem 4.1. *Let W^B and W^T be weight functions defined as above. Then the relation (4.2) defines a one-to-one correspondence between an orthonormal basis of $\mathcal{V}_{2n}^d(W^B, \mathbb{Z}_2^d)$ and an orthonormal basis of $\mathcal{V}_n^d(W^T)$.*

This theorem allows us to obtain results on orthogonal polynomials on T^d from those on B^d. See [31, 32] for some of its applications. We are interested in the generalized classical orthogonal polynomials, which we define as polynomials orthogonal with respect to the weight function $W_{\kappa,\mu}^T$ in (1.7),

$$W_{\kappa,\mu}^T(\mathbf{x}) = h_\kappa^2(\sqrt{x_1}, \ldots, \sqrt{x_d})(1 - |\mathbf{x}|_1)^{\mu-1/2}/\sqrt{x_1 \cdots x_d}\,, \qquad \mu > -1/2,$$

where h_κ is a weight function as defined in (1.5) associated to a reflection group G and we require that h_κ is invariant under $\mathbb{Z}_2^d$. In other words, we require that the reflection group G is a semi-direct product of another reflection group G_0 and $\mathbb{Z}_2^d$. Essentially, this limits G to two classes, $G = \mathbb{Z}_2^d$ itself, or $G = \mathcal{B}_d$, the hyperoctahedral group. We list the corresponding weight functions below.

Case 4.1. The abelian group $\mathbb{Z}_2^d$:

$$W_{\kappa,\mu}^T(\mathbf{x}; \mathbb{Z}_2^d) = x_1^{\kappa_1 - 1/2} \ldots x_d^{\kappa_d - 1/2}(1 - |\mathbf{x}|_1)^{\mu - 1/2}, \tag{4.3}$$

which corresponds to the weight function $W_{\kappa,\mu}^B(\mathbf{x}; \mathbb{Z}_2^d)$ in (2.5). The polynomials orthogonal with respect to this weight function are the classical orthogonal polynomials.

Case 4.2. The hyperoctahedral group $\mathcal{B}_d$:

$$W_{\kappa,\mu}^T(\mathbf{x}) = \prod_{i=1}^d x_i^{\kappa_1 - 1/2} \prod_{1 \le i < j \le d} |x_i - x_j|^{\kappa_0 - 1/2}(1 - |\mathbf{x}|_1)^{\mu - 1/2}, \tag{4.4}$$

which corresponds to the weight function $W_{\kappa,\mu}^B(\mathbf{x})$ in (2.17). They are related to the Laguerre type polynomials, which are the eigenfunctions of $\mathcal{H}_B$ in (1.3).

Using Theorem 4.1, we can derive a differential-difference operator for the orthogonal polynomials on T^d in the above two cases. We denote orthogonal polynomials with respect to $W_{\kappa,\mu}^T$ by $P_\alpha^n(W_{\kappa,\mu}^T; \mathbf{x})$. By (4.2), these polynomials correspond to orthogonal polynomials of degree $2n$ on B^d that are even in each variable, and we can change variables $x_i \to \sqrt{z_i}$ in the corresponding equation (2.14). This leads to

$$\frac{\partial}{\partial x_i} = 2\sqrt{z_i}\frac{\partial}{\partial z_i} \qquad \text{and} \qquad \frac{\partial^2}{\partial x_i^2} = 2\Big[\frac{\partial}{\partial z_i} + 2z_i\frac{\partial^2}{\partial z_i^2}\Big].$$

Using these formulae, and the explicit formula of the h-Laplacian, we can write down the differential-difference operator in the two cases. In the Case 4.1, Dunkl's operator takes a particular simple form

$$\mathcal{D}_i = \frac{\partial}{\partial x_i} + \kappa_i \frac{1 - \sigma_i}{x_i}, \qquad 1 \le i \le d.$$

Since we are dealing with (2.14) acting on functions that are even in each of their variables, the difference part disappears from the equation. Hence, upon changing variables $x_i \to \sqrt{x_i}$, we obtain a differential equation in this case.

Theorem 4.2. *The classical orthogonal polynomials associated with $W^T_{\kappa,\mu}(\mathbf{x};\mathbb{Z}_2^d)$ in (4.3) satisfy the partial differential equation*

$$\sum_{i=1}^d x_i(1-x_i)\frac{\partial^2 P}{\partial x_i^2} - 2\sum_{1\le i<j\le d} x_i x_j \frac{\partial^2 P}{\partial x_i \partial x_j}$$
$$+\sum_{i=1}^d \Big((\kappa_i+1/2) - \big(|\kappa|_1+\mu+\frac{d-1}{2}\big)x_i\Big)\frac{\partial P}{\partial x_i} = -n(n+|\kappa|_1+\mu+(d-1)/2)P.$$

This equation was known already in [1] (see also [14, p. 269]) at least for $d=2$, where it was derived using a special bi-orthogonal basis. For $d=1$, it reduces to the classical differential equation for the Jacobi polynomials. From the above discussion and Theorems 2.1 and 4.1, this differential equation is obtained from the h-Laplacian. The way it is derived in the above explains the role played by the symmetry of T^d. In the Case 4.2, we use the explicit formula of Δ_h in (2.18). The fact that we apply (2.14) to functions that are even in each of their variables allows us to drop the first difference part in (2.18) and combine the other two difference parts. The result is the following equation.

Theorem 4.3. *The orthogonal polynomials associated with $W^T_{\kappa,\mu}$ in (4.4) satisfy the differential-difference equation*

$$\Big[\sum_{i=1}^d x_i(1-x_i)\frac{\partial^2}{\partial x_i^2} - 2\sum_{1\le i<j\le d} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^d (\kappa_1+1-2(2\gamma_\kappa+2\mu+d)x_i)\frac{\partial}{\partial x_i}$$
$$+\kappa_0 \sum_{1\le i<j\le d}\frac{1}{x_i-x_j}\Big(2\Big(x_i\frac{\partial}{\partial x_i} - x_j\frac{\partial}{\partial x_j}\Big) - \frac{x_i+x_j}{x_i-x_j}(1-\sigma_{ij})\Big)\Big]P$$
$$= -n\Big(n+\gamma_\kappa+\mu+\frac{d-1}{2}\Big)P.$$

In the following section, we will show that the left hand side of this equation is related to the operator $\mathcal{H}_B$ in (1.3). Theorem 4.1 also allows us to derive an orthonormal basis for $W^T_{\kappa,\mu}$ from the basis for $W^B_{\kappa,\mu}$. For late use, we again denote the normalized weight function by $\tilde W^T_{\kappa,\mu}$. We have

$$\tilde W^T_{\kappa,\mu}(\mathbf{x};\mathbb{Z}_2^d) = w^T_{\kappa,\mu}\tilde h^2_\kappa(\sqrt{x_1},\ldots,\sqrt{x_d})(1-|\mathbf{x}|_1)^{\mu-1/2}/\sqrt{x_1\cdots x_d}\,, \tag{4.5}$$

where $w^T_{\kappa,\mu} = w^B_{\kappa,\mu}$. Since G is a semi-direct product of G_0 and $\mathbb{Z}_2^d$, the weight function h_κ is invariant under $\mathbb{Z}_2^d$. Let $\mathcal{H}^d_{2n}(h^2_\kappa,\mathbb{Z}_2^d)$ denote the space of h-harmonics of degree $2n$ that are invariant under $\mathbb{Z}_2^d$. Let $\{S^h_{2n,\beta}\}$ denote a basis of $\mathcal{H}^d_{2n}(h^2_\kappa,\mathbb{Z}_2^d)$. Then each element $S^h_{2n,\beta}$ is even in each of its variables. Since h-harmonics are homogeneous, the function $S^h_{2n,\beta}(\sqrt{x_1},\ldots,\sqrt{x_d})$ is also a homogeneous polynomial. The correspondence (4.2) and formula (2.22) shows that orthogonal polynomials for $W^T_{\kappa,\mu}$ are given by

$$P^n_{j,\beta}(W^T_{\kappa,\mu};\mathbf{x}) = c^T_{j,n}\tilde C_{2j}^{(\mu,2n-2j+\gamma_\kappa+\frac{d-1}{2})}(|\mathbf{x}|_1)S^h_{2n-2j,\beta}(\sqrt{x_1},\ldots,\sqrt{x_d}) \tag{4.6}$$

where $c^T_{j,n} = c^B_{j,2n}$. Moreover, just as in Proposition 2.5, we also have

Proposition 4.4. *An orthonormal basis of* $\mathcal{V}_n^d(\tilde W^T_{\kappa,\mu})$ *is given by*

$$\Big\{P^n_{j,\beta}(W^T_{\kappa,\mu};\mathbf{x}) : 0 \le j \le n, S^h_{2n-2j} \in \mathcal{H}^d_{2n-2j}(h^2_\kappa, \mathbb{Z}^d_2)\Big\}.$$

In the case of $\mathbb{Z}_2^d$, we can derive from (2.19) closed formulae of an orthonormal basis for the weight function $W^T_{\kappa,\mu}(\mathbf{x};\mathbb{Z}_2^d)$. We state the formulae below for future reference. Let $\mathbf{x}_j$ and α^j be defined as in (2.24). Then the elements of the basis are given by

$$\begin{aligned}R_\alpha(W^T_{\kappa,\mu};\mathbf{x}) =& A^{2n}_{2\alpha}\prod_{j=1}^{d-1}\tilde C_{2\alpha_j}^{(2|\alpha^{j+1}|_1+|\kappa^{j+1}|_1+\mu+\frac{d-j}{2},\kappa_j)}\Big(\frac{\sqrt{x_j}}{\sqrt{1-|\mathbf{x}_{j-1}|_1}}\Big)\\ &\times\Big(\frac{1-|\mathbf{x}_j|_1}{1-|\mathbf{x}_{j-1}|_1}\Big)^{|\alpha^{j+1}|_1}\tilde C^{(\mu,\kappa_d)}_{2\alpha_d}\Big(\frac{\sqrt{x_d}}{\sqrt{1-|\mathbf{x}_{d-1}|_1}}\Big)\end{aligned} \tag{4.7}$$

for all α such that $|\alpha|_1 = n$, where the constant $A^{2n}_{2\alpha}$ is defined as in (2.25) and we write $2\alpha = (2\alpha_1, \dots, 2\alpha_d)$.

The correspondence also allows us to derive a compact formula for the reproducing kernel $P_n(W^T_{\kappa,\mu})$, defined as in (2.31) with B replaced by T. Let us introduce the notation $\{\mathbf{x}\}^{1/2} = (\sqrt{x_1},\dots,\sqrt{x_d})$ for $\mathbf{x}\in T^d$.

Theorem 4.5. *Let* V_κ *be the intertwining operator associated to* h_κ*. Then*

$$P_n(W^T_{\kappa,\mu};\mathbf{x},\mathbf{y}) = \frac{n+\gamma_\kappa+\mu+(d-1)/2}{\gamma_\kappa+\mu+(d-1)/2}c_\mu\times$$
$$\int_{-1}^1 V_\kappa\Big[C_{2n}^{(\gamma_\kappa+\mu+(d-1)/2)}\big(\langle\{\mathbf{x}\}^{1/2},\cdot\rangle + t\sqrt{1-|\mathbf{x}|_1}\sqrt{1-|\mathbf{y}|_1}\,\big)\Big](\{\mathbf{y}\}^{1/2})(1-t^2)^{\mu-1}dt.$$

Proof. Using Theorem 4.1, we see that the kernel $P_n(W^T_{\kappa,\mu})$ is related to the reproducing kernel $P_n(W^B_{\kappa,\mu})$ on the unit ball by

$$P_n(W^T_{\kappa,\mu};\mathbf{x},\mathbf{y}) = \frac{1}{2^d}\sum_{\varepsilon\in\mathbb{Z}_2^d}P_{2n}\big(W^B_{\kappa,\mu};(\varepsilon_1\sqrt{x_1},\dots,\varepsilon_d\sqrt{x_d}\,),(\sqrt{y_1},\dots,\sqrt{y_d}\,)\big). \tag{4.8}$$

In fact, since the reproducing kernel is unique, it is sufficient to show that the right hand side has the reproducing property; that is, if we denote the right hand side temporarily by $Q_n(\mathbf{x},\mathbf{y})$, then

$$\int_{T^d} P(\mathbf{y})Q_n(\mathbf{x},\mathbf{y})W^T_{\kappa,\mu}(\mathbf{y})d\mathbf{y} = P(\mathbf{x})$$

for any polynomial $P\in\mathcal{V}_n^d(W^T_{\kappa,\mu})$. We can take $P = P^n_{j,\beta}(W^T_{\kappa,\mu})$. Upon changing variable $y_i\mapsto y_i^2$, this equation follows from the reproducing property of $P_n(W^B_{\kappa,\mu})$; see also [32]. Hence, we can use the formula (2.32) to get the compact formula. Since G is a semi-direct product of G_0 and $\mathbb{Z}_2^d$ and $R(\sigma)V_\kappa = V_\kappa R(\sigma)$, where $R(\sigma)f(\mathbf{x}) := f(\mathbf{x}\sigma)$, for every $\sigma\in G$, it follows that $R(\varepsilon)V_\kappa = V_\kappa R(\varepsilon)$ for $\varepsilon \in \mathbb{Z}_2^d$. Therefore, the summation over $\mathbb{Z}_2^d$ does not appear in the final formula. $\square$

In the Case 4.1, the intertwining operator V_κ enjoys a compact formula (2.29), so that the reproducing kernel can be written down explicitly. The result is the following remarkable formula for the classical orthogonal polynomials on T^d,

$$P_n(W_{\kappa,\mu}^T, \mathbf{x}, \mathbf{y}) = \frac{2n + |\kappa|_1 + \mu + (d-1)/2}{|\kappa|_1 + \mu + (d-1)/2} \times$$
$$\int_{-1}^1 \int_{[-1,1]^d} C_{2n}^{(|\kappa|_1+\mu+\frac{d-1}{2})} \left(\sqrt{x_1 y_1}\, t_1 + \ldots + \sqrt{x_d y_d} t_d + s\sqrt{1-|\mathbf{x}|_1}\sqrt{1-|\mathbf{y}|_1}\right)$$
$$\times \prod_{i=1}^d c_{\kappa_i} (1-t_i^2)^{\kappa_i - 1} d\mathbf{t} (1-s^2)^{\mu-1} ds.$$

This formula was derived and used to study the summability of the Fourier orthogonal expansion in [32]. Notice that the left hand side is in fact an multiple summation.

In the Case 4.2, using the correspondence (4.2), we can also study the non-symmetric orthogonal polynomials on T^d, which correspond to the non-symmetric orthogonal polynomials associated to the hyperoctahedral group on B^d, as indicated in Section 2.5. These polynomials are closely related to the Laguerre type polynomials defined as eigenfunctions of (1.3), as we shall see in the following section.

5. Generalized Laguerre polynomials

We showed in Section 3 that the generalized Hermite polynomials can be derived as the limit of the generalized classical orthogonal polynomials on B^d. In this section we show that orthogonal polynomials on the simplex T^d converge to the generalized Laguerre polynomials which are orthogonal with respect to the weight function

$$W_\kappa^L(\mathbf{x}) = h_\kappa^2(\sqrt{x_1}, \ldots, \sqrt{x_d}) e^{-|\mathbf{x}|_1} / \sqrt{x_1 \cdots x_d}, \qquad \mathbf{x} \in \mathbb{R}_+^d,$$

where $\mathbb{R}_+^d = \{\mathbf{x} \in \mathbb{R}^d : x_1 \geq 0, \ldots, x_d \geq 0\}$ and h_κ subject to the same assumption as in the previous section, that is, it is invariant under the semi-direct product of a reflection group G_0 and $\mathbb{Z}_2^d$. Hence, corresponding to the two cases studied in Section 4, we consider the following two cases.

Case 5.1. The abelian group $\mathbb{Z}_2^d$:

$$W_\kappa^L(\mathbf{x}, \mathbb{Z}_2^d) = x_1^{\kappa_1} \ldots x_d^{\kappa_d} e^{-|\mathbf{x}|_1}. \tag{5.1}$$

This is the multiple of the Laguerre weight function in one variable. The polynomials orthogonal with respect to it are the multiple Laguerre polynomials.

Case 5.2. The hyperoctahedral group $\mathcal{B}_d$:

$$W_\kappa^L(\mathbf{x}) = \prod_{i=1}^d x_i^{\kappa_1} \prod_{1 \leq i < j \leq d} |x_i - x_j|^{\kappa_0} e^{-|\mathbf{x}|_1}, \tag{5.2}$$

which is the weight function in (1.4). The orthogonal polynomials are the generalized Laguerre polynomials associated to (1.4).

We should emphasize that the parameter κ in (5.1) and (5.2) is chosen so that the notation is in line with the usual one for multiple Laguerre polynomials and in line with (1.4); these weight functions correspond to $W^T_{\kappa+1/2,\mu}(\mathbf{x})$ on T^d, where we write $\kappa+1/2=(\kappa_1+1/2,\ldots,\kappa_d+1/2)$. We again denote the normalized weight function by $\tilde W^L_\kappa$. Then

$$\tilde W^L_\kappa(\mathbf{x}) = 2w^L_\kappa \tilde h^2_\kappa(\sqrt{x_1},\ldots,\sqrt{x_d})e^{-|\mathbf{x}|_1}/\sqrt{x_1\cdots x_d}, \tag{5.3}$$

where $w^L_\kappa = w^H_\kappa$. The orthogonal polynomials on $\mathbb{R}^d_+$ can be derived from those orthogonal on $\mathbb{R}^d$ using an analogy of Theorem 4.1. In fact, let $V^d_{2n}(W^H_\kappa,\mathbb{Z}^d_2)$ denote the space consisting of polynomials in $V^d_{2n}(W^H_\kappa)$ that are invariant under $\mathbb{Z}^d_2$; then (4.2) establishes a one-to-one correspondence between an orthonormal basis of $V^d_{2n}(\tilde W^H_\kappa,\mathbb{Z}^d_2)$ and an orthonormal basis of $\mathcal{V}^d_n(\tilde W^L_\kappa)$. Let $\{S^h_{2n,\beta}\}$ denote a basis of $\mathcal{H}^d_{2n}(\tilde h^2_\kappa;\mathbb{Z}^d_2)$. The orthonormal polynomials associated to the $\tilde W^L_\kappa$ are then given by

$$P^n_{j,\beta}(W^L_\kappa;\mathbf{x}) = c^L_{j,n}\tilde H^{2n-2j+\gamma_\kappa+\frac{d-1}{2}}_{2j}(\sqrt{|\mathbf{x}|_1})S^h_{2n-2j,\beta}(\sqrt{x_1},\ldots,\sqrt{x_d}) \tag{5.4}$$

where $c^L_{j,n}=c^H_{j,2n}$. This basis can be derived from (3.5). We also have

Proposition 5.1. *An orthonormal basis of $\mathcal{V}^d_n(\tilde W^L_{\kappa,\mu})$ is given by*

$$\left\{P^n_{j,\beta}(W^L_\kappa;\mathbf{x}) : 0\le j\le n, S^h_{2n-2j}\in\mathcal{H}^d_{2n-2j}(h^2_\kappa,\mathbb{Z}^d_2)\right\}.$$

We denote by L^α_n the usual Laguerre polynomials (cf. [26]), and denote the orthonormal basis of $L^2(\tilde w^{(\mu)}_L dx,\mathbb{R}_+)$, where $\tilde w^{(\mu)}_L(x)=x^\mu e^{-x}/\Gamma(\mu+1)$, by $\tilde L_n(x)$. Then

$$\tilde L^\mu_n(x) = L^\mu_n(x)/\sqrt{L^\mu_n(0)}, \qquad \text{where} \quad L^\mu_n(0)=\binom{n+\mu}{n}.$$

The formula (3.1) shows that we can replace $H^{(\lambda)}_{2k}$ in (5.4) by Laguerre polynomials. In the Case 5.1, an orthonormal basis can be given by a product of Laguerre polynomials,

$$\tilde L^\kappa_\alpha(x) := \tilde L^{\kappa_1}_{\alpha_1}(x_1)\cdots\tilde L^{\kappa_d}_{\alpha_d}(x_d). \tag{5.5}$$

We also use the notation L^κ_α for the usual multiple Laguerre polynomials.

The generalized Laguerre polynomials can be derived from the generalized Hermite polynomials, as we indicated above, just as the polynomials $P^n_{j,\beta}(W^T_{\kappa,\mu})$ are derived from those $\mathbb{Z}^d_2$ invariant orthogonal polynomials on B^d. Another approach to the generalized Laguerre polynomials is to show that they are the limits of the generalized classical polynomials on the simplex. Indeed, using the limit relation (3.12), we can conclude that

Theorem 5.2. *For $P_{j,\beta}^n(W_{\kappa,\mu}^T)$ defined in (4.6) and $P_{j,\beta}^n(W_\kappa^L)$ defined in (5.4), we have*

$$\lim_{\mu\to\infty} P_{j,\beta}^n(W_{\kappa+1/2,\mu}^T;\mathbf{x}/\mu) = P_{j,\beta}^n(W_{\kappa,\mu}^L;\mathbf{x}). \tag{5.6}$$

In Case 5.1, we can work with the basis (4.7) of product type. We have

Theorem 5.3. *The multiple Laguerre polynomials (5.5) associated to W_κ^L in (5.1) are the limit of the product type basis in (4.7),*

$$\lim_{\mu\to\infty} R_\alpha(W_{\kappa+1/2,\mu}^T;\mathbf{x}/\mu) = \tilde L_\alpha^\kappa(\mathbf{x}) \tag{5.7}$$

Thus, the classical orthogonal polynomials on T^d converge to the multiple Laguerre polynomials. As a consequence of the limit relation, we can derive a differential-difference equation for the generalized Laguerre polynomials from the equation satisfied by the generalized classical polynomials on the simplex. To derive the equation, we make a change of variables $\mathbf{x}\mapsto\mathbf{x}/\mu$ in the equations in Theorems 4.2 and 4.3 and then divide by μ and take limit $\mu\to\infty$, as in the proof of Theorem 3.3. This way, we get from the equation in Theorem 4.2 the following result.

Theorem 5.4. *The multiple Laguerre polynomials associated to W_κ^L in (5.1) satisfy the partial differential equation*

$$\sum_{i=1}^d x_i\frac{\partial^2 P}{\partial x_i^2} + \sum_{i=1}^d \Big((\kappa_i+1)-x_i)\frac{\partial P}{\partial x_i}\Big) = -nP.$$

This is the classical differential equation satisfied by the multiple Laguerre polynomials. In the Case 5.2, we obtain from the equation in Theorem 4.3 the following result.

Theorem 5.5. *The orthogonal polynomials associated with W_κ^L in (5.2) satisfy the differential-difference equation*

$$\begin{aligned}\Big[&\sum_{i=1}^d\Big(x_i\frac{\partial^2}{\partial x_i^2}+(\kappa_1+1-x_i)\frac{\partial}{\partial x_i}\Big)\\ &+\kappa_0\sum_{1\le i<j\le d}\frac{1}{x_i-x_j}\Big(2\Big(x_i\frac{\partial}{\partial x_i}-x_j\frac{\partial}{\partial x_j}\Big)-\frac{x_i+x_j}{x_i-x_j}(1-\sigma_{ij})\Big)\Big]P=-nP.\end{aligned}$$

The left hand side of this equation is exactly the operator $\mathcal{H}_B$ in (1.3). These equations can also be obtained from the correspondence between $\mathcal{V}_{2n}^d(W_\kappa^H,\mathbb{Z}_2)$ and $\mathcal{V}_n^d(W_\kappa^L)$, similar to the way we derived Theorems 4.2 and 4.3 from (2.14). From the limit relation we can also derive the Mehler type formula for the Laguerre polynomials. We derive such a formula from the compact formula for the reproducing kernel of $\mathcal{V}_n^d(W_{\kappa,\mu}^T)$ in Theorem 4.5, following the proof of Theorem 3.6.

Theorem 5.6. *Let $\{P^n_\alpha(W^L_\kappa)\}$, $|\alpha|_1 = n$ and $n \in \mathbb{N}_0$, denote an orthonormal basis with respect to W^L_κ on $\mathbb{R}^d_+$. Then for $0 < z < 1$ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d_+$*

$$\sum_{n=0}^{\infty} \sum_{|\alpha|_1=n} P^n_\alpha(W^L_\kappa; \mathbf{x}) P^n_\alpha(W^L_\kappa; \mathbf{y}) z^n$$
$$= \frac{1}{(1-z)^{\gamma_\kappa + d/2}} \exp\Big\{ - \frac{z(|\mathbf{x}|_1 + |\mathbf{y}|_1)}{1-z} \Big\} V_\kappa \Big[\exp\Big\{ \frac{2\sqrt{z}\langle \{\mathbf{x}\}^{1/2}, \cdot \rangle}{1-z} \Big\} \Big] (\{\mathbf{y}\}^{1/2})$$

Proof. The proof follows the same line as that of Theorem 3.6. We merely point out the difference. Since we have $C^{(\lambda)}_{2n}$ in the formula of Theorem 4.5, we need to expand

$$\tilde{K}_\mu(\mathbf{x}, \mathbf{y}, \mathbf{u}, z) = [K_\mu(\mathbf{x}, \mathbf{y}, \mathbf{u}, z) + K_\mu(\mathbf{x}, \mathbf{y}, \mathbf{u}, -z)]/2$$

in terms of the Gegenbauer series, where K_μ is defined from the formula for the reproducing kernel of $\mathcal{V}^d_n(W^T_{\kappa,\mu})$, similar to the kernel in the proof of Theorem 3.6. Since $\tilde{K}_\mu$ is even in z, the coefficients of $C^{(\lambda)}_{2n+1}$ in the expansion vanishes. This way, we can write $\tilde{K}_\mu$ in terms of an infinite sum in z^{2n} with coefficients given by $P_n(W^T_{\kappa,\mu}; \mathbf{x}, \mathbf{y})$. Rest of the proof follows just as in the proof of Theorem 3.6. We omit the details. □

In the Case 5.1, we can write the formula in Theorem 5.6 explicitly out, using the compact formula of the intertwining operator in (2.29). Moreover, introducing the modified Bessel function I_τ of order τ,

$$I_\tau(x) = \frac{1}{\sqrt{\pi}\Gamma(\tau + 1/2)} \Big(\frac{x}{2}\Big)^\tau \int_{-1}^{1} e^{-xt}(1-t^2)^{\tau - 1/2} dt,$$

which can be written as $I_\tau(x) = e^{-\tau\pi i/2} J_\tau(x e^{\pi i/2})$, where J_τ is the standard Bessel function of order τ, we see that the formula takes the form

$$\sum_{n=0}^{\infty} \sum_{|\alpha|_1=n} \tilde{L}^\kappa_\alpha(\mathbf{x}) \tilde{L}^\kappa_\alpha(\mathbf{x}) z^n = \prod_{i=1}^{d} \frac{(\Gamma\kappa_i + 1)}{1-z} \exp\Big\{ - \frac{z(x_i + y_i)}{1-z} \Big\} (x_i y_i z)^{-\kappa_i/2} I_{\kappa_i}\Big(\frac{2\sqrt{z x_i y_i}}{1-z} \Big),$$

which is the product of the classical formula for the Laguerre polynomials ([14, Vol. 2, p. 189]).

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