LOWER BOUND FOR THE NUMBER OF NODES OF CUBATURE FORMULAE ON THE UNIT BALL

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ABSTRACT. A lower bound for positive cubature formulae on the unit ball is given. For the Chebyshev weight function on the ball in $\mathbb{R}^2$, the new bound shows that a positive cubature formula of degree s with all nodes inside the ball will need at least $N_s \ge 0.13622 s^2(1+\mathcal{O}(s^{-1}))$ number of nodes, in comparison with the classical lower bound of $N_s \ge 0.125 s^2(1+\mathcal{O}(s^{-1}))$.

1. INTRODUCTION

Let W be a nonnegative weight function defined on $\mathbb{R}^d$ and denote by $\mathcal{L}(f) = \int_{\mathbb{R}^d} f(x)W(x)dx$. A cubature formula of degree s with respect to W is a linear functional

$$\mathcal{I}_s(f) = \sum_{k=1}^{N_s} \lambda_k f(x_k), \qquad \lambda_k \in \mathbb{R}, \quad x_k \in \mathbb{R}^d,$$

such that $\mathcal{I}_s(f) = \mathcal{I}(f)$ for all polynomials f of degree at most s, and there exists at least one polynomial f^* of degree exactly s for which $\mathcal{I}_s(f^*) \ne \mathcal{L}(f^*)$. If all cubature weights λ_k are positive, we call $\mathcal{I}_s$ a *positive cubature*. The purpose of this paper is to provide a lower bound for N_s for the weight function

$$W_\mu(x) = w_\mu(1-\|\mathbf{x}\|^2)^{\mu-1/2}, \qquad w_\mu = \frac{\Gamma(\mu+\frac{d+1}{2})}{\pi^{d/2}\Gamma(\mu+\frac{1}{2})}$$

on the unit ball $B^d = \{x : \|x\| \le 1\}$, where $\|x\|^2 = x_1^2 + \ldots + x_d^2$ and $\mu \ge 0$. In the case $\mu = 0$, W_0 is called the Chebyshev weight function on the unit ball. The weight function W_μ is normalized so that it has unit integral over B^d. As a consequence, $\sum_{k=1}^{N_s} \lambda_k = 1$ for every cubature formula.

Let Π_n^d denote the space of polynomials of degree n in d variables. It is known that the number of nodes N_s of any cubature formula of degree s satisfies

$$N_s \ge \dim \Pi^d_{[s/2]} = \binom{[s/2]+d}{d}. \tag{1.1}$$

Furthermore, for many weight functions, including W_μ and all other centrally symmetric weight functions, N_s satisfies Möller's lower bound if s is an odd integer. For $s = 2m-1$ and $d = 2$, this bound states

$$N_{2m-1} \ge \dim \Pi^2_{m-1} + \left[\frac{m}{2}\right] = \frac{m(m+1)}{2} + \left[\frac{m}{2}\right]. \tag{1.2}$$

The bound (1.1) is classical (cf. [7]). Möller's bound appeared in [6]; it shows that the bound (1.1) is not sharp for W_μ and s being odd. Furthermore, the bound

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(1.2) is not sharp for W_μ if m is an odd integer ([9]). For cubature formula of even degree, the bound (1.1) is the only known lower bound .

In the next section we give a new lower bound for N_s that holds for both s being even and odd. The method of deriving the bound can be traced back to the work of Delsarte, Goethals and Seidel [3] for spherical designs, which has been extended and used by many authors for cubature formulae on the unit sphere; see the book of [2] and the references therein. However, as far as we know, this method has not been used for cubature formulae on any other domains. On the other hand, there is a close relation between cubature formulae on the sphere and those on the unit ball (cf. [10]). This relation suggests the extension of the method to the ball.

The method depends on a choice of an extreme function. One such function was constructed by Yudin for μ being an half integer and used in [12] to get a lower bound for the spherical designs. We extend the definition to all μ with an elementary approach. Using this function, a new lower bound for the Chebyshev weight function is derived which shows that the lower bound (1.1) and (1.2) are not sharp for s large. In fact, for $d = 2$, the new lower bound shows $N_s \ge 0.13622s^2(1+\mathcal{O}(s^{-1}))$ in contrast with $N \ge 0.125s^2(1+\mathcal{O}(s^{-1}))$ of (1.1) and (1.2). In particular, this shows that (1.1) and (1.2) are far from sharp for s large. Numerical computation shows that the new bound is better for $s \ge 31$. Further study indicates, however, that Yudin's function appears to yield improved lower bound only for W_μ with μ sufficiently small. In particular, it does not seem to give better bound for the unit weight function.

The paper is organized as follows. In the next section we derive the lower bound in general. Section 3 discusses Yudin's construction of extreme function. The lower bounds derived using Yudin's function are discussed in Section 4.

2. Lower bound for positive cubature formulae

Let $\mathcal{V}_n^d(W_\mu)$ denote the space of orthogonal polynomials of degree n with respect to W_μ on B^d. Making use of the orthogonality, it follows that a cubature formula $\mathcal{I}_s(f)$ for W_μ holds if and only if

$$\mathcal{I}_s(f) := \sum_{k=1}^{N_s} \lambda_k f(x_k) = 0, \qquad f \in \mathcal{V}_n^d(W_\mu), \quad 1 \le n \le s,$$

and there is a $f^* \in \mathcal{V}_{s+1}^d$ such that $\mathcal{I}_s(f^*) \ne 0$. For a multiindex $\alpha \in \mathbb{N}_0^d$, let $|\alpha| = \alpha_1 + \cdots + \alpha_d$. Let $\{P_\alpha : |\alpha| = n\}$ denote an orthonormal basis of $\mathcal{V}_n^d(W_\mu)$. Such a basis can be given explicitly in terms of Gegenbauer polynomials (see, for example, [4]). The reproducing kernel of $\mathcal{V}_n^d$ is denoted by $P_n(W_\mu; x, y)$ and it can be given in terms of the orthonormal basis $\{P_\alpha\}$,

$$P_n(W_\mu; x, y) = \sum_{|\alpha|=n} P_\alpha(x) P_\alpha(y). \tag{2.1}$$

Let C_n^λ denote the Gegenbauer polynomials of degree n, which are orthogonal polynomials with respect to $w_\lambda(t) = (1-t^2)^{\lambda-1/2}$ and normalized so that $C_n^\lambda(1) =$

$\binom{n+2\lambda-1}{n}$. Then the reproducing kernel $P_n(W_\mu; x, y)$ has a compact formula ([11])

$$P_n(W_\mu; x, y) = c_\mu \frac{n + \mu + \frac{d-1}{2}}{\mu + \frac{d-1}{2}} \times \int_{-1}^{1} C_n^{\mu+\frac{d-1}{2}}(\langle x, y\rangle + \sqrt{1-\|x\|^2}\sqrt{1-\|y\|^2}\, t)(1-t^2)^{\mu-1} dt, \tag{2.2}$$

where $\langle x, y\rangle$ denotes the usual Euclidean inner product of $x, y \in \mathbb{R}^d$ and the constant c_μ is given by

$$c_\mu = \left[\int_{-1}^{1} (1-t^2)^{\mu-1} dt\right]^{-1} = \frac{\Gamma(\mu + 1/2)}{\sqrt{\pi}\Gamma(\mu)}.$$

For $\mu = 0$, the formula (2.2) holds under the limit

$$\lim_{\mu\to 0} c_\mu \int_{-1}^{1} f(t)(1-t^2)^{\mu-1} dt = \frac{1}{2}[f(1) + f(-1)]. \tag{2.3}$$

We will need to consider the Fourier expansion of a function f in terms of the Gegenbauer polynomials. For $f \in L^2(w_\lambda)$, its Gegenbauer expansion is

$$f(t) = \sum_{n=0}^{\infty} \widehat{f}_n^\lambda C_n^\lambda(t), \qquad \widehat{f}_n^\lambda = h_n^\lambda \int_{-1}^{1} f(t) C_n^\lambda(t)(1-t^2)^{\lambda-\frac{1}{2}} dt,$$

in which $[h_n^\lambda]^{-1} = \int_{-1}^{1}[C_n^\lambda(t)]^2(1-t^2)^{\lambda-1/2} dt$. We need one more definition.

Definition 2.1. *For $\lambda \geq 0$ and $s \geq 1$, define*

$$\mathcal{F}_s(\lambda) = \{f : f \in C[-1,1], \quad f(t) \geq 0, \quad t \in [-1,1], \quad \text{and} \quad f(t) = \sum_{k=0}^{\infty} \widehat{f}_k^\lambda C_k^\lambda(t) \quad \text{with} \quad \widehat{f}_k^\lambda \leq 0 \ \text{for} \ k > s\}.$$

Furthermore, for a continuous function F on $[-1,1]$, we define

$$\sigma(F; r) = c_\mu \int_{-1}^{1} F(r + (1-r)t)(1-t^2)^{\mu-1} dt, \qquad 0 \leq r \leq 1. \tag{2.4}$$

Theorem 2.2. *Let F be defined on $[-1,1]$ such that $F \in \mathcal{F}_s(\mu + (d-1)/2)$. Then the number of nodes, N_s, of a positive cubature formula with respect to W_μ whose nodes are all inside B^d satisfies*

$$N_s \geq \min_{0\leq r\leq 1} \sigma(F; r)/\widehat{F}_0^\lambda, \qquad \lambda = \mu + \frac{d-1}{2}.$$

Proof. Let F be given as in the statement. Consider the function $\Phi(x, y)$ defined by

$$\Phi(x, y) = c_\mu \int_{-1}^{1} F(\langle x, y\rangle + \sqrt{1-\|x\|^2}\sqrt{1-\|y\|^2}\, t)(1-t^2)^{\mu-1} dt.$$

In this proof we write $\widehat{F}_n = \widehat{F}_n^\lambda$. Expanding F in terms of the Gegenbauer polynomials C_n^λ and using the formula (2.2), we conclude that

$$\Phi(x, y) = \sum_{n=0}^{\infty} \widehat{F}_n \frac{\lambda}{n+\lambda} P_n(W_\mu; x, y).$$

By the definition of the reproducing kernel (2.1) in terms of an orthonormal basis,

$$\sum_{k,l=1}^{N_s} \lambda_k \lambda_l P_n(W_\mu; x_k, x_l) = \sum_{|\alpha|=n} \left(\sum_{k=1}^{N_s} \lambda_k P_\alpha(x_k) \right)^2 \ge 0, \qquad n \in \mathbb{N}_0.$$

Consequently, using the fact that the cubature formula is of degree s so that $\sum_{k=1}^{N_s} P_n(W_\mu; x_k, x) = 0$ for $1 \le n \le s$ by (2.1), it follows from the assumption $\widehat{F}_n \le 0$ for $n > s$ that

$$\begin{aligned} I := \sum_{k,l=1}^{N_s} \lambda_k \lambda_l \Phi(x_k, x_l) &= \sum_{n=0}^{\infty} \widehat{F}_n \frac{\lambda}{n+\lambda} \sum_{k,l=1}^{N_s} \lambda_k \lambda_l P_n(W_\mu; x_k, x_l) \\ &= \widehat{F}_0 \sum_{k,l=1}^{N_s} \lambda_k \lambda_l + \sum_{n=s+1}^{\infty} \widehat{F}_n \frac{\lambda}{n+\lambda} \sum_{k,l=1}^{N_s} \lambda_k \lambda_l P_n(W_\mu; x_k, x_l) \\ &\le \widehat{F}_0 \sum_{k,l=1}^{N_s} \lambda_k \lambda_l = \widehat{F}_0. \end{aligned}$$

Since $F(t) \ge 0$, it follows that $\Phi(x, y) \ge 0$. Hence, for the positive cubature formula we have by the Cauchy inequality that

$$\begin{aligned} I = \sum_{k,l=1}^{N_s} \lambda_k \lambda_l \Phi(x_k, x_l) &\ge \sum_{k=1}^{N_s} \lambda_k^2 \Phi(x_k, x_k) \\ &= \sum_{k=1}^{N_s} \lambda_k^2 \sigma(F, \|x_k\|^2) \ge \min_{0 \le r \le 1} \sigma(F, r) \sum_{k=1}^{N_s} \lambda_k^2 \\ &\ge \min_{0 \le r \le 1} \sigma(F, r) \left(\sum_{k=1}^{N_s} \lambda_k \right)^2 \Big/ N_s = \min_{0 \le r \le 1} \sigma(F, r)/N_s \end{aligned}$$

These two inequalities give the stated estimate on N_s. □

If F is an increasing function, then $\sigma(F, r)$ is an increasing function on $[0, 1]$. In this case, we have the following corollary:

Corollary 2.3. *If, in addition, F is an increasing function on $[-1, 1]$, then*

$$N_s \ge \sigma(F, 0)/\widehat{F}_0^\lambda = \frac{c_\mu \int_{-1}^1 F(t)(1-t^2)^{\mu-1} dt}{c_{\lambda+1/2} \int_{-1}^1 F(t)(1-t^2)^{\lambda-1/2} dt}, \qquad \lambda = \mu + \frac{d-1}{2}.$$

As we mentioned in the introduction, the idea of the proof goes back to [3]. It has been used for spherical designs, which are cubature formulae on the unit sphere with $\lambda_k = 1$ for all k, the so-called Chebyshev cubature formulae (see [2] and the reference therein).

3. Yudin's function in $\mathcal{F}(\lambda)$

Clearly, any polynomial of degree s that is nonnegative on $[-1, 1]$ is an element of $\mathcal{F}_s(\lambda)$, but not every choice gives a good lower bound for N_s. In the case of the spherical designs, many authors have discussed choices of proper polynomials in $\mathcal{F}_s(\lambda)$. See [2] and the reference therein.

We examine a function in $\mathcal{F}_s(\lambda)$ constructed by Yudin [12] for $\lambda = (d-2)/2$. The definition in [12] is given in terms of integration on the unit sphere and the proof uses properties of spherical harmonics. In the following, we shall give a more elementary proof that works for all $\lambda \ge 0$.

Let $\gamma_s = \gamma_s^\lambda$ be the largest zero of $C_{s-1}^{\lambda+1}(t)$. Define function f and g by

$$f(t) = \begin{cases} C_s^\lambda(t) - C_s^\lambda(\gamma_s), & t \ge \gamma_s \\ 0, & t < \gamma_s \end{cases} \qquad \text{and} \qquad g(t) = \begin{cases} 1, & t \ge \gamma_s \\ 0, & t < \gamma_s. \end{cases}$$

Proposition 3.1. *Let f and g be defined as in the above. Then the function*

$$F(x) = \int_{-1}^1 g(u) c_\lambda \int_{-1}^1 f\left(ux + t\sqrt{1-u^2}\sqrt{1-x^2}\right)(1-t^2)^{\lambda-1} dt (1-u^2)^{\lambda-\frac12} du$$

is an element of $\mathcal{F}_{s-1}(\lambda)$. In fact, $\widehat{F}_s^\lambda = 0$ and for $k \ne s$,

$$\widehat{F}_k^\lambda = (h_k^\lambda)^2 \frac{s(s+2\lambda)}{(k-s)(k+s+2\lambda)} C_s^\lambda(\gamma_s) \frac{\lambda}{h_0^\lambda (k+\lambda)} \left(\int_{\gamma_s}^1 C_k^\lambda(y)(1-y^2)^{\lambda-\frac12} dy \right)^2.$$

The proof of this proposition depends on the following lemma.

Lemma 3.2. *For $s \ne k$ and $x \in [-1,1]$,*

$$\int_x^1 C_s^\lambda(y) C_k^\lambda(y)(1-y^2)^{\lambda-1/2} dy - a_{s,k} C_s^\lambda(x) \int_x^1 C_k^\lambda(y)(1-y^2)^{\lambda-1/2} dy$$
$$= b_{s,k}(1-x^2)^{\lambda+1/2} C_k^\lambda(x) \frac{d}{dx} C_s^\lambda(x),$$

where

$$a_{s,k} = \frac{k(k+2\lambda)}{(k-s)(k+s+2\lambda)}, \qquad b_{s,k} = -\frac{1}{(k-s)(k+s+2\lambda)}.$$

Proof. First we recall that the Gegenbauer polynomials satisfy a differential equation [8, p. 80]

$$\frac{d}{dx}\left[(1-x^2)^{\lambda+1/2} \frac{d}{dx} C_n^\lambda(x)\right] = -n(n+2\lambda)(1-x^2)^{\lambda-1/2} C_n^\lambda(x). \tag{3.1}$$

Integrating the differential equation (3.1) gives

$$\int_x^1 C_k^\lambda(y)(1-y^2)^{\lambda-1/2} dy = \frac{1}{k(k+2\lambda)} (1-x^2)^{\lambda+1/2} \frac{d}{dx} C_k^\lambda(x), \quad k > 0. \tag{3.2}$$

Using (3.2) the derivative of the left hand side of the stated equation is

$$(a_{s,k}-1) C_s^\lambda(x) C_k^\lambda(x)(1-x^2)^{\lambda-1/2} - \frac{a_{s,k}}{k(k+2\lambda)} \frac{d}{dx} C_s^\lambda(x) \frac{d}{dx} C_k^\lambda(x)(1-x^2)^{\lambda+1/2}$$
$$= -b_{s,k}(1-x^2)^{\lambda-1/2}\left[s(s+2\lambda) C_s^\lambda(x) C_k^\lambda(x) - (1-x^2) \frac{d}{dx} C_s^\lambda(x) \frac{d}{dx} C_k^\lambda(x)\right].$$

Using (3.1) again, the last expression is easily seen to be the derivative of the right hand side of the stated equation. □

Proof of Proposition 3.1. Since f is a continuous function (in fact, f is differentiable), it is evident that F is continuous for $x \in [-1,1]$. The fact that γ_s is the largest zero of $(d/dx) C_s^\lambda(x) = 2\lambda C_{s-1}^{\lambda+1}(x)$ shows that $f(x) \ge 0$ on $[-1,1]$, so that $F(x) \ge 0$. Let $a_{s,k}$ be as in the previous lemma. Then

$$a_{s,k} - 1 = \frac{s(s+2\lambda)}{(k-s)(k+s+2\lambda)}.$$

Consequently, the identity in the previous lemma with $x = \gamma_s$ shows that for $k \neq s$,

$$\begin{aligned}\widehat{f}_k &= h_k^\lambda \int_{\gamma_s}^1 (C_s^\lambda(y) - C_s^\lambda(\gamma_s))C_k^\lambda(y)(1-y^2)^{\lambda-\frac{1}{2}}dy \\ &= h_k^\lambda \frac{s(s+2\lambda)}{(k-s)(k+s+2\lambda)} C_s^\lambda(\gamma_s) \int_{\gamma_s}^1 C_k^\lambda(y)(1-y^2)^{\lambda-\frac{1}{2}}dy.\end{aligned}$$

Moreover, using the fact that $\int_0^1 |C_k^\lambda(t)|(1-t^2)^{\lambda-1/2}dt = \mathcal{O}(k^{\lambda-1})$ ([8, (7.34.1), p. 173]) it is easy to see that $|\widehat{f}_k| = \mathcal{O}(k^{-\lambda-1})$; hence, using the fact that $C_k^\lambda(t) = \mathcal{O}(k^{\lambda-1})$ for $-1 < t < 1$, it follows that the expansion of f converges uniformly on every compact set inside $[-1,1]$. The definition of $g(x)$ shows that the Fourier coefficients of g is

$$\widehat{g}_k = h_k^\lambda \int_{\gamma_s}^1 C_k^\lambda(y)(1-y^2)^{\lambda-\frac{1}{2}}dy$$

In particular, the equation (3.2) shows that $\widehat{g}_s = 0$. The Gegenbauer polynomials satisfy a product formula

$$C_k^\lambda(u)C_k^\lambda(x) = C_k^\lambda(1)c_\lambda \int_{-1}^1 C_k^\lambda(ux + t\sqrt{1-u^2}\sqrt{1-x^2})(1-t^2)^{\lambda-1}dt.$$

Using this formula, we get

$$\begin{aligned}F(x) &= \int_{-1}^1 g(u) \sum_{k=0}^\infty \widehat{f}_k \frac{1}{C_k^\lambda(1)} C_k^\lambda(u)C_k^\lambda(x)(1-u^2)^{\lambda-1/2}du \\ &= \sum_{k=0}^\infty \widehat{f}_k \widehat{g}_k \frac{\lambda}{h_0^\lambda(k+\lambda)} C_k^\lambda(x),\end{aligned}$$

since $h_k^\lambda C_k^\lambda(1) = h_0^\lambda(k+\lambda)/\lambda$ ([7, (4.7.3) and (4.7.15)]), from which the formula for $\widehat{F}_k$ follows. That $F \in \mathcal{F}_{s-1}(\lambda)$ follows from the explicit formula of $\widehat{F}_k$ and from the fact that $C_s^\lambda(\gamma_s) < 0$. □

For $\lambda = (d-1)/2$, the function F is defined in [12] by

$$F(\langle a, b\rangle) = \int_{S^{d-1}} f(\langle a, x\rangle)g(\langle x, b\rangle)d\omega(x)$$

with the same f and g. The proof in [12] uses the properties of the harmonic functions.

The definition in Proposition 3.1 allows us to derive the following properties of F, which shows in particular that F is supported on a small interval, since $\gamma_s \to 1$ as $s \to \infty$.

Proposition 3.3. *Let F be defined as in the previous proposition. Then $F(x) = 0$ if $x \le 2\gamma_s^2 - 1$ and $F(x)$ is nondecreasing on $[-1, \gamma_2]$.*

Proof. Let $\phi(t, u, x) = ux + t\sqrt{1-u^2}\sqrt{1-x^2}$ and $\phi(u, x) = \phi(1, u, x)$. Since $g(u) = 0$ for $u \le \gamma_s$ and $g(u) = 1$ for $u \ge \gamma_s$, we can write

$$F(x) = \int_{\gamma_s}^1 c_\lambda \int_{-1}^1 f(\phi(t, u, x))(1-t^2)^{\lambda-1}dt(1-u^2)^{\lambda-1/2}du. \tag{3.3}$$

Assume $x \le \gamma_s \le u \le 1$. Then it is easy to verify that $\partial\phi/\partial u < 0$ for this range of x and u, so that $\phi(u, x) \le \phi(\gamma_s, x)$ for $x \le \gamma_s$. Furthermore, solving the inequality $\phi(\gamma_s, x) \le \gamma_s$ gives $x \le 2\gamma_s^2 - 1$. Clearly, $2\gamma_s^2 - 1 \le \gamma_s$. Therefore, if $x \le 2\gamma_s^2 - 1$, then

$\phi(t,u,x) \le \phi(u,x) \le \gamma_s$, so that $f(\phi(t,u,x)) = 0$ for $-1 < t < 1$ and, consequently, $F(x) = 0$.

Next we show that F is nondecreasing for $x \in [-1, \gamma_s]$. Since $f'(x) = \dfrac{d}{dx} C_s^\lambda(x)$ for $x \ge \gamma_s$, it follows that f' is nonnegative on $[-1,1]$. Taking derivative of F in (3.3) gives

$$F'(x) = \int_{\gamma_s}^1 c_\lambda \int_{-1}^1 f'(\phi(t,u,x)) \frac{\partial}{\partial x} \phi(t,u,x)(1-t^2)^{\lambda-1} dt (1-u^2)^{\lambda-1/2} du$$

The assumption that $x \le \gamma_s \le u$ implies that $\dfrac{\partial}{\partial x}\phi(t,u,x) = u - tx\sqrt{1-u^2}/\sqrt{1-x^2}$ ≥ 0 which shows that $F'(x) \ge 0$ and F is nondecreasing. $\square$

In fact, numerical evidence indicates that the function F is increasing on $[-1,1]$.

4. Lower bound for special weight functions

Using Yudin's function in Theorem 2.2 gives an explicit lower bound for positive cubature formulae. The bound takes a particular simple form for the Chebyshev weight function.

Theorem 4.1. *Every positive cubature formula for the Chebyshev weight function $(1-\|x\|^2)^{-1/2}$ on the ball B^d whose nodes are inside B^d satisfies*

$$N_s \ge \int_0^1 (1-t^2)^{(d-2)/2} dt \Big/ \int_{\gamma_s}^1 (1-t^2)^{(d-2)/2} dt, \tag{4.1}$$

where γ_s is the largest zero of $C_s^{(d+1)/2}(t)$.

Proof. We use F in the previous proposition, except that we take $F \in \mathcal{F}_s(\lambda)$ instead of $\mathcal{F}_{s-1}(\lambda)$, where $\lambda = (d-1)/2$. For the Chebyshev weight, we use the limit (2.3) to conclude that

$$\min_{0\le r\le 1} \sigma(F,r) = \min_{0\le r\le 1} (F(1) + F(2r-1))/2 = F(1)/2.$$

We have $\widehat{F}_0 = \widehat{f}_0 \widehat{g}_0 / h_0^\lambda$ and $F(1) = \int_{\gamma_s}^1 f(u)(1-u^2)^{\lambda-1/2} du = \widehat{f}_0/h_0^\lambda$. Consequently,

$$N_s \ge \sigma(F,0)/\widehat{F}_0 = F(1)/(2\widehat{F}_0) = 1/(2\widehat{g}_0) = \frac{1}{2h_0^\lambda \int_{\gamma_s}^1 (1-y^2)^{\lambda-1/2} dy}$$

and the estimate follows from the definition of h_0^λ. $\square$

In the case of $d = 1$, we are looking at the quadrature formula for Chebyshev weight on the interval $[-1,1]$. In this case, γ_s is the largest zero of the Chebyshev polynomial of the second kind $C_s^1(t) = U_s(t)$. It follows that $\gamma_s = \cos(\pi/(s+1))$. Consequently, the bound in the theorem states that $N_s \ge (s+1)/2$, which is sharp. The case $d = 2$ is stated in the following corollary.

Corollary 4.2. *For the Chebyshev weight function on B^2,*

$$N_s \ge (j_1/2)s^2(1 + \mathcal{O}(s^{-1})) \ge 0.13622 s^2 (1 + \mathcal{O}(s^{-1})),$$

where j_1 is the first positive zero of the Bessel function $J_1(x)$.

Proof. We use the asymptotic formula for the largest zero $\cos\theta_s$ of C_s^α ([1, p. 787] or [5, Corollary 2]),

$$\theta_s = \frac{j_{\alpha-1/2}}{s} + \mathcal{O}(s^{-3}),$$

where j_α is the first positive zero of the Bessel function $J_\alpha(t)$. In the case of $d=2$, the lower bound in the previous theorem takes the form $N_s \ge 1/(1-\gamma_s)$, where γ_s is the largest zero of $C_s^{3/2}(t)$. It follows that

$$N_s \ge \frac{1}{1-\cos\theta_s} = \frac{2s^2}{j_1^2}(1+\mathcal{O}(s^{-1})) = 0.136221...\, s^2(1+\mathcal{O}(s^{-1}))$$

for the Chebyshev weight function on B^2. □

For s sufficiently large, this lower bound improves the lower bound given in (1.1) and (1.2), since those classical bounds give

$$N_s \ge (s^2/8)(1+\mathcal{O}(s^{-1})) = 0.125s^2(1+\mathcal{O}(s^{-1})).$$

In particular, this shows that the lower bound in (1.1) and (1.2) is far off for s large. The numerical computation of the bound in Corollary 4.2 shows that the bound in (4.1) starts to get better than that of (1.1) and (1.2) when $s \ge 31$. We give the numerical result of the first few improved bounds in the following table, in which we let

$$N_s = \text{ Bound in (1.1) and (1.2)} \qquad \text{and} \qquad N_s^* = \text{Bound in (4.1)}$$

s	29	30	31	32	33	34	35
N_s	127	136	144	153	161	171	180
N_s^*	126.85	135.3	144.07	153.007	162.27	171.8	181.6

Let us point out that, in terms of asymptotics behavior in s, the lower bounds given in Theorem 4.1 also work for weight functions W_m with m being an integer. Let us consider the case of $d=2$ and $m=1$. If $\sum_{k=1}^N \lambda_k f(x_k)$, $x_k \in B^d$, is a positive cubature formula of degree s for $W_0(x) = 1/\sqrt{1-\|x\|^2}$, then N satisfies the lower bound in Corollary 4.2. Setting $f(x) = (1-\|x\|)g(x)$ leads to a cubature formula $\sum_{k=1}^N \lambda_k^* g(x_k)$ of degree $s-2$ for $W_1(x) = \sqrt{1-\|x\|^2}$. Clearly every positive cubature formula for W_1 can be derived this way, which shows that the number of nodes N_s for W_1 is bounded below by $0.13622(s+2)^2(1+\mathcal{O}(s^{-1}))$, which is the same as $0.13622s^2(1+\mathcal{O}(s^{-1}))$ for s sufficiently large.

The lower bound for the Chebyshev weight function is obtained as the limit of the lower bound in Theorem 2.2 for $\mu \to 0$. Hence, we can expect improved lower bounds for the weight function W_μ for μ small. The most interesting case is naturally the case $\mu = 1/2$ for which the weight function W_μ is a constant. In this case, $\lambda = 1$, and the graph of the function for small s indicates that F is an increasing function and that the lower bound in Corollary 2.3 takes the form

$$N_s \ge \frac{1}{2}\int_{-1}^1 F(t)(1-t^2)^{-1/2}dt \Big/ \int_{-1}^1 F(t)(1-t^2)^{1/2}dt. \tag{4.2}$$

However, it is rather surprising that this lower bound appears to be worse than the classical bound. We need an explicit formula for the function F. Let $\gamma_s = \cos\theta_s$.

By Proposition 3.3, $F(\cos\theta) = 0$ for $\theta \ge 2\theta_s$. Let $f_s(x) = C_s^\lambda(x) - C_d^\lambda(\gamma_s)$ and define

$$H_\lambda(\theta,\phi,\xi) = (\cos(\theta-\phi) - \cos\xi)^{\lambda-1}(\cos\xi - \cos(\theta+\phi))^{\lambda-1}.$$

Since $f(\cos\xi) = 0$ if $\xi \ge \theta_s$, it follows that for $0 \le \theta \le \theta_s$

$$F(\cos\theta) = \frac{c_\lambda}{(\sin\theta)^{2\lambda-1}}\Big[\int_0^{\theta_s-\theta}\int_{|\theta-\phi|}^{\theta+\phi} f_s(\cos\xi)H_\lambda(\theta,\phi,\xi)\sin\xi d\xi \sin\phi d\phi + \int_{\theta_s-\theta}^{\theta_s}\int_{|\theta-\phi|}^{\theta_s} f_s(\cos\xi)H_\lambda(\theta,\phi,\xi)\sin\xi d\xi \sin\phi d\phi\Big]$$

and for $\theta_s \le \theta \le 2\theta_s$

$$F(\cos\theta) = \frac{c_\lambda}{(\sin\theta)^{2\lambda-1}}\int_{\theta-\theta_s}^{\theta_s}\int_{\theta-\phi}^{\theta_s} f_s(\cos\xi)H_\lambda(\theta,\phi,\xi)\sin\xi d\xi \sin\phi d\phi.$$

For $\mu = 1/2$ we have $\lambda = 1$ and $H_\lambda(\theta,\phi,\xi) = 1$, and C_s^1 is the Chebyshev polynomial of the second kind, so that we can derive an explicit formula of F. The formula is still complicated, but it allows us to draw the graph of F which shows that it is indeed increasing and compute the lower bound in (4.2) using a computer. The numerical computation for s up to 200 shows that this bound is much worse than the classical bound in (1.1) and (1.2). It appears that Yudin's function only gives improved bound for μ in a small neighbourhood of 0. An interesting question is to construct another family of increasing function F that will maximize the lower bound in (4.2).

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