ON POLYNOMIAL INTERPOLATION OF TWO VARIABLES

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Abstract. Polynomial interpolation of two variables based on points that are located on multiple circles is studied. First, the poisedness of a Birkhoff interpolation on points that are located on several concentric circles is established. Second, using a factorization method, the poisedness of a Hermite interpolation based on points located on various circles, not necessarily concentric, is established. Even in the case of Lagrange interpolation, this gives many new sets of poised interpolation points.

1. Introduction

We study polynomial interpolation of two variables for points located on several circles (not necessarily concentric). This is a continuation of our recent study in [2]. For the background of this study we refer to the introduction section of [2], below we recall the basic definition and the result there.

Let Π_n^2 denote the space of polynomials P of two variables of total degree n,

$$P(x,y) = \sum_{k=0}^{n} \sum_{j=0}^{k} c_{j,k} x^j y^{k-j}.$$

It is known that $\dim \Pi_n^2 = (n+1)(n+2)/2$. In this paper we only consider the case that the number of interpolation conditions matches the dimension of Π_n^2. If there is a *unique* solution to an interpolation problem, we say that the problem is *poised.*

Throughout this paper, let $\partial/\partial r$ denote the normal derivative

$$\frac{\partial P}{\partial r} = \frac{\partial P}{\partial x}\cos\theta + \frac{\partial P}{\partial y}\sin\theta, \qquad (x,y) = (r\cos\theta, r\sin\theta).$$

For a positive number $r > 0$, denote the circle of radius r centered at the origin by $S(r) = \{(x,y) : x^2+y^2 = r\}$. Let $[t]$ denote the integer part of t. In [2] we studied the following Hermite interpolation problem

Problem 1. *Let n be a positive integer. Let $0 < r_1 < r_2 < \ldots < r_\lambda \le 1$ and let $\mu_1, \mu_2, \ldots, \mu_\lambda$ be non-negative integers such that*

$$\mu_1 + \mu_2 + \cdots + \mu_\lambda = \left[\frac{n}{2}\right] + 1.$$

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Denote by $\{(x_{l,j}, y_{l,j}) : 0 \le j \le 2m\}$ *distinct points on the circle* $S(r_l)$*, where* $m = [(n+1)/2]$ *and* $1 \le l \le \lambda$*. Characterize the points for which the interpolation problem*

$$\left(\frac{\partial^k P}{\partial r^k}\right)(x_{l,j}, y_{l,j}) = f_{j,l,k}, \quad 0 \le k \le \mu_l - 1, \quad 1 \le l \le \lambda, \quad 0 \le j \le 2m, \tag{1.1}$$

has a unique solution in Π_n^2 *for any given data* $\{f_{j,l,k}\}$*.*

A natural choice of points on the circle is the equidistant points. For a real number α, define

$$\Theta_{\alpha,m} = \{\theta_j^\alpha : \theta_j^\alpha = (2j+\alpha)\pi/(2m+1), \quad j = 0, 1, \ldots, 2m\}, \tag{1.2}$$

which is the set of $2m+1$ equally spaced points on the unit circle. The main result in [2] shows that the interpolation problem is poised for these points.

Theorem 1.1. *The interpolation problem* (1.1) *based on the equidistant points*

$$(x_{l,j}, y_{l,j}) = (r_l \cos\theta_j, r_l \sin\theta_j), \quad \theta_j \in \Theta_{0,m},$$

on the circles $S(r_l)$*,* $1 \le l \le \lambda$*, is poised.*

The proof exploits the structure of polynomials in two variables and reduces the problem to a Hermite-Birkhoff interpolation in one variable. There is also examples in [2] showing that the interpolation problem is not poised for arbitrary points on the circle.

We will extend the above result in two directions. The first extension comes from a strengthening of the method in [2], which allows us to include gaps in the directional derivatives in Problem 1, the so-called Birkhoff interpolation, instead of the consecutive derivatives. The Birkhoff interpolation problem is usually described using the notion of *incidence matrices*, which are matrices whose entries are 0 and 1. Let $X = \{r_1, \ldots, r_\lambda\}$ be a set of distinct real numbers. We assume that $r_1 < r_2 < \ldots < r_\lambda$. Let $E = (e_{l,k})$ be an incidence matrix with λ rows and n columns. Let $|E|$ denote the number of 1's in E; $|E| = \sum_l \sum_k e_{l,k}$. Together E and X define a Birkhoff interpolation problem in one variable

$$P^{(k)}(r_l) = f_{l,k}, \qquad e_{l,k} = 1. \tag{1.3}$$

The Birkhoff interpolation problem is said to be *poised* if there is a unique polynomial of degree at most $|E| - 1$ that satisfies the above interpolation conditions. With the notion of the incidence matrix, we consider the following generalization of Problem 1.

Problem 2. *Let* n *be a positive integer. Let* $0 < r_1 < r_2 < \ldots < r_\lambda$ *and let* E *be an incidence matrix of* λ *rows with* $|E| = ([\frac{n}{2}] + 1)(2[\frac{n+1}{2}] + 1)$*. Denote by* $\{(x_{l,j}, y_{l,j}) : 0 \le j \le 2m\}$ *distinct points on the circle* $S(r_l)$*, where* $m = [(n+1)/2]$ *and* $1 \le l \le \lambda$*. Find proper points and the incidence matrix* E *for which the interpolation problem*

$$\left(\frac{\partial^k P}{\partial r^k}\right)(x_{l,j}, y_{l,j}) = f_{j,l,k}, \quad e_{l,k} = 1, \quad 0 \le j \le 2m, \tag{1.4}$$

has a unique solution in Π_n^2 *for any given data* $\{f_{j,l,k}\}$*.*

We will show that for equidistant points on the circles and E being a matrix with no odd *sequence supported from the right* (see the definition in Section 2), the above problem has a unique solution.

The second extension of the result in [2] is in the direction of factorization in the sense of Bezout's theorem. In the case of interpolation on one circle (that is, $\lambda = 1$), an independent proof of Theorem 1.1 is given later by Hakopian and Ismail in [3]. They proved the following factorization theorem which not only yields the case $\lambda = 1$ of Theorem 1.1 but also leads to some other interesting extensions.

Theorem 1.2. *Let s, m be positive integers and $m \le s \le 2m$. If $P \in \Pi_s^2$ satisfies*

$$\Big(\frac{\partial^k P}{\partial r^k}\Big)(\cos\theta_j, \sin\theta_j) = 0, \quad 0 \le k \le s-m, \quad \theta_j \in \Theta_{0,m},$$

then there is a polynomial $Q \in \Pi_{2m-s-2}^2$ such that

$$P(x,y) = (x^2+y^2-1)^{s+1-m} Q(x,y).$$

In particular, $Q = 0$ if $s = 2m$ or $s = 2m-1$ and Q is a constant if $s = 2m-2$.

For the Lagrange interpolation (no derivatives), this is the well-known Bezout's theorem, which holds for arbitrary points on the circle (see Theorem 3.7 below). The fact that the interpolation points are equidistant on the circle play essential roles for the Hermite data. One naturally asks if the factorization theorem holds for the more general setting of several circles. The proof in [3] does not seem to apply to the interpolation problem with more than one circles. It turns out, however, that the method in [2] can be used to prove a factorization theorem for the general setting. The result is the following theorem:

Theorem 1.3. *Let n, s be positive integers, $n = 2m$ or $n = 2m-1$, and $[n/2] \le s \le n$. Let $r_1, r_2, \ldots, r_\lambda$ be distinct positive real numbers, and let $\tau_1, \tau_2, \ldots, \tau_\lambda$ be positive integers such that*

$$\tau_1 + \tau_2 + \cdots + \tau_\lambda = s - m + 1.$$

Denote by $\{(x_{l,j}, y_{l,j}) : 0 \le j \le 2m\}$ the equidistant points on the circle $S(r_l)$,

$$(x_{l,j}, y_{l,j}) = (r_l \cos\theta_j, r_l \sin\theta_j), \quad \theta_j \in \Theta_{\alpha,m}, \tag{1.5}$$

$1 \le l \le \lambda$, for a fixed $\alpha \in [0,2)$. Assume that $P \in \Pi_s^2$ satisfies

$$\Big(\frac{\partial^k P}{\partial r^k}\Big)(x_{l,j}, y_{l,j}) = 0, \quad 0 \le k \le \tau_l - 1, \quad 1 \le l \le \lambda, \quad 0 \le j \le 2m.$$

Then there is a polynomial $Q \in \Pi_{2m-s-2}^2$ such that

$$P(x,y) = \prod_{l=1}^{\lambda} (x^2+y^2-r_l^2)^{\tau_l} Q(x,y).$$

In particular, $Q = 0$ if $s = n$.

This theorem can be used to prove a number of results on the poisedness of polynomial interpolation. One important feature is that the factorization process can be used repeatedly to obtain a total factorization of a polynomial. This gives, for example, the poisedness of the Hermite interpolation on equidistant points on several groups of circles, in which the number of points on the same group of circles is the same but different from those in different groups; moreover, the circles from different groups no longer have to be concentric. Even in the case of the Lagrange interpolation, many new sets of poised interpolation points can be obtained this way (see Theorem 3.4 and Example 3.6). For the Lagrange interpolation on one circle, the factorization theorem holds for arbitrary points on the circle by the classical Bezout theorem (see Theorem 3.7 below). For several circles, however, the factorization is not a simple consequence of Bezout's theorem, the result depends on the fact that the interpolation points are equidistant on the circles.

The paper is organized as follows. The result on the Birkhoff interpolation is proved in Section 2. The factorization theorem and its consequences are presented in Section 3.

2. Birkhoff interpolation

For the Birkhoff interpolation (1.3) we recall the following notion. A sequence in an incidence matrix is a sequence of consecutive 1's in a row of E, say $e_{l,k} = 1$ for $k = i+1, \ldots, i+j$, $e_{l,i} = 0$ and $e_{l,i+j+1} = 0$; it's an odd sequence if j is odd. A supported sequence is a sequence such that there are nonzero elements of E in both its upper and lower left sides, that is, there are $e_{i_1,j_1} = 1$ and $e_{i_2,j_2} = 1$ with $i_1 < l$, $i_2 > l$, $j_1 \le i$ and $j_2 \le i$; a sequence is *supported from the right* if there are nonzero elements of E belolw and to the left of the begining of the sequence (it may be more proper to say supported from upper side, but we have in mind interpolation points on the real line). An incidence matrix E of $\lambda \times n$ is said to satisfy the Pólya condition if

$$\sum_{k=1}^{j} \sum_{l=1}^{\lambda} e_{l,k} \ge j+1, \qquad j = 0, 1, \ldots, n.$$

By the Atkinson-Sharma theorem ([1]), a Birkhoff interpolation is poised if E has no odd supported sequence and E satisfies the Pólya condition. Our main result in this section is the following.

Theorem 2.1. *Let E be an incidence matrix that satisfies the conditions in Problem 1 and has no odd sequence supported from the right. Assume that E satisfies the Pólya condition. Then the interpolation problem* (1.4) *based on the equidistant points*

$$(x_{l,j}, y_{l,j}) = (r_l \cos\theta_{l,j}, r_l \sin\theta_{l,j}), \quad \theta_{l,j} \in \Theta_{0,m},$$

on the circles $S(r_l)$, $1 \le l \le \lambda$, is poised.

For example, there is a unique polynomial of degree 2 that satisfies the interpolation condition

$$P\Big(\cos\frac{2j\pi}{3}, \sin\frac{2j\pi}{3}\Big) = f_{0,j}, \quad \frac{d^2P}{dr^2}\Big(\cos\frac{2j\pi}{3}, \sin\frac{2j\pi}{3}\Big) = f_{1,j}, \quad j = 0, 1, 2,$$

for any given data $\{f_{0,j}, f_{1,j}\}$. Here E contains only one row, it has one odd sequence (of 1 element) which is clearly not supported from the right.

If the interpolation condition is given on consecutive directional derivatives (that is, the Hermite interpolation), then E clearly satisfies the Pólya condition. In that case, Theorem 2.1 reduces to Theorem 1.1.

Remark 2.1. In [2], Theorem 1.1 was stated with $\theta \in \Theta_{\alpha_l,m}$; that is, equidistant points on different circles can differ by an angle. A careful examine of the proof shows, however, that this is not the case. We thank Hakop Hakopian for pointing this out to us.

For the proof we first recall some elementary lemmas in [2]. For our purpose, it is often more convenient to use the polar coordinates

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta, \qquad r \ge 0, \quad 0 \le \theta \le 2\pi.$$

For a polynomial $P \in \Pi_n^2$, we shall write $\widetilde{P}(r,\theta) = P(r\cos\theta, r\sin\theta)$. The following lemma gives the expression of a polynomial in polar coordinates ([2]).

Lemma 2.2. *For $n \ge 0$, in polar coordinates, every polynomial $P_n \in \Pi_n^2$ can be written as*

$$\widetilde{P}_n(r,\theta) = A_0(r^2) + \sum_{j=1}^{n} \left[r^j A_j(r^2)\cos j\theta + r^j B_j(r^2)\sin j\theta \right],$$

where $A_j(t)$ and $B_j(t)$ are polynomials of degree $[(n-j)/2]$.

More importantly, on the equidistant points of the circle of radius r, the expression can be simplified ([2]). Let $\Theta_{\alpha,m}$ be defined as in (1.2).

Lemma 2.3. *For $\theta \in \Theta_{\alpha,m}$ and $P_n \in \Pi_n^2$ with $n = 2m$ or $n = 2m-1$,*

$$\widetilde{P}_n(r,\theta) = A_0(r^2) + \sum_{j=1}^{m} \Big[\big(r^j A_j(r^2) + r^{2m-j+1} A_{2m-j+1}(r^2)\big)\cos j\theta + \big(r^j B_j(r^2) - r^{2m-j+1} B_{2m-j+1}(r^2)\big)\sin j\theta \Big],$$

and we assume that $A_{2m} = B_{2m} = 0$ if $n = 2m-1$.

In order to show that the Birkhoff interpolation problem (1.4) is poised at the equidistant points $\{(x_{l,j}, y_{l,j})\}$, we need to show that if $P \in \Pi_n^2$ and

$$\frac{d^k \widetilde{P}}{dr^k}(r_l,\theta) = 0, \quad e_{l,k} = 1, \quad \theta \in \Theta_{0,m}, \tag{2.1}$$

then P is a zero polynomial. Using the expression in Lemma 2.3 for $\widetilde{P}$, the interpolation conditions imply the following

Lemma 2.4. *Let P_n be as in Lemma 2.3, $n = 2m$ or $n = 2m-1$. If $\widetilde{P}_n$ satisfies (2.1), then for $0 \le j \le m$,*

$$\frac{d^k}{dr^k}\Big[r^j A_j(r^2) + r^{2m-j+1} A_{2m-j+1}(r^2) \Big]_{r=r_l} = 0, \quad e_{l,k} = 1,$$

$$\frac{d^k}{dr^k}\Big[r^j B_j(r^2) - r^{2m-j+1} B_{2m-j+1}(r^2) \Big]_{r=r_l} = 0, \quad e_{l,k} = 1,$$

where we assume $A_j = B_j = 0$ for $j > n$ and $B_0 = 0$.

Proof. Let $n = 2m$. The expression of P in Lemma 2.3 shows that (2.1) can be written as

$$\frac{d^k}{dr^k} A_0(r^2)\Big|_{r=r_l} + \sum_{j=1}^{m} \Big[\frac{d^k}{dr^k}\big(r^j A_j(r^2) + r^{2m-j+1} A_{2m-j+1}(r^2)\big)\Big|_{r=r_l} \cos j\theta + \frac{d^k}{dr^k}\big(r^j B_j(r^2) - r^{2m-j+1} B_{2m-j+1}(r^2)\big)\Big|_{r=r_l} \sin j\theta \Big] = 0$$

for $\theta \in \Theta_{\alpha,m}$ and $e_{l,k} = 1$. As a trigonometric polynomial of degree m that takes the value zero at $2m+1$ distinct points in $[0, 2\pi)$, the uniqueness of the trigonometric interpolation ([7, vol. 2, p. 1]) shows that the coefficients of $\cos j\theta$ and $\sin j\theta$ are zero, which gives the stated equations. The case $n = 2m-1$ is similar. $\square$

To prove Theorem 2.1 it is sufficient to show that the two systems in the above lemma implies that $A_j \equiv 0$ and $B_j \equiv 0$ for $0 \le j \le m$. It turns out that this can be derived from a special Birkhoff interpolation problem of one variable. We first prove the following result.

Lemma 2.5. *Let $m_1, \ldots, m_N$ be distinct non-negative integer numbers and let E be an incidence matrix with λ rows and $|E| = N$. Assume that E has no odd sequences supported from the right, and E satisfies the Pólya condition. If $P \in$ span$\{t^{m_1}, \ldots, t^{m_N}\}$ and*

$$P^{(k)}(r_l) = 0, \qquad e_{l,k} = 1$$

for $0 < r_1 < \cdots < r_\lambda$, then $P \equiv 0$.

Proof. Let P satisfy the conditions of the lemma. Let $M = \max\{m_1, \ldots, m_N\}$. Then P has exact degree M and P has $M + 1 - N$ zero coefficients. It follows that

$$P^{(k)}(0) = 0, \qquad k \in \{0, 1, \ldots, M\} \setminus \{m_1, \ldots, m_N\}.$$

Let $E_0 = \{e_{0,k}\}$ be the incidence matrix of one row that describe the above Birkhoff interpolation, that is, E is a row vector such that $e_{0,k} = 1$ if $k \in \{0, 1, \ldots, M\} \setminus \{m_1, \ldots, m_N\}$ and $e_{0,k} = 0$ otherwise. Let $E_0 \cup E$ be the incidence matrix that has E_0 as the first row, followed by the rows of E in its original order. Clearly $|E_0 \cup E| = M + 1$. The incidence matrix $E_0 \cup E$ describes a Birkhoff interpolation problem on $\lambda + 1$ points, $0, r_1, \ldots, r_\lambda$, and P satisfies the condition that

$$P^{(k)}(r_l) = 0, \qquad e_{l,k} = 1, \quad e_{l,k} \in E_0 \cup E, \quad P \in \Pi^1_M.$$

By the assumption on E, this new incidence matrix has no odd supported sequence and it clearly satisfies the Pólya condition. Consequently, it is poised and we conclude that $P \equiv 0$. □

For the proof of our theorem, we need the following corollary of the above lemma.

Lemma 2.6. *Let E be an incidence matrix that has no odd sequence supported from the right. Assume that E satisfies the Pólya condition and $|E| = s + 1$. Let $p(t)$ be a polynomial of degree $s - j$, $q(t)$ be a polynomial of degree $j - 1$ and $q(t) = 0$ if $j = 0$. If*

$$\frac{d^k}{dr^k}\Big[r^j p(r^2) + r^{2m-j+1} q(r^2)\Big]_{r=r_l} = 0, \quad e_{l,k} = 1,$$

then $p \equiv 0$ and $q \equiv 0$.

Proof. We assume that the polynomials p and q take the following form:

$$p(t) = \sum_{i=0}^{d} a_i t^i \quad \text{and} \quad q(t) = \sum_{i=0}^{s-m-d-1} b_i t^i.$$

Set

$$\varphi(r) := r^j p(r^2) + r^{2m-j+1} q(r^2).$$

Then a_i is the coefficient of r^{2i+j} and b_i is the coefficient of $r^{2m-j+1+2i}$ of the polynomial φ. Since $2i + j$ and $2m - j + 1 + 2i$ are integers of different parity, it follows that the number of non-zero coefficients in the polynomial φ is equal to s. So that we can apply Lemma 2.5 with $N = s - 1$ to conclude that $\varphi = 0$, which implies that $p \equiv 0$ and $q \equiv 0$. □

Proof of Theorem 2.1. If $n = 2m$, then A_l in Lemma 2.4 is of degree $m - [(l+1)/2]$ and A_{2m-l+1} is of degree $[(l-1)/2]$. Hence, Lemma 2.6 can be used with $l = 2j$ or $l = 2j - 1$ and $s = m$ to prove that the equations in Lemma 2.4 imply $A_j = 0$ and $A_{2m-j+1} = 0$ for $0 \le j \le m$, and similarly $B_j = 0$ and $B_{2m-j+1=0}$ for $0 \le j \le m$. Consequently, we conclude that $P \equiv 0$ and the theorem is proved. The case $n = 2m - 1$ is proved similarly. ☐

3. Factorization theorem and interpolation

In this section we prove the factorization in Theorem 1.3 and use it to derive a number of results for the polynomial interpolation. The proof of the factorization is based on the following lemma, another corollary of Lemma 2.5, which we stated here more as a way of fixing the notation.

Lemma 3.1. *Let s, d, j, m be non-negative integers such that $j \le 2m+1$, $s \ge m+1+d$. Let $\tau_1, \ldots, \tau_\lambda$ be positive integers satisfying $\tau_1 + \cdots + \tau_\lambda = s - m + 1$. Let $p(t)$ be a polynomial of degree d, $q(t)$ be a polynomial of degree $s - m - d - 1$. If p and q satisfy*

$$\frac{d^k}{dr^k}\left[r^j p(r^2) + r^{2m-j+1} q(r^2)\right]_{r=r_l} = 0, \quad 0 \le k \le \tau_l - 1, \quad 1 \le l \le \lambda,$$

then $p \equiv 0$ and $q \equiv 0$.

Proof. We define the polynomial φ as in Lemma 2.6. Then φ is of degree $M = \max\{j+2d, 2s-2d-j-1\}$, and the number of non-zero coefficients in the polynomial φ is equal to $s - m + 1$. The interpolation conditions are described by the $\lambda \times M$ incidence matrix $E = (e_{l,k})$ defined by

$$e_{l,k} = 1, \qquad 1 \le l \le \lambda, \quad 0 \le k \le \tau_l - 1.$$

Then $|E| = s - m + 1$ and E clearly satisfies the Pólya condition and has no supported sequences. We can apply Lemma 2.5 with $N = s - m + 1$ to conclude that $\varphi = 0$, which implies that $p \equiv 0$ and $q \equiv 0$. ☐

The lemma can be proved directly, without relying on Lemma 2.5 since in this case the matrix E is Hermitian. The condition $|E| = s - m + 1$ means that $\varphi(r)$ has $s - m + 1$ nonzero coefficients and then, by Descartes' rule of signs, φ has less than $s - m + 1$ positive zeros (counting multiplicities) or is identically zero.

We are ready to prove the factorization theorem:

Proof of Theorem 1.3. Up to a rotation, we can assume that $\alpha = 0$ in the proof. By Lemma 2.2, we can take P in the form

$$\widetilde{P}(r,\theta) = A_0(r^2) + \sum_{j=1}^{s} \left[r^j A_j(r^2)\cos j\theta + r^j B_j(r^2)\sin j\theta\right],$$

where $A_j(t)$ and $B_j(t)$ are polynomials of degree $[(s-j)/2]$. By Lemma 2.3, we have that for $\theta \in \Theta_{0,m}$,

$$\begin{aligned}\widetilde{P}_n(r,\theta) = & A_0(r^2) + \sum_{j=1}^{m} \left[r^j A_j(r^2)\cos j\theta + r^j B_j(r^2)\sin j\theta\right] \\ & + \sum_{j=2m+1-s}^{m} \left[r^{2m-j+1} A_{2m-j+1}(r^2)\cos j\theta - r^{2m-j+1} B_{2m-j+1}(r^2)\sin j\theta\right].\end{aligned}$$

Using Lemma 2.4, the interpolation conditions implies the following two cases.

Case 1: $0 \le j \le 2m - s$.

$$\frac{d^k}{dr^k}\left[r^j A_j(r^2)\right]_{r=r_l} = 0, \frac{d^k}{dr^k}\left[r^j B_j(r^2)\right]_{r=r_l} = 0, \quad 0 \le k \le \tau_l - 1, \quad 1 \le l \le \lambda.$$

Since the derivatives are taking consecutively, this shows that $A_j^{(k)}(r_l^2) = 0$ and $B_j^{(k)}(r_l^2) = 0$ for $0 \le k \le \tau_l - 1$ and $1 \le l \le \lambda$. Consequently, we conclude that

$$A_j(r^2) = \prod_{l=1}^{\lambda}(r^2 - r_l^2)^{\tau_l} A_j^*(r^2), \qquad B_j(r^2) = \prod_{l=1}^{\lambda}(r^2 - r_l^2)^{\tau_l} B_j^*(r^2),$$

where A_j^* and B_j^* are polynomials of degree $[(s-j)/2] - (s-m+1)$ and we define $A_j^* = B_j^* = 0$ if $[(s-j)/2] - (s-m+1) < 0$.

Case 2: $2m - s + 1 \le j \le m$.

$$\frac{d^k}{dr^k}\left[r^j A_j(r^2) + r^{2m-j+1} A_{2m-j+1}(r^2)\right]_{r=r_l} = 0, \quad 0 \le k \le \tau_l - 1, \quad 1 \le l \le \lambda,$$

$$\frac{d^k}{dr^k}\left[r^j B_j(r^2) - r^{2m-j+1} B_{2m-j+1}(r^2)\right]_{r=r_l} = 0, \quad 0 \le k \le \tau_l - 1, \quad 1 \le l \le \lambda.$$

Using the lemmas with $d = [(s-j)/2]$, we conclude that $A_j = B_j = 0$ and $A_{2m-j+1} = B_{2m-j+1} = 0$.

Together, the two cases show that

$$\widetilde{P}(r,\theta) = \prod_{l=1}^{\lambda}(r^2 - r_l^2)^{\tau_l}\left[A_0^*(r^2) + \sum_{j=1}^{2m-s}\left(r^j A_j^*(r^2)\cos j\theta + r^j B_j^*(r^2)\sin j\theta\right)\right],$$

which completes the proof. $\square$

Let $X = \{x_1, \ldots, x_s\}$ be a subset of $\mathbb{R}^2$. Let L_X be a linear map, $L_X : \Pi^2 \mapsto \mathbb{R}^N$, that describes an Hermite interpolation problem $L_X P = L_X f$ for a given function f. For example, for the Lagrange interpolation, $L_X P = \{P(x_1), \ldots, P(x_N)\}$. Each component of $L_X P$ is either P or one of its consecutive directional derivative (or a sum of consecutive derivatives of P) evaluated at some point in X, where by consecutive derivatives we mean that *at least* one directional derivative is included for each consecutive degree involved (Hermite interpolation). We say that (X, L_X) is *regular for the Hermite interpolation in* Π_n^2, if $N = \dim \Pi_n^2$ and there is a unique polynomial $P \in \Pi_n^2$ such that $L_X P = L_X f$ for any given function f.

An immediate consequence of the previous theorem is the following result:

Theorem 3.2. *Let m, s be positive integers and $m \le s \le 2m$. Let $r_1, r_2, \ldots, r_\lambda$ be distinct positive real numbers, and $\tau_1, \tau_2, \ldots, \tau_\lambda$ be positive integers such that*

$$\tau_1 + \tau_2 + \cdots + \tau_\lambda = s - m + 1.$$

Denote by $\{(x_{l,j}, y_{l,j}) : 0 \le j \le 2m\}$ the equidistant points on the circle $S(r_l)$, $1 \le l \le \lambda$, as in (1.5). Let $X \subset \mathbb{R}^2 \setminus \bigcup_{l=1}^{\lambda} S(r_l)$, and (X, L_X) be regular for a Hermite interpolation in Π_{2m-s-2}^2. Then for any given function f, there is a unique interpolation polynomial $P \in \Pi_{2m-s}^2$ such that $L_X P = L_X f$ and

$$\left(\frac{\partial^k P}{\partial r^k}\right)(x_{l,j}, y_{l,j}) = \left(\frac{\partial^k f}{\partial r^k}\right)(x_{l,j}, y_{l,j}), \quad 0 \le k \le \tau_l - 1, \quad 1 \le l \le \lambda, \quad 0 \le j \le 2m.$$

Proof. Again we only need to prove that $f(x) = 0$ implies $P(x) = 0$. By the factorization theorem $\widetilde{P}(r,\theta) = \prod_{l=1}^{\lambda}(r^2 - r_l^2)^{\tau_l}\widetilde{Q}(r,\theta)$, where Q is a polynomial in Π^2_{2m-s-2}. Since L_X assigns the Hermite interpolation conditions, which means consecutive derivatives, it follows that Q satisfies $L_X Q = 0$. Hence, since (X, L_X) is regular for the Hermite interpolation in Π^2_{2m-s-2}, it follows that $Q \equiv 0$, hence $P \equiv 0$. □

In other words, the theorem states that we can combine two interpolation processes to make up a new interpolation process. For example, we can take X and L_X to be the same type of Hermite interpolation as in our Theorem 1.1, which leads to a Hermite interpolation on two groups of circles, on one group of circles we interpolate at $2m+1$ points, on the other we interpolate at $2m - 2[(s+1)/2] + 1$ points. In particular, we can take (X, L_X) as the Lagrange interpolation.

Even more important, however, is the fact that we can iterate several times of the above theorem to get an interpolation scheme that consists of several groups of circles, such that the number of interpolation points on different groups of circles are all different; furthermore, the circles from different group need not to be concentric. Denote by

$$S(\mathbf{a}, r) = \{(x_1, x_2) : (x - a_1)^2 + (x - a_2)^2 = r^2\}, \quad \mathbf{a} = (a_1, a_2) \in \mathbb{R}^2, \quad r \geq 0,$$

the circle of radius r with center at $\mathbf{a}$. Then we can consider interpolation based on points that are equally spaced on the circles $S(\mathbf{a}_k, r_{l,k})$ and the number of points on the circles $S(\mathbf{a}_k, r_{1,k}), S(\mathbf{a}_k, r_{2,k}), \ldots$ are the same but different from the number of points on $S(\mathbf{a}_{k'}, r_{1,k'}), S(\mathbf{a}_{k'}, r_{2,k'}), \ldots$ for $k \neq k'$. Perhaps the most interesting case is the Lagrange interpolation, that is, no derivatives involved, which we state below.

Theorem 3.3. *Let $s, \sigma, \lambda_1, \lambda_2, \ldots, \lambda_\sigma$ be positive integers such that $s > \lambda_1 > \lambda_2 > \ldots > \lambda_\sigma$ and define*

$$s_k = s - 2\lambda_1 - \cdots - 2\lambda_{k-1} - \lambda_k + 1.$$

For $1 \leq k \leq \sigma$, let $\mathbf{a}_k = (a_{1,k}, a_{2,k}) \in \mathbb{R}^2$ and $r_{l,k}$ be distinct non-negative real numbers, $1 \leq l \leq \lambda_k$. For $\alpha_1, \ldots, \alpha_\sigma \in [0, 2)$, let

$$(x_{j,l,k}, y_{j,l,k}) = (a_{1,k} + r_{l,k}\cos\theta_j, a_{2,k} + r_{l,k}\sin\theta_j), \quad \theta_j \in \Theta_{\alpha_k, s_k}, \tag{3.1}$$

where $0 \leq j \leq 2s_k$, denote the equidistant points on the circle $S(\mathbf{a}_k, r_{l,k})$ and assume that all points are distinct. If $P \in \Pi^2_s$ satisfies

$$P(x_{j,l,k}, y_{j,l,k}) = 0, \qquad 0 \leq j \leq 2s_k, \quad 1 \leq l \leq \lambda_k, \quad 1 \leq k \leq \sigma,$$

then there is a polynomial $Q \in \Pi^2_{s-2\lambda_1-\ldots-2\lambda_\sigma}$ such that

$$P(x, y) = \prod_{k=1}^{\sigma}\prod_{l=1}^{\lambda_k}\big((x - a_{1,k})^2 + (y - a_{2,k})^2 - r_{l,k}^2\big)Q(x, y).$$

In particular, if $s < 2\lambda_1 + \cdots + 2\lambda_\sigma$, then $Q = 0$.

Proof. Applying an affine transformation, it is easy to see that the previous theorem holds if the circles $S(r_l)$ are replaced by $S(\mathbf{a}, r_l)$ for any fixed $\mathbf{a} \in \mathbb{R}$. Since

$P(x_{j,l,1}, y_{j,l,1}) = 0$, we can apply Theorem 1.3 with $m = s - \lambda_1 + 1$ and the circles center at $\mathbf{a}_1$ to conclude that there is a polynomial $Q_1 \in \Pi^2_{s-2\lambda_1}$ such that

$$P(x,y) = \prod_{l=1}^{\lambda_1} \big((x - a_{1,1})^2 + (y - a_{2,1})^2 - r_{l,1}^2\big) Q_1(x,y).$$

Since $r_{l,k}$ are distinct and all points $(x_{j,l,k}, y_{j,l,k})$ are distinct, we have $Q_1(x_{j,l,2}, y_{j,l,2}) = 0$. Hence the previous theorem with $m = s - 2\lambda_1 - \lambda_2 - 1$ and the circles center at $\mathbf{a}_2$ implies that there is a polynomial $Q_2 \in \Pi^2_{s-2\lambda_1-2\lambda_2}$ such that

$$P(x,y) = \prod_{l=1}^{\lambda_1} \big((x-a_{1,1})^2+(y-a_{2,1})^2-r_{l,1}^2\big) \prod_{l=1}^{\lambda_2} \big((x-a_{1,2})^2+(y-a_{2,2})^2-r_{l,2}^2\big) Q_2(x,y).$$

Continuing this process for $k = 3, 4, \ldots, \sigma$ completes the proof. □

Remark 3.1. The equidistant points for circles in the same group have the same angles; that is, α_k in Θ_{α_k, s_k} is a constant for the circles $S(\mathbf{a}_k, r_{1,k}), \ldots, S(\mathbf{a}_k, r_{\lambda_k,k})$. However, the equidistant points for circles in different groups can differ by a constant; that is, α_k may not be equal to α_l for $k \neq l$.

We remark that $r_{l,k}$ are distinct non-negative numbers, so that at most one $r_{l,k}$ can be zero. If $r = 0$, the corresponding circle $S(\mathbf{a}, r)$ is just the center $\mathbf{a}$. Also, the centers $\mathbf{a}_k$ of the circles are not necessarily distinct. One interesting case is when all $\mathbf{a}_k = 0$, that is, all circles are concentric at the origin.

An immediate consequence of the above theorem is the following result on the Lagrange interpolation.

Theorem 3.4. *Let $n, \sigma, \lambda_1, \lambda_2, \ldots, \lambda_\sigma$ be positive integers, such that $n > \lambda_i$ and*

$$\lambda_1 + \cdots + \lambda_\sigma = [n/2] + 1. \tag{3.2}$$

Let $\mathbf{a}_k = (a_{1,k}, a_{2,k})$ and let $r_{l,k}$ be distinct non-negative real numbers, $1 \le l \le \lambda_k$, $1 \le k \le \sigma$. Define

$$n_1 = n - \lambda_1 + 1, \quad \textit{and} \quad n_k = n - 2\lambda_1 - \cdots - 2\lambda_{k-1} - \lambda_k + 1, \quad k \ge 2.$$

Let $(x_{j,l,k}, y_{j,l,k})$, $0 \le j \le 2n_k$, denote the equidistant points on the circle $S(\mathbf{a}_k, r_{l,k})$ as in (3.1) and assume that all points are distinct. Then for any given data $\{f_{j,l,k}\}$, there is a unique polynomial $P \in \Pi^2_n$ that satisfies

$$P(x_{j,l,k}, y_{j,l,k}) = f_{j,l,k}, \qquad 0 \le j \le 2n_k, 1 \le l \le \lambda_k, 1 \le k \le \sigma.$$

The Lagrange interpolation described in the theorem has exactly $(n+1)(n+2)/2$ points, that is, the number of interpolation conditions matches the dimension of Π^2_n. The formula

$$\dim \Pi^2_s = \dim \Pi^2_{s-2\lambda} + \lambda\big(2(s - \lambda + 1) + 1\big)$$

can be used repeatedly to verify this fact. Hence, the theorem is a consequence of Theorem 3.3 with $n = s$ and the assumption (3.2) ensures that $n < 2\lambda_1 + \cdots + 2\lambda_\sigma$.

The interpolation points in the theorem are equidistant points located on several groups of circles (there are σ groups, circles in the k-th group concentric at $\mathbf{a}_k$), the number of points on the circles is the same for circles in the same group, but may differ for different groups. The simplest case is $n = 2$, in which we interpolate 6 points with polynomials in Π^2_2. Recall that if 6 points are all located on one circle, then the interpolation in Π^2_2 does not have a unique solution.

Example 3.5. $n = 2$. *Since* $n > \lambda_i$, *there is only one integer solution for* $\lambda_1 + \lambda_2 = 2$, *namely,* $\sigma = 2$, $\lambda_1 = \lambda_2 = 1$, *which gives a poised Lagrange interpolation of 5 points on one circle and another point not on that circle.*

For each fixed n the theorem includes a number of poised Lagrange interpolation schemes. In fact, every integer solution of the equation (3.2) corresponds to one poised Lagrange interpolation. To illustrate the result, we list all cases for $n = 6$ below.

Example 3.6. $n = 6$. *Since* λ_i *are positive integers, there are 7 solutions to the equation* $\lambda_1 + \cdots + \lambda_\sigma = [6/2] + 1 = 4$. *Each leads to a poised interpolation problem according to the theorem. We list them below:*

(1) $\sigma = 1$, $\lambda_1 = 4$: *4 concentric circles with 7 points each;*
(2) $\sigma = 2$,
  (a) $\lambda_1 = 3, \lambda_2 = 1$: *3 concentric circles with 9 points each and one additional point;*
  (b) $\lambda_1 = 1, \lambda_2 = 3$: *1 circle with 13 points and 3 concentric circles with 5 points each;*
(3) $\sigma = 3$,
  (a) $\lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 1$: *2 concentric circles with 11 points each, one circle with 5 points and one additional point;*
  (b) $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 1$: *1 circle with 13 points, 2 concentric circles with 7 points each and one additional point;*
  (c) $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 2$: *1 circle with 13 points, 1 circle with 9 points and 2 concentric circles with 3 points each;*
(4) $\sigma = 4$, $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1, \lambda_4 = 1$: *3 circles with 13, 9, 5 points, respectively, and one additional point.*

In the above list we only give the number of points on each circle. The circles may not be concentric unless specifically stated, and it should be understood that points on each circle are equidistant and all points are distinct. The 'one additional point' in some of the cases corresponds to the degenerate circle of radius 0.

In general, for each n, the number of poised Lagrange interpolation schemes contained in Theorem 3.3 is equal to the integer solutions of the equation (3.2), which is the number of ways that $[n/2] + 1$ can be written as a sum of positive integers. This number grows exponentially as n increases.

In the last case of the Example 3.6, we have 4 circles and each has different number of nodes. It turns out that in this case we do not have to restrict to equidistant points on the circles, as seen from the next proposition.

Theorem 3.7. *Let* m *be a positive integer and* $r_0, r_1, \ldots, r_{[m/2]}$ *be distinct positive real numbers. For each* l, *let* $\{\theta_{l,j} : 0 \le j \le 2(m-2l)\}$ *be a set of* $2(m-2l)+1$ *distinct points in* $[0, 2\pi)$. *Then there is a unique interpolation polynomial* $P \in \Pi_m^2$ *such that*

$$P(r_l \cos\theta_{l,j}, r_l \sin\theta_{l,j}) = f_{l,j}, \qquad 0 \le l \le [m/2], \quad 0 \le j \le 2(m-2l),$$

for any given numbers $\{f_{l,j}\}$.

Proof. The number of interpolation conditions matches the dimension of Π_m^2, since

$$\sum_{l=0}^{[m/2]} (2(m-2l)+1) = ([m/2]+1)(2m+1-2[m/2]) = (m+1)(m+2)/2.$$

It suffices to show that $f_{l,j} = 0$ implies that $P \equiv 0$. By Lemma 2.2 we take P as

$$\widetilde{P}(r,\theta) = A_0(r^2) + \sum_{j=1}^{m} \left[r^j A_j(r^2) \cos j\theta + r^j B_j(r^2) \sin j\theta \right].$$

By the uniqueness of the trigonometric interpolation, the condition $P(r_0, \theta_{j,0}) = 0$, $0 \le j \le 2m$, shows that $A_j(r_0^2) = B_j(r_0^2) = 0$, so that

$$\widetilde{P}(r,\theta) = (r^2 - r_0^2)\widetilde{P}_1(r,\theta), \quad P_1 \in \Pi^2_{m-2}. \tag{3.3}$$

Using the same argument to interpolation on the circles of radii $r_1, r_2, \ldots$ gives

$$\widetilde{P}(r,\theta) = \prod_{l=0}^{[m/2]} (r^2 - r_l^2)\widetilde{Q}(r,\theta).$$

Since $2[m/2] + 1 > m$, $\widetilde{Q} = 0$ which shows that $\widetilde{P}(r,\theta) = 0$. □

In fact, this theorem is a consequence of Bezout's theorem; we give a complete proof here because the proof is independent and very simple. Indeed, one version of the classical Bezout's theorem states (see, for example, [6, p. 59]) that if two algebraic curves, of order m and n, have more than mn common points, then they have a common component. Under the assumption of the theorem, the polynomial P that vanishes on all the interpolation conditions and the polynomial $q(x) = x_1^2 + x_2^2 - r_0^2$ (the circle with radius r_0) have $2m+1$ common zeros, so that they have a common component. But q is irreducible, it follows that $P(x) = q(x)P_1(x)$, which is (3.3). Using this process repeatedly proves the theorem. Bezout's theorem has been used in polynomial interpolation of several variables by many authors; see, for example, the recent survey [5] and the references there in. Theorem 3.7, however, does not seem to have been stated before.

One natural question is whether Theorem 3.4 holds true for arbitrary points on the circles. Despite Theorem 3.7, we believe that this is not the case. The factorization theorem given in Theorem 1.3 is unlikely to hold for arbitrary points on the circles. In fact, in the case of one circle with Hermite data, there are examples for $n = 3, 4$ in [2] showing that the interpolation is not unique for the arbitrary points. Some results concerning interpolation at arbitrary points on the circles can be found in the recent papers of Ismail [4].

4. Further remarks

It is well-known that Lagrange interpolation by polynomials of two variables is poised for almost all set of points. However, checking the poisedness of a given set of points is often difficult. Our main result provides many sets of points that admit unique solution for polynomial interpolation in two variables, as demonstrated by Theorem 3.4 and Example 3.6. The method used in proving the main result can also be used for other sets of points. First of all, the following theorem is an immediate consequence of Bezout's theorem as discussed after the proof of Theorem 3.7.

Theorem 4.1. *Let m be a positive integer and let $(x_{l,j}, y_{l,j})$, $0 \le j \le 2(m-2l)$, be points on $[m/2]+1$ conics, $C_1, C_2, \ldots, C_{[m/2]+1}$ in $\mathbb{R}^2$, and assume that points on C_l do not belong to $C_1 \cup \cdots \cup C_{l-1}$ for $2 \le l \le [m/2]+1$. Then there is a unique interpolation polynomial $P \in \Pi^2_m$ such that*

$$P(x_{l,j}, y_{l,j}) = f_{l,j}, \qquad 0 \le l \le [m/2], \quad 0 \le j \le 2(m-2l),$$

for any given numbers $\{f_{l,j}\}$.

This holds for arbitrary points on the conics. Our main result in Theorem 1.3 and Theorem 3.4, however, depends on the use of equidistant points on the circles; it is not clear if the result can be generalized to arbitrary conics. What is clear, however, is the following obvious extension of our results: instead of points on the circles, we can assume that the points are on the ellipses. Indeed, if we use the coordinates (r,θ) defined by

$$x = ar\cos\theta, \quad y = br\cos\theta, \qquad r \ge 0, \quad 0 \le \pi \le 2\pi,$$

where a and b are nonzero constants, then we can consider analogous of Problem 1 and Problem 2 with normal derivatives

$$\frac{\partial P}{\partial r} = a\frac{\partial P}{\partial x}\cos\theta + b\frac{\partial P}{\partial y}\sin\theta, \qquad (x,y) = (ar\cos\theta, br\sin\theta).$$

Straightforward extension shows that Theorem 2.1 for the Birkhoff interpolation and Theorem 1.3 of the factorization theorem hold for interpolation points

$$(x_{l,j}, y_{l,j}) = (a\,r_l\cos\theta_j, b\,r_l\sin\theta_j), \quad \theta_j \in \Theta_{0,m},$$

in which circles $x^2+y^2-r_l^2$ are replaced by the ellipses $a^{-2}x^2+b^{-2}y^2-r_l^2$. Consequently, all results in Section 3 hold with circles replaced by ellipses; that is, we can consider interpolation on 'equidistant' points on several groups of ellipses, in which the number of points on the same group of ellipses is the same but different from those in different groups, and the ellipses in the same group are concentric.

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