ON A HERMITE INTERPOLATION BY POLYNOMIALS OF TWO VARIABLES *

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Abstract. A problem of Hermite interpolation by polynomials of two variables is studied. The interpolation matches preassigned data of function values and consecutive normal derivatives on a set of points on several circles centered at the origin. It includes Lagrange interpolation as a special case. The uniqueness of the interpolation is established when the points are equidistant on the circles, while the points on different circles may differ by arbitrary rotations. This leads to a cubature formula on the unit disc, which can be given explicitly without knowing the explicit formula of the interpolation polynomial.

Key words. Hermite interpolation, polynomials, two variables

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1. Introduction and Main Result. We consider a problem of Hermite interpolation by polynomials of two variables. The interpolation points are located on several circles centered at the origin and the interpolation matches preassigned data of function values and certain consecutive normal derivatives. When no derivative values are interpolated, it becomes a problem of Lagrange interpolation.

Let Π_n^2 denote the space of polynomials P of two variables of total degree n,

$$P(x,y) = \sum_{k=0}^{n} \sum_{j=0}^{k} c_{j,k} x^j y^{k-j}.$$

It is known that $\dim \Pi_n^2 = (n+1)(n+2)/2$. We consider the case that the number of interpolation conditions matches the dimension of Π_n^2. If there is a *unique* solution to an interpolation problem, we say that the problem is *poised*. Unlike polynomial interpolation in one variable, Hermite or Lagrange interpolation in several variables is not always poised. A typical example is the Lagrange interpolation of 6 points on the unit circle by polynomials of degree 2 in two variables, since the points lie on the curve x^2+y^2-1. Similar examples hold for the Hermite interpolation; for example, let $\{(x_j,y_j) : 1 \le j \le 18\}$ be distinct points on the unit circle; then the problem of interpolating $P(x_i,y_i)$ and the normal derivatives $(\partial P/\partial r)(x_i,y_i)$ by polynomial of degree 7 does not have a unique solution, since the polynomial $(x^2+y^2-1)^2$ vanishes on all conditions. In both cases, the interpolation points (multiple points for the Hermite interpolation) lie on an algebraic curve. Both, however, are examples of *almost poised* interpolation problems, meaning that the problem is poised for almost all choice of points (except points from a set of measure zero). There are general results available for almost poisedness (see, for example, [2, 7, 10] and the references therein). However, even when an interpolation problem is almost poised, how to choose a particular set of points so that the interpolation problem is poised remains difficult. The purpose of this paper is to provide an example of poised Hermite interpolation in two variables.

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Throughout this paper, let $\partial/\partial r$ denote the normal derivative

$$\frac{\partial P}{\partial r} = \frac{\partial P}{\partial x}\cos\theta + \frac{\partial P}{\partial y}\sin\theta, \qquad (x,y) = (r\cos\theta, r\sin\theta).$$

For a positive number $r > 0$, denote the circle of radius r centered at the origin by $S(r) = \{(x,y) : x^2 + y^2 = r^2\}$. Let $[t]$ denote the integer part of t. Our interpolation problem is as follows:

Problem: *Let n be a positive integer. Let $0 < r_1 < r_2 < \ldots < r_\lambda \le 1$ and let $\mu_1, \mu_2, \ldots, \mu_\lambda$ be positive integers such that*

$$\mu_1 + \mu_2 + \ldots + \mu_\lambda = \left[\frac{n}{2}\right] + 1.$$

Denote by $\{(x_{l,j}, y_{l,j}) : 0 \le j \le 2m\}$ distinct points on the circle $S(r_l)$, where $m = [(n+1)/2]$ and $1 \le l \le \lambda$. Characterize the points for which the interpolation problem

$$\Big(\frac{\partial^k P}{\partial r^k}\Big)(x_{l,j}, y_{l,j}) = f_{j,l,k}, \quad 0 \le k \le \mu_l - 1, \quad 1 \le l \le \lambda, \quad 0 \le j \le 2m, \qquad (1.1)$$

has a unique solution in Π_n^2 for any given preassigned data $\{f_{j,l,k}\}$.

In other words, the problem requires interpolating up to $(\mu_l - 1)$-th order normal derivatives at $2m+1$ distinct points on the l-th circle, $l = 1, 2, \ldots, \lambda$. We note that the number of interpolation conditions (1.1) equals the $\dim \Pi_n^2$, as

$$\left(2\left[\frac{n+1}{2}\right] + 1\right)\left(\left[\frac{n}{2}\right] + 1\right) = \frac{(n+1)(n+2)}{2}.$$

If $\mu_l = 1$ for all l, then $\lambda = [n/2] + 1$. In this case, the interpolation problem becomes the Lagrange interpolation on $[n/2] + 1$ circles. If $\lambda = 1$, then there is only one circle, which we choose to have radius 1, and the problem becomes Hermite interpolation on the unit circle.

The most natural choice of the interpolation points on the circle $S(r_l)$ is the equidistant points,

$$(x_{l,j}, y_{l,j}) = (r_l\cos\theta_{l,j}, r_l\sin\theta_{l,j}), \qquad \theta_{l,j} = \frac{(2j + \alpha_l)\pi}{2m+1}, \quad 0 \le j \le 2m, \qquad (1.2)$$

where α_l is a number independent of j. Here we include the number α_l to indicate the fact that equidistant points on different circles can differ by a rotation. The main result of the paper is as following:

THEOREM 1.1. *The Hermite interpolation problem* (1.1) *based on the equidistant points* (1.2) *is poised.*

This result is proved in Section 2, where we also discuss some of its consequences and various properties of the unique interpolation polynomial. Let us mention that the result is new even for Lagrange interpolation (see Corollary 2.6). Lately another solution of Theorem 1.1 in the particular case of $\lambda = 1$ was given independently by Hakopian and Ismaeil [6].

Integrating the interpolation polynomial P in the theorem gives a cubature formula on the unit disc $B^2 = \{(x,y) : x^2 + y^2 \le 1\}$. Although the explicit formula of P is hard to find, the integral mean of P over a circle of radius r can be derived as a solution of an interpolation problem of one variable. This allows us to give explicit formulae for the cubature formula. These results are discussed in Section 3, where several cubature formulae are given explicitly.

In Section 4, we consider the interpolation problem (1.1) for arbitrary set of points on the circle. For the special case of $\lambda = 1$ (interpolating on the unit circle), we show that the problem is poised for arbitrary points only when $n = 2$. Necessary and sufficient conditions on the points for the poisedness are given for $n = 3, 4$.

2. The equidistant points. We start with the following observation. Let $\mathcal{P}_n$ denote the homogeneous polynomials of degree n in two variables; that is, $\mathcal{P}_n = \mathrm{span}\{x^k y^j : k+j = n, k \ge 0, j \ge 0\}$. Because the interpolation points are on the unit circle, it is often more convenient to use the polar coordinates

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta, \qquad r \ge 0, \quad 0 \le \theta \le 2\pi.$$

Using the elementary trigonometric identities, it follows that in polar coordinates,

$$\mathcal{P}_{2n} = \mathrm{span}\{r^{2n}, r^{2n}\cos 2k\theta, r^{2n}\sin 2k\theta, \quad 1 \le k \le n\},$$

and

$$\mathcal{P}_{2n+1} = \mathrm{span}\{r^{2n+1}\cos(2k+1)\theta, r^{2n+1}\sin(2k+1)\theta, \quad 0 \le k \le n\}.$$

For a polynomial $P \in \Pi_n^2$, we shall write $\widetilde{P}(r,\theta) = P(r\cos\theta, r\sin\theta)$. Then every polynomial $P \in \Pi_n^2$ can be expressed as

$$\widetilde{P}(r,\theta) = P(r\cos\theta, r\sin\theta) = \sum_{k=0}^{n} q_k(\theta) r^k \tag{2.1}$$

where $r^k q_k(\theta) \in \mathcal{P}_k$ and $q_k(\theta)$ are trigonometric polynomials of degree k given by

$$q_{2j}(\theta) = a_{0,2j} + \sum_{l=1}^{j}(a_{l,2j}\cos 2l\theta + b_{l,2j}\sin 2l\theta), \tag{2.2}$$

$$q_{2j-1}(\theta) = \sum_{l=1}^{j}(a_{l,2j-1}\cos(2l-1)\theta + b_{l,2j-1}\sin(2l-1)\theta). \tag{2.3}$$

This expression is useful for interpolation problem of arbitrary points on the circle. For the proof of Theorem 1.1, we use a different expression.

Lemma 2.1. *For $n \ge 0$, in polar coordinates, every polynomial $P_n \in \Pi_n^2$ can be written as*

$$\widetilde{P}_n(r,\theta) = A_0(r^2) + \sum_{j=1}^{n}\left[r^j A_j(r^2)\cos j\theta + r^j B_j(r^2)\sin j\theta\right],$$

where $A_j(t)$ and $B_j(t)$ are polynomials of degree $[(n-j)/2]$.

This comes from rewriting (2.1) as a sum of $\cos j\theta$ and $\sin j\theta$. Note that the coefficients $r^j A_j(r^2)$ and $r^j B_j(r^2)$ are lacunary polynomials in r. Furthermore, the fact that the interpolation points are equidistant can be used to simplify the expression at these points. Let

$$\Theta_\alpha = \{\theta_j^\alpha : \theta_j^\alpha = (2j+\alpha)\pi/(2m+1), \quad j = 0, 1, \ldots, 2m, \quad 0 \le \alpha < 2\}. \tag{2.4}$$

LEMMA 2.2. *For $\theta \in \Theta_\alpha$ and $P_n \in \Pi_n^2$ with $n = 2m$ or $n = 2m-1$,*

$$\widetilde{P}_n(r,\theta) = A_0(r^2) + \sum_{j=1}^{m} \Big[\big(r^j A_j(r^2) + r^{2m-j+1} C_{2m-j+1}(r^2)\big)\cos j\theta + \big(r^j B_j(r^2) - r^{2m-j+1} D_{2m-j+1}(r^2)\big)\sin j\theta\Big],$$

where

$$\begin{aligned} C_{2m-j+1}(r^2) &= A_{2m-j+1}(r^2)\cos\alpha\pi + B_{2m-j+1}(r^2)\sin\alpha\pi, \\ D_{2m-j+1}(r^2) &= A_{2m-j+1}(r^2)\sin\alpha\pi - B_{2m-j+1}(r^2)\cos\alpha\pi, \end{aligned}$$

and we assume that $A_{2m} = B_{2m} = 0$ if $n = 2m-1$.

Proof. Using the expression in the previous lemma, the proof follows from the fact that

$$\begin{aligned} \cos(2m-j+1)\theta &= \cos(\alpha\pi - j\theta) = \cos\alpha\pi\cos j\theta + \sin\alpha\pi\sin j\theta, \\ \sin(2m-j+1)\theta &= \sin(\alpha\pi - j\theta) = \sin\alpha\pi\cos j\theta - \cos\alpha\pi\sin j\theta \end{aligned}$$

for $\theta \in \Theta_\alpha$. □

In order to prove that the Hermite interpolation problem (1.1) is regular, we need to show that if $P \in \Pi_n^2$ and

$$\frac{d^k\widetilde{P}}{dr^k}(r_l,\theta) = 0, \quad 0 \le k \le \mu_l - 1, \quad 1 \le l \le \lambda, \quad \theta \in \Theta_{\alpha_l}, \tag{2.5}$$

then P is a zero polynomial. Using the expression in Lemma 2.2 for $\widetilde{P}$, the interpolation conditions imply the following.

LEMMA 2.3. *Let P_n be as in Lemma 2.2, $n = 2m$ or $n = 2m-1$. If $\widetilde{P}_n$ satisfies (2.5), then for $0 \le j \le m$,*

$$\frac{d^k}{dr^k}\Big[r^j A_j(r^2) + r^{2m-j+1} C_{2m-j+1}(r^2)\Big]_{r=r_l} = 0, \quad 0 \le k \le \mu_l - 1, \quad 1 \le l \le \lambda,$$

$$\frac{d^k}{dr^k}\Big[r^j B_j(r^2) - r^{2m-j+1} D_{2m-j+1}(r^2)\Big]_{r=r_l} = 0, \quad 0 \le k \le \mu_l - 1, \quad 1 \le l \le \lambda,$$

where we assume $A_j = B_j = 0$ for $j > n$ and $B_0 = 0$.

Proof. Let $n = 2m$. Using the expression of P in Lemma 2.2, the interpolation conditions (2.5) can be written as

$$\frac{d^k}{dr^k}A_0(r^2)\Big|_{r=r_l} + \sum_{j=1}^{m}\Big[\frac{d^k}{dr^k}\big(r^j A_j(r^2) + r^{2m-j+1}C_{2m-j+1}(r^2)\big)\Big|_{r=r_l}\cos j\theta + \frac{d^k}{dr^k}\big(r^j B_j(r^2) - r^{2m-j+1}D_{2m-j+1}(r^2)\big)\Big|_{r=r)l}\sin j\theta\Big] = 0$$

for $\theta \in \Theta_{\alpha_l}$ and $0 \le k \le m$. The above expression is a trigonometric polynomial of degree m, which takes the value zero at $2m+1$ distinct points in $[0, 2\pi)$. It then follows from the uniqueness of the trigonometric interpolation ([12, vol. 2, p. 1]) that the coefficients of $\cos j\theta$ and $\sin j\theta$ are zero, which gives the stated equations. The case $n = 2m-1$ is similar. □

The proof of the uniqueness of the Hermite interpolation depends on the following result of polynomials of one variable.

LEMMA 2.4. *Fix integer j, $0 \le j \le s \le 2m+1$. Let $\mu_1, \dots, \mu_\lambda$ satisfy $\mu_1 + \dots + \mu_\lambda = s+1$. Let $p(t)$ be a polynomial of degree $s-j$, $q(t)$ be a polynomial of degree $j-1$ and $q(t) = 0$ if $j = 0$. If*

$$\frac{d^k}{dr^k}\left[r^j p(r^2) + r^{2m-j+1} q(r^2)\right]_{r=r_l} = 0, \quad 0 \le k \le \mu_l - 1, \quad 1 \le l \le \lambda, \tag{2.6}$$

then $p = 0$ and $q = 0$.

Proof. We assume that the polynomials p and q take the following form:

$$p(t) = \sum_{i=0}^{s-j} a_i t^i \quad \text{and} \quad q(t) = \sum_{i=0}^{j-1} b_i t^i.$$

Set

$$\varphi(r) := r^j p(r^2) + r^{2m-j+1} q(r^2).$$

Clearly φ is a polynomial of degree

$$M := \max\{2s-j, 2m+j-1\}.$$

Note that a_i is the coefficient of r^{2i+j}, while b_i is the coefficient of $r^{2m-j+1+2i}$ of φ. Since $2i+j$ and $2m-j+1+2i$ are integers of different parity, $\{a_i\}$ and $\{b_i\}$ are coefficients (up to a non-zero factor) in the Taylor expansion of φ at zero. The rest of the Taylor coefficients are zero by the definition of φ. Thus, we have

$$\varphi^{(k)}(0) = 0, \quad k \notin \{2i+j : 0 \le i \le s-j\} \cup \{2m+1-j+2i : 0 \le i \le j-1\}$$

and $k \le M$. This gives $M + 1 - (s-j+1) - j = M - s$ equations. Together with (2.6) which consists of $s+1$ interpolation conditions for φ, we see that φ satisfies a set of Hermite-Birkhoff interpolation conditions at $\lambda+1$ points. There are total $M+1$ interpolation conditions and φ is a polynomial of degree at most M. Since $j \le s$, the incidence matrix of this problem satisfies the Pólya condition and has no supported sequence, therefore it is non-singular (see [1]). Thus $\varphi \equiv 0$. Consequently, $a_i = b_i = 0$, which implies that $p \equiv 0$ and $q \equiv 0$. The proof is complete. □

We are now in position to prove our main result Theorem 1.1, which we restated below.

THEOREM 2.5. *For any $\alpha_1, \dots, \alpha_l$, the interpolation problem* (1.1) *based on the equidistant points*

$$(x_{l,j}, y_{l,j}) = (r_l \cos\theta_j, r_l \sin\theta_j), \quad \theta_j \in \Theta_{\alpha_l},$$

on the circles $S(r_l)$, $1 \le l \le \lambda$, is poised.

Proof. If $n = 2m$, then A_l in Lemma 2.3 is of degree $m - [(l+1)/2]$ and C_{2m-l+1} is of degree $[(l-1)/2]$. Hence, Lemma 2.4 can be used with $l = 2j$ or $l = 2j-1$ and $s = m$ to prove that the equations in Lemma 2.3 imply $A_j = 0$ and $C_{2m-j+1} = 0$ for $0 \le j \le m$, and similarly $B_j = 0$ and $D_{2m-j+1=0}$ for $0 \le j \le m$. By definition of C_j and D_j, this gives

$$A_{2m-j+1}(r^2)\cos\alpha\pi + B_{2m-j+1}(r^2)\sin\alpha\pi = 0$$

$$A_{2m-j+1}(r^2)\sin\alpha\pi - B_{2m-j+1}(r^2)\cos\alpha\pi = 0,$$

which implies that $A_{2m-j+1} = 0$ and $B_{2m-j+1} = 0$ since the determinant of the coefficients is -1. Consequently, we conclude that $P \equiv 0$ and the theorem is proved. If $n = 2m-1$, then A_l in Lemma 2.3 is of degree $m - [(l+2)/2]$ and C_{2m-l+1} is of degree $[(l-3)/2]$. Hence, Lemma 2.4 can be used with $l = 2j$ or $l = 2j+1$ and $s = m-1$ to prove that the equations in Lemma 2.3 imply $A_j = 0$ and $C_{2m-j+1} = 0$ for $0 \le j \le m-1$, and similarly $B_j = 0$ and $D_{2m-j+1} = 0$ for $0 \le j \le m-1$. Again, it follows that $A_{2m-j+1} = B_{2m-j+1} = 0$ and the proof is complete. □

We emphasize that α_l can be different for different circles. Two cases may be of particular interesting. The first one is all $\alpha_l = 0$, in which case the interpolation points are also located on rays starting from the origin. The second case is when points on neighboring circle differ by a rotation of $\pi/(2m+1)$ (for example, $\alpha_{2l} = 0$ and $\alpha_{2l+1} = 1$).

When all $\mu_l = 1$, the Hermite interpolation becomes the Lagrange interpolation. We state this case as the following corollary.

COROLLARY 2.6. *Let $0 < r_0 < r_1 < \cdots < r_{[n/2]} = 1$ be the radii of the circles $S(r_l)$, $0 \le l \le [n/2]$. Let Θ_{α_l} be the point set on $S(r_l)$ defined in* (2.4). *Then the Lagrange interpolation problem*

$$P(r_l\cos\theta_j, r_l\sin\theta_j) = f_{j,l}, \quad 0 \le l \le [n/2], \quad \theta_j \in \Theta_{\alpha_l},$$

is poised in Π_n^2.

As a special case of this corollary, when the points on the neighboring circles differ by a rotation of $\pi/(2m+1)$ and the radii of the circles take special values $r_l = \cos\frac{m\pi}{2m+1}/\cos\frac{l\pi}{2m+1}$, $l = 1, 2, \ldots, m$ and $n = 2m-1$, the interpolation points form a special geometric formation. The cases of $m = 2, 3$ (that is, $n = 3, 5$) are depicted in Fig. 1.

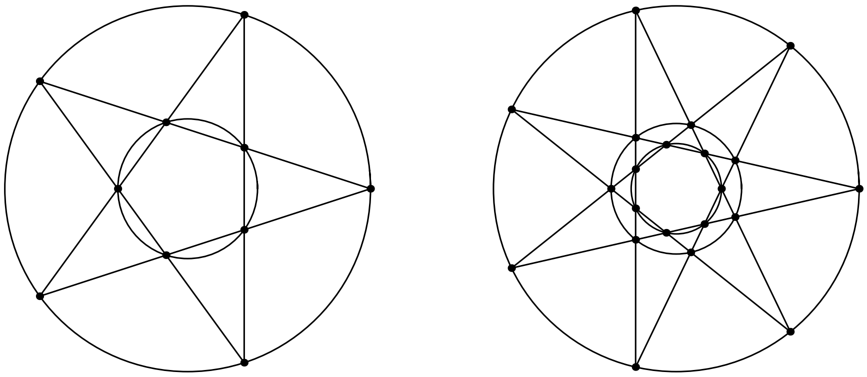

Figure 1

These special points were discovered in [4] (see also [11]). They serve as a nice example of the *natural lattices* of Chung and Yao [3]. Let $H_1, \ldots, H_{n+d}$ be a collection of hyperplanes in $\mathbb{R}^d$ such that any distinct d hyperplanes chosen among them intersect at one point and different choices give different points; the set of all intersection points is called the n-th order natural lattice in $\mathbb{R}^d$ generated by $H_1, \ldots, H_{n+d}$. As it was shown in [3], the interpolation at natural lattices by polynomials of degree n is poised. For $d = 2$, the hyperplanes become lines. As indicated in Fig. 1, each point is the intersection of exactly two lines.

For $0 \le k \le \mu_l - 1$, $1 \le l \le \lambda$ and $0 \le j \le 2m$, let $L_{j,l,k}$ denote the fundamental interpolation polynomial at point $(x_{l,j}, y_{l,j})$ defined by

$$\Big(\frac{\partial^v L_{j,l,k}}{\partial r^v}\Big)(r_u \cos\theta_{u,i}, r_u \sin\theta_{u,i}) = \delta_{j,i}\delta_{l,u}\delta_{k,v}, \qquad 0 \le v \le \mu_l - 1, \quad 1 \le u \le \lambda, \quad 0 \le i \le 2m. \tag{2.7}$$

Let P be the unique interpolation polynomial that satisfies (1.1). Then P can be written as

$$P(x,y) = \sum_{l=1}^{\lambda} \sum_{k=0}^{\mu_l - 1} \sum_{j=0}^{2m} f_{j,l,k} L_{j,l,k}(x,y).$$

It turns out that all $L_{j,l,k}$ can be obtained from $L_{0,l,k}$ by a rotation.

THEOREM 2.7. *For $0 \le k \le \mu_l - 1$, $1 \le l \le \lambda$ and $0 \le j \le 2m$,*

$$L_{j,l,k}(r\cos\theta, r\sin\theta) = L_{0,l,k}(r\cos(\theta - \theta_j^0), r\sin(\theta - \theta_j^0)), \quad \textit{if } \alpha_l = 0$$

and with $\beta = \pi/(2m+1)$,

$$L_{j,l,k}(r\cos\theta, r\sin\theta) = L_{0,l,k}(r\cos(\theta - \theta_j^1 + \beta), r\sin(\theta - \theta_j^1 + \beta)), \quad \textit{if } \alpha_l = 1.$$

Furthermore, the polynomials $L_{0,l,k}(x,y)$ are even in y when $\alpha_l = 0$.

Proof. The stated formulae follow directly from the rotational invariance of the interpolation points.

To prove that $L_{0,l,k}$ is even in y for $\alpha = 0$, we consider the polynomial $[L_{0,l,k}(x,y) + L_{0,l,k}(x,-y)]/2$, which is clearly even in y. It is easy to see that this polynomial satisfy the defining interpolation conditions of (2.7) for $L_{0,l,k}$, since the only nonzero interpolation condition appears at $\theta = 0$, for which $y = r\sin\theta$ takes value zero. By the uniqueness of the interpolation, it must equal to $L_{0,l,k}$. □

This theorem allows us to reduce the problem of finding explicit formulae for $L_{j,l,k}$ to the case of $j = 0$. But finding explicit formula for $L_{0,l,k}$ is not easy. Even in the case of $\lambda = 1$, that is, interpolation on the unit circle, we do not know the general form of the formulae. Below we give the formulae for the first two cases for $\lambda = 1$. Since there is only one circle, we choose $\alpha = 0$ and write $L_k(x,y) := L_{0,1,k}(x,y)$. The formulae are given in polar coordinates.

The case $n = 2$ with $\lambda = 1$:

$$\begin{aligned} L_0(x,y) &= (1 + 4r\cos\theta - 2r^2\cos 2\theta)/3, \\ L_1(x,y) &= (-1 + r^2 - 4r\cos\theta + 4r^2\cos 2\theta)/6. \end{aligned}$$

The case $n = 3$ with $\lambda = 1$:

$$\begin{aligned} L_0(x,y) &= \big(1 + (3r - r^3)\cos\theta + 6r^2\cos 2\theta - 4r^3\cos 3\theta\big)/5, \\ L_1(x,y) &= -(1 - r^2 + 2r(1 - r^2)\cos\theta + 4r^2\cos 2\theta - 4r^3\cos 3\theta)/10. \end{aligned}$$

Notice that there is no $\sin j\theta$ terms in the above formulae. In fact, that $L_k(x,y)$ is even in y means that there is no $\sin j\theta$ terms in the polar coordinates.

3. Integral Mean and Cubature formula. Although we do not know the explicit formula of the interpolation polynomial in general, the fact that the equidistant points on the circle have a rotational invariance property allows us to compute the integral of the interpolation polynomial. In fact, the mean of the polynomial over any circle of radius r can be computed from an interpolation polynomial of one variable.

Let $0 < r_1 < \ldots < r_\lambda \le 1$ and $\mu_1 + \ldots \mu_\lambda = [n/2] + 1$ as in (1.1). We consider the following interpolation problem of one variable: finding a polynomial q of degree $2[n/2] + 1$ such that

$$q^{(k)}(r_l) = g_{l,k}, \quad 0 \le k \le \mu_l - 1, \quad 1 \le l \le 2\lambda, \tag{3.1}$$

where $r_{2\lambda-j+1} = -r_j$, $1 \le l \le \lambda$, we also assume that $\mu_{2\lambda-j+1} = \mu_j$ for $1 \le j \le \lambda$. This is a Hermite interpolation problem that has $2([n/2]+1)$ interpolation conditions. It has a unique solution. The unique polynomial that satisfies (3.1) can be written as

$$q(r) = \sum_{l=1}^{2\lambda} \sum_{k=0}^{\mu_l - 1} g_{l,k} \ell_{l,k}(r), \tag{3.2}$$

where $\ell_{l,k}$ are the fundamental interpolation polynomials defined by $\ell_{l,k}^{(v)}(r_u) = \delta_{k,v}\delta_{l,u}$, $0 \le v \le \mu_u - 1$, $1 \le u \le 2\lambda$. Let $\Omega(x) = (x-r_1)^{\mu_1} \ldots (x-r_\lambda)^{\mu_\lambda}$. Then the polynomials $\ell_{l,k}$ are given by

$$\ell_{l,k}(r) = \frac{\Omega(-r)\Omega(r)}{k!(r-r_l)^{\mu_l - 1}} \sum_{j=0}^{\mu_l - k - 1} \frac{1}{j!} \frac{d^j}{dr^j} \left[\frac{(r-r_l)^{\mu_l}}{\Omega(r)\Omega(-r)} \right] \Big|_{r=r_l} (r-r_l)^j$$

for $1 \le l \le \lambda$ (see, for example, [2]). Since the interpolation points are symmetric, it is easy to see that $\ell_{2\lambda-l+1,k}(r) = (-1)^k \ell_{l,k}(-r)$.

Let $P(x,y)$ denote the unique solution in Theorem 2.5. Let

$$p(r) = \frac{1}{2\pi} \int_0^{2\pi} P(r\cos\theta, r\sin\theta) d\theta, \qquad r \ge 0. \tag{3.3}$$

Since $p(r)$ is a polynomial of r, its definition can be extended to $r \in \mathbb{R}$. It turns out that p can be written in terms of fundamental polynomials of (3.1).

THEOREM 3.1. *Let $\ell_{l,k}$ be defined as in (3.2). Then the mean of the unique interpolation of Theorem 2.5 is given by*

$$p(r) = \sum_{l=1}^{\lambda} \sum_{k=0}^{\mu_l - 1} F_{l,k} \left[\ell_{l,k}(r) + \ell_{l,k}(-r) \right]$$

where, for $n = 2m$ or $n = 2m-1$, the coefficients $F_{l,k}$ are defined by

$$F_{l,k} = \frac{1}{2m+1} \sum_{j=0}^{2m} f_{j,l,k}, \quad 0 \le k \le \mu_l - 1, \quad 1 \le l \le \lambda.$$

Proof. Assume that the polynomial $\widetilde{P}(r,\theta)$ is of the form (2.1); that is,

$$\widetilde{P}(r,\theta) = \sum_{k=0}^{n} r^k q_k(\theta),$$

where $q_k(\theta)$ are trigonometric polynomials of degree k given in (2.2) and (2.3). Since $q_{2j-1}(\theta)$ contains only trigonometric polynomials of odd degree, see (2.3), its integral over $[0, 2\pi]$ is zero. Consequently, as a polynomial of r, $p(r)$ is an even function. For a fixed r, $\widetilde{P}(r,\theta)$ is a trigonometric polynomial of degree at most $2m$. Thus, in order to evaluate the univariate integral of P, we can use the quadrature formula

$$\frac{1}{2\pi}\int_0^{2\pi} \tau(t)dt = \frac{1}{2m+1}\sum_{j=0}^{2m} \tau(\theta_j^\alpha), \qquad \theta_j^\alpha = \frac{(2j+\alpha)\pi}{2m+1}. \tag{3.4}$$

which is known to hold for every trigonometric polynomial of degree $2m$. For $\alpha = 0$ this is the classical result in [12, Vol.2, p. 8], and for $\alpha \neq 0$ it follows from

$$\int_0^{2\pi} \tau(\theta + t)dt = \int_0^{2\pi} \tau(t)dt,$$

which holds for every θ and for every trigonometric polynomial τ. Since $q_k(\theta)$ in (2.1) are of degree $k \le 2m$, this gives

$$p(r) = \frac{1}{2m+1}\sum_{j=0}^{2m} \widetilde{P}(r, \theta_j^\alpha).$$

Taking derivatives with respect to r and using the interpolation conditions of $P(x,y)$ it follows that

$$p^{(k)}(r_l) = \frac{1}{2m+1}\sum_{j=0}^{2m} \frac{d^k\widetilde{P}}{dr^k}(r_l,\theta_j) = F_{l,k}, \qquad 0 \le k \le \mu_l - 1, \quad 1 \le l \le \lambda.$$

Since p is an even polynomial, it follows that $p^{(k)}(-r_l) = (-1)^k F_{l,k}$, and these conditions determine p uniquely. Evidently, the polynomial p satisfies the interpolation conditions (3.1) with $g_{l,k} = F_{l,k}$ and $g_{2\lambda-l+1,k} = (-1)^k F_{l,k}$ for $0 \le k \le \mu_l - 1$ and $1 \le l \le \lambda$. Consequently, we have

$$p(r) = \sum_{l=1}^{\lambda}\sum_{k=0}^{\mu_l-1} F_{l,k}\left[\ell_{l,k}(r) + \ell_{l,k}(-r)\right],$$

using the fact that $\ell_{2\lambda-l+1,k}(r) = (-1)^k\ell_l(-r)$. □

The polynomials $\ell_{l,k}(r)$ are interpolation polynomials of one variable, they can often be written down explicitly. We give two cases below. The first one is when $\mu_l = 1$ for all l, that is, the case of Lagrange interpolation.

COROLLARY 3.2. *Let $P(x,y)$ denote the unique Lagrange interpolation polynomial in Corollary 2.6. Then the mean $p(r)$ of (3.3) is given by*

$$p(r) = \sum_{l=0}^{[n/2]} F_l\left[\ell_l(r) + \ell_l(-r)\right], \quad \ell_l(r) = \prod_{j=0}^{[n/2]}\left(\frac{r+r_j}{r_l+r_j}\right)\prod_{j=0, j\neq l}^{[n/2]}\left(\frac{r-r_j}{r_l-r_j}\right),$$

where, for $n = 2m$ or $n = 2m-1$, F_l are defined by

$$F_l = \frac{1}{2m+1}\sum_{j=0}^{2m} f_{j,l}, \qquad 0 \le l \le [n/2].$$

Proof. This follows from the fact that $\mu_l = 1$, so that the fundamental polynomials for (3.1) are given by ℓ_l given in the statement. □

The second case is when $\lambda = 1$, that is, interpolation on one circle which we choose to be the the unit circle. We choose $\alpha = 0$.

COROLLARY 3.3. *Let $P(x,y)$ be the unique Hermite interpolation polynomials satisfying*

$$\frac{\partial^k P}{\partial r^k}\left(\cos\theta_j^0, \sin\theta_j^0\right) = f_{j,k}, \quad 0 \le k \le \left[\frac{n}{2}\right], \quad 0 \le j \le 2m.$$

Then the mean $p(r)$ of (3.3) *is given by*

$$p(r) = \sum_{k=0}^{[n/2]} \frac{(-1)^k F_k}{k!} \sum_{j=0}^{[n/2]-k} \binom{[n/2]+j}{j} \times \left[\left(\frac{1-r}{2}\right)^{[n/2]}\left(\frac{1+r}{2}\right)^{k+j} + \left(\frac{1+r}{2}\right)^{[n/2]}\left(\frac{1-r}{2}\right)^{k+j}\right].$$

where F_k are given formally by the same formula as in the previous corollary.

Proof. It follows from the theorem that $p(r)$ is an even polynomial, uniquely determined by $p^{(k)}(1) = F_k$ and $p^{(k)}(-1) = (-1)^k F_k$. By the formula for the Hermite interpolation of two points (cf. [5]), we have with $s = [n/2]$

$$p(r) = \sum_{k=0}^{s} h_{k,0}(r) p^{(k)}(-1) + \sum_{k=0}^{s} h_{k,1}(r) p^{(k)}(1),$$

where $h_{k,0}$ is given by the formula

$$h_{k,0}(r) = \left(\frac{1-r}{2}\right)^s \frac{1}{k!} \sum_{j=0}^{s-k} \binom{s+j}{j} \left(\frac{1+r}{2}\right)^{k+j},$$

and $h_{k,1}(r) = (-1)^k h_{k,0}(-r)$. This gives the stated formula of $p(r)$. □

The explicit formula for the mean of the interpolation polynomial can be used to derive cubature formulae. This comes from the following formula,

$$\int_{B^2} f(x,y)\,dxdy = \int_0^1 r \int_0^{2\pi} f(r\cos\theta, r\sin\theta)\,d\theta dr, \tag{3.5}$$

which shows, in particular, that the integral of the interpolation polynomial $P(x,y)$ over B^2 is reduced to the integral of the mean $p(r)$.

THEOREM 3.4. *Let $W(r)$ be an even nonnegative weight function defined on $[-1,1]$. Let $\ell_{l,k}(r)$ be the fundamental interpolation polynomial in* (3.1). *Then*

$$\int_{B^2} f(x,y) W(\sqrt{x^2+y^2})\,dxdy = \sum_{l=0}^{\lambda} \sum_{k=0}^{\mu_l - 1} \Lambda_{l,k} \sum_{j=0}^{2m} \frac{d^k f}{dr^k}\left(r_l \cos\theta_j^\alpha, r_l \sin\theta_j^\alpha\right)$$

for $f \in \Pi_n^2$, where the weights are given by

$$\Lambda_{l,k} = \frac{2\pi}{2m+1} \int_{-1}^{1} \ell_{l,k}(r)|r|W(r)\,dr.$$

Proof. As a projection operator, the interpolation polynomial preserves polynomials in Π_n^2. The cubature formula follows from integrating the unique interpolation polynomial $P(x,y)$ in Theorem 2.5. Using the formula (3.5), we have

$$\int_{B^2} P(x,y)W(\sqrt{x^2+y^2})dxdy = 2\pi \int_0^1 p(r)rW(r)dr. \tag{3.6}$$

Hence, by Theorem 3.1, the cubature formula has the stated form with cubature weights given by

$$\Lambda_{l,k} = \frac{2\pi}{2m+1}\int_0^1 [\ell_{l,k}(r) + \ell_{l,k}(-r)]\, rW(r)dr.$$

Changing integration from $[0,1]$ to $[-1,1]$ gives the stated result. □

One interesting special case is when $\mu_l = 1$ for all l, for which no derivative values appear in the formula. This corresponds to the Lagrange interpolation discussed in Corollary 3.2.

THEOREM 3.5. *The following cubature formula holds for $f \in \Pi_n^2$,*

$$\int_{B^2} f(x,y)W(\sqrt{x^2+y^2})dxdy = \sum_{l=0}^{[n/2]} \Lambda_l \sum_{j=0}^{2m} f\left(\cos\theta_j^{\alpha_l}, \sin\theta_j^{\alpha_l}\right),$$

where the cubature weights λ_k are given by

$$\Lambda_l = \frac{2\pi}{2m+1}\int_{-1}^1 \frac{g(r)}{(r-r_l)g'(r)}|r|W(r)dr, \qquad g(r) = \prod_{j=0}^{[n/2]} (r^2+r_j^2).$$

Proof. Since $\ell_l(r)$ in Corollary 3.2 is equal to $g(r)/((r-r_l)g'(r))$, this is a special case of Theorem 3.4. □

This is like the classical cubature formula on B^2, which can be derived from the formula (3.5) by using the quadrature (3.4) for the trigonometric polynomials, then use the quadrature formula on $[-1,1]$,

$$\int_{-1}^1 f(r)|r|W(r)dr = \sum_{l=0}^{[n/2]} \Lambda_l f(r_l) + \sum_{l=0}^{[n/2]} \Lambda_l f(-r_l), \quad \Lambda_l = \int_{-1}^1 \frac{g(r)}{(r-r_l)g'(r)}|r|W(r)dr,$$

which holds for all polynomials of degree n. Let us point out, however, that the usual product formula on the ball uses even number of points on the circles and the radii of the circles are chosen according to certain Gaussian quadrature formula (see, for example, [8] and [9]). The above theorem shows that it is not necessary to choose radii in the case that the number of nodes on the circle is odd. This left open the question of choosing the radii to obtain cubature formulae of higher order.

Another special case that may be of interesting comes from the Hermite interpolation on the unit circle, for which the explicit formula of the mean $p(r)$ is given in Corollary 3.3.

THEOREM 3.6. *The following cubature formula holds for $f \in \Pi_n^2$:*

$$\int_{B^2} f(x,y)dxdy = \sum_{k=0}^{[n/2]} \Lambda_k \sum_{j=0}^{2m} \frac{d^k f}{dr^k}\left(\cos\theta_j^0, \sin\theta_j^0\right),$$

where the coefficients Λ_k are given by

$$\Lambda_k = \frac{(-1)^k}{k!}\sum_{j=0}^{s-k}\binom{s+j}{j}\sum_{i=0}^{[(s-k-j)/2]}\binom{s-k-j}{2i}\frac{\Gamma(k+j+1)\Gamma(i+1)}{\Gamma(k+j+i+2)}.$$

Proof. As in Theorem 3.4, using the formula (3.6) with $W(r)=1$, the cubature formula comes from integrating the interpolation polynomial in Corollary 3.3. The cubature weights are given by

$$\Lambda_{l,k} = (-1)^k\int_0^1 [h_{k,1}(r)+h_{k,1}(-r)]rdr.$$

Let $s=[n/2]$. The main computation is as follows:

$$\begin{aligned}&\int_0^1\left[\left(\frac{1-r}{2}\right)^s\left(\frac{1+r}{2}\right)^{k+j}+\left(\frac{1+r}{2}\right)^s\left(\frac{1-r}{2}\right)^{k+j}\right]rdr\\ &=\int_0^1 r(1-r^2)^{k+j}\sum_{i=0}^{s-k-j}\binom{s-k-j}{i}r^i(1+(-1)^i)dr\\ &=\sum_{i=0}^{[\frac{s-k-j}{2}]}\binom{s-k-j}{2i}\frac{\Gamma(k+j+1)\Gamma(i+1)}{\Gamma(k+j+i+2)},\end{aligned}$$

where the last step follows from changing variable $t\to r^2$ and using the standard Beta integral. □

4. Arbitrary distinct points on the circle. In this section we consider the interpolation problem (4.1) for arbitrary set of points on the unit circle. The discussion is limited to the case of $\lambda=1$, so that we are considering the interpolation problem on the unit circle: show that there is a unique polynomial $P\in\Pi_n^2$ satisfying

$$\frac{\partial^k P}{\partial r^k}(\cos\theta_j,\sin\theta_j)=f_{j,k},\quad 0\le k\le\left[\frac{n}{2}\right],\quad 0\le j\le 2m=2\left[\frac{n+1}{2}\right],\qquad (4.1)$$

for any preassigned data $\{f_{j,k}\}$, where $\{(x_j,y_j)=(\cos\theta_j,\sin\theta_j): 0\le j\le 2m\}$ is a set of distinct points on the unit circle.

The interpolation problem (4.1) uses the same number of points for $n=2m$ or $n=2m-1$. It follows that for the poisedness of the interpolation, we only need to consider the case of $n=2m-1$.

PROPOSITION 4.1. *Let (x_j,y_j), $0\le j\le 2m$, be a set of distinct points on the unit circle. If the interpolation problem* (4.1) *has a unique solution for $n=2m-1$, then it has a unique solution for $n=2m$.*

Proof. Assume that the interpolation problem (4.1) does not have a unique solution for $n=2m$. Let P be a nonzero polynomial such that

$$\left(\frac{\partial^k P}{\partial r^k}\right)(x_j,y_j)=0,\quad 0\le k\le m,\quad 0\le j\le 2m.$$

We may assume that P is of the form (2.1). Then it is easy to verify that the polynomial $Q=nP-r\partial P/\partial r$ is a polynomial of degree $n-1=2m-1$, and Q is not zero since otherwise $P=r^n q_n(\theta)$, which cannot satisfy the above interpolation condition. Evidently, Q and its consecutive normal derivatives up to $m-1$ vanish at (x_j, y_j) for $0 \le j \le 2m$, which shows that the interpolation problem (4.1) is not poised for $n = 2m-1$. ☐

Thus, in the following we only need to consider $n = 2m-1$. For arbitrary distinct points, we use an expression for the polynomials P, similar to that in (2.1),

$$P(r,\theta) = \sum_{l=0}^{2m-1} q_l(\theta) r^l / l!.$$

Let $\Theta = \{\theta_j : 0 \le j \le 2m\}$. In order to show that the problem (4.1) is poised, we need to show that $P \in \Pi^2_{2m-1}$ satisfying

$$\frac{d^k P}{dr^k}(1,\theta_j) = \sum_{l=k}^{2m-1} q_l(\theta)/(l-k)! = 0, \quad 0 \le k \le m-1, \quad \theta \in \Theta, \tag{4.2}$$

implies $P \equiv 0$. In the case of $m = 1$, this system of equations corresponds to the Lagrange interpolation by a trigonometric polynomial of degree 1 at three distinct points, hence it has unique solution.

Proposition 4.2. *Let $\{(\cos\theta_j, \sin\theta_j),\ 0 \le j \le 2\}$, be a set of three distinct points on the unit circle. Then the interpolation problem* (4.1) *has a unique solution for $n = 1$ and 2.*

For $n \ge 3$, additional conditions on the points are needed for the poisedness of the interpolation. In the case of $n = 3$ and 4, we have

Proposition 4.3. *Let $\{(\cos\theta_j, \sin\theta_j),\ 0 \le j \le 4\}$ be a set of distinct points on the unit circle. The interpolation problem* (4.1) *has a unique solution for $n = 3$ and 4 if and only if $\Theta = \{\theta_j : 0 \le j \le 4\}$ admits a unique trigonometric interpolation in the space* $\operatorname{span}\{1, \cos t, \sin t, \cos 3t, \sin 3t\}$.

The proof amounts to solve the system of equations (4.2) in the case of $m = 2$. If we assume that the polynomial P is of the form (2.1) with $n = 3$, then an easy computation shows that (4.2) is uniquely solvable if and only if

$$2c_0 + c_1 \cos\theta + c_2 \sin\theta + c_3 \cos 3\theta + c_4 \sin 3\theta = 0, \quad \theta \in \Theta,$$

has a unique solution, where Θ contains 5 distinct points.

The trigonometric interpolation by $a_0 + a_1\cos\theta + a_2 \sin\theta + a_3 \cos 3\theta + a_4 \sin 3\theta$ at 5 distinct points is poised if and only if the interpolation matrix M, formed by rows $\{1, \cos\theta_j, \sin\theta_j, \cos 3\theta_j, \sin 3\theta_j\}$, is invertible . Using $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$ and $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$, it follows that the determinant of M is equal to the determinant of the matrix whose rows are $\{1, e^{-i\theta_j}, e^{i\theta_j}, e^{-3i\theta_j}, e^{3i\theta_j}\}$. Multiplying each column by $e^{3i\theta_j}$ and write $t_j = e^{i\theta_j}$, we conclude that the determinant of M is equal to a multiple of the determinant of $\widetilde{M}$, where the rows of $\widetilde{M}$ are $\{1, t_j^2, t_j^3, t_j^4, t_j^6\}$. It follows that

$$\det \widetilde{M} = \prod_{1 \le l < j \le 5} (t_l - t_j)(\sigma_1\sigma_4 + t_1\sigma_1\sigma_3 + t_1^2\sigma_3),$$

where $\sigma_1 = t_2 + t_3 + t_4 + t_5$, $\sigma_3 = t_2t_3t_4 + t_3t_4t_5 + t_2t_3t_5 + t_2t_4t_5$, $\sigma_4 = t_2t_3t_4t_5$. In particular, it follows that if $t_1 = 1$, $t_2 = -t_4$, $t_3 = -t_5$, then the trigonometric interpolation is not unique; that is, the interpolation is not unique if $\theta_4 = \pi + \theta_1$ and $\theta_3 = \pi + \theta_2$.

Similar results also hold for $n = 5$. We state the result for the next case.

PROPOSITION 4.4. *Let* $(\cos\theta_j, \sin\theta_j)$, $0 \le j \le 6\}$, *be a set of distinct points on the unit circle. The interpolation problem* (4.1) *has a unique solution for* $n = 5$ *and* 6 *if the trigonometric interpolation on* $\Theta = \{\theta_j : 0 \le j \le 6\}$ *admits unique solution in the spaces* $\mathrm{span}\{1, \cos\theta, \sin\theta, \cos 2\theta, \sin 2\theta, \cos 4\theta, \sin 4\theta\}$ *and* $\mathrm{span}\{1, \cos\theta, \sin\theta, \cos 3\theta, \sin 3\theta, \cos 5\theta, \sin 5\theta\}$.

One naturally expects that similar results should hold for $n \neq 7$; that is, the uniqueness of the interpolation problem (4.1) requires that the interpolation points on the unit circle admit unique lacunary trigonometric interpolation of several types. It may be of interest to determine the types of the lacunary trigonometric interpolation for all n. More interesting is perhaps to characterize the points that lead to unique interpolation.

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