In Memoriam

Hubert A. Berens (1936 - 2015)

   Hubert Anton Berens passed away in Erlangen, Germany, on February 9, 2015. He was 78 years old. He is survived by his wife, Ursula, sons Christian & Harald and daughter Eve, and eight grandchildren.

   Berens was born on May 6, 1936 in Suttrop (Warstein), Germany. After receiving his Abitur in Rüthen in 1957, he went to RWTH (Rheinisch-Westfälische Technische Hochschule) in Aachen to study physics. He received his diploma in 1962 and was granted his doctorate in mathematics at RWTH in 1964. After his habilitation in 1968, Berens joined the University of California at Santa Barbara as an assistant professor. He took a leave to University of Texas at Austin to work with George Lorentz in 1970 and left his position in UCSB for UT Austin in 1972. He was appointed the Wilhelm Specht professor at the Mathematical Institute of the University of Erlangen-Nürnberg, Germany, in 1973, a position he held until his retirement in 2001. He had seven Ph.D students.

   The author of seventy papers and two books, Berens worked in approximation theory, functional analysis and Fourier analysis. He received his doctoral degree from Paul Butzer, their book on semigroups of operators [B1] became a classic (1268 citations in Google Scholar). He wrote his habilitation thesis on interpolation

methods for dealing with approximation processes in Banach spaces. His work with G. G. Lorentz [19] on Bernstein polynomials is deep and influential (244 citations in Google Scholar) and his work with Butzer and Pawelke [14] is one of the very first on the characterization of the best approximation on the unit sphere. After he returned to Germany in the 1970s, he switched to nonlinear approximation, worked on Chebyshev sets and continuous selection of metric projections. He returned to classical approximation theory and Fourier analysis in the 1990s, revisited some of his favorite topics and worked extensively in ℓ_1 summability of Fourier integrals. A renowned mathematician in the approximation theory community, Berens made original and substantial contributions that lived up to his high standards.

Below is a collection of reminiscences from Berens' colleagues, friends, and students.

Reminiscences of Hubert Berens

Alexey Alimov[1]

I first met Professor Hubert Berens in Erlangen in 1997, when he invited me right after I defended my Ph.D. thesis at the Moscow State University. My thesis, written under the supervision of Prof. Stechkin and Prof. Tsar'kov, dealt with geometric approximation theory, I was happy to find a group of people in Erlangen who shared my enthusiasm for this subject. My second visit in Erlangen was in 1999–2000, when Prof. Berens secured for me a visiting position at the Mathematical Institute. Each time he was a perfect host, organizing my accommodation and transportation, and also making sure that I had enough free time to become familiar with the Bavarian people, their lifestyle, culture, and history. Thanks to Hubert, I enjoyed Erlangen, its surroundings, riding as far as Pottenstein in Fränkische Schweiz on Hubert's nice bike that he gave me on my very first day at Erlangen. Strange as it may seem, but he made me refrain from studying German, for this, he believed, could hinder my research work. However, I dared to circumvent his injunction and eventually came to enjoy, *inquantum*, the music of Hochdeutsch.

My first joint work with Hubert Berens culminated in a 1999 paper [62] on the Chebyshev approximation in matrix spaces, which generalizes and brings together a good deal of the previous efforts on this subject. In 1999, during my second visit to Prof. Berens, he posed the problem of intrinsic characterization of Chebyshev sets in the space $\ell^\infty(n)$ equipped with the uniform norm. Later, I managed to solve this problem, demonstrating, in addition, that the Chebyshev sets in $\ell^\infty(n)$ are extremal Chebyshev, which means that if an intersection of a Chebyshev set M with a bar Π (an intersection of hyperstrips generated by extreme functionals) is nonempty, but the interjection with the relative interior of Π with M is empty, then $\Pi \cap M$ is a singleton. (One gets the standard definition of a Chebyshev set when Π is a closed ball, which is, of course, a bar.) Hubert was very glad to learn about this result, which complements his intrinsic characterization of suns in $\ell^\infty(n)$.

I should mention one of the most important breakthroughs of Prof. Berens in the geometric approximation theory. Let me first recall some notation adopted in this field. If Q denotes some property (for example, 'connected') we say that a closed set M is:

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P-Q if for all $x \in X$ the set $P_M x$ of nearest points in M to x is nonempty and has property Q;

B-Q if the intersection of M with the closed ball $B(x, r)$ has property Q for all $x \in X$, $r > 0$;

For example, a closed subset of a finite-dimensional space if P-nonempty or is an existence set (proximinal).

According to a well-known result of L. P. Vlasov, a boundedly compact P-acyclic set M in a Banach space is a sun (the latter means that for any $x \notin M$ there exists a point $y \in P_M x$ such that y is a nearest point to all points in the ray emanating from y and passing through x). 'Suns' have important characteristic features. For example, they can be characterized by separability properties of one kind or another: a ball can be separated from such a set by means of a larger ball or a supporting cone. These properties are akin to the conventional separability properties of convex sets in terms of half-spaces (hyperplanes) and hence can be formulated in terms of the Kolmogorov criterion of best approximation..

A long open question asks whether the converse of Vlasov's result is true. In their 1983 paper [36], Berens and Hetzelt gave a positive answer to this problem in an arbitrary two-dimensional Banach space X_2. In particular, they characterized the suns in X_2 in terms of metric convexity and completely described the ranges of the metric projection operator in this setting.

For Banach spaces X_n of dimension $n \geq 3$ it is still unknown whether a sun in X_n is even P-connected. (Of course, the answer is positive in the trivial cases of smooth or locally uniformly rotund spaces.)

A profound breakthrough on this problem was made by Berens and Hetzelt in their 1984 paper [37], where they put forward an intrinsic characterization of suns in $\ell^\infty(n)$ in terms of ℓ^1-convexity. As a corollary, they showed that a sun in $\ell^\infty(n)$ is B-cell-like (and in particular, P- and B-acyclic). In this paper they applied the machinery of metric convexity in the geometric approximation theory for the first time. It is not very often that a single problem brings to life an entire theory. But this is just the case with the theory of metric convexity (Menger-connectedness) in approximation, which was further developed by A. L. Brown, C. Franchetti and S. Roversi, A. R. Alimov, etc. In particular, the machinery of ℓ^1-convexity (in the Menger sense) in the context of best approximation has led to the concept of monotone path-connectedness, which has proved instrumental and altogether quite natural in problems on approximation of functions and in general development of the theory.

I saw Hubert one last time in Erlangen on July 2010, when I delivered a talk on the connectedness of suns in Banach spaces with an emphasis of their monotone path-connectedness. Hubert was glad to hear about the advancements on what he brought to life.

God rest his soul.

Elena Berdysheva[2]

In summer 1999, I happened to be in Erlangen. I was a PhD student at the Ural State University shortly before my PhD defense.

Some friends introduced me to Berens during the coffee time before a seminar talk at the Mathematical Institute. I knew who he was. The first time I had heard

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his name – in connection with his joint book with Paul Butzer on semigroups of operators [B1] – I still was a student. I also had heard about his approximation theory group in Erlangen. At our first meeting, Berens was very brief and definite – typical of him, I learned later: "Do you want to give a talk at my seminar?" I agreed. A week later, after my talk, he asked again just one question: "Do you want to work with me?" I agreed. This was the beginning of a collaboration and a friendship that lasted until Berens' death in February 2015.

My PhD advisor – I must say, I prefer the German word "Doktorvater" – was Vitalii Arestov, and it was he who built a mathematician in me. He shared his taste for mathematics with me. However, I also consider Berens my teacher. I was very young as I met him and as we began a very close, very intensive collaboration. I still feel his influence in everything I do in mathematics. Can one say "Postdoc-Vater"? I do have two fathers in mathematics.

I stayed in Erlangen until fall 2002. Berens was somebody who took care of his people, including in formal, practical issues like getting fellowships, positions, etc. He immediately managed to organize a fellowship of the Free Sate of Bavaria for me, followed by an assistant position for one semester, and then followed by our big project, my Alexander-von-Humboldt Fellowship. At that time, the group around Berens consisted of Tomas Sauer (who then left for Gießen), Martina Finzel, Wolfgang zu Castell, and the lovely secretary Jutta Zintchenko. "This is a family", Wolfgang once said. Indeed, I was surrounded by open, friendly, supportive people – people I could count on. When I think back on this time, I feel deep gratitude towards all of them. Thanks to them, I learned how to live in Germany, which became my Heimat. And they thought me German! In fact, I just did not have a chance not to speak German. Every day, all of us had lunch together, and these lunches were a place for conversations about politics, history, culture, and the like. At my first lunch, the company spoke English – because of me. On the second day, there was a mixture of English and German. From the third day on, they switched to German, though being very, very patient with me. My son did the rest, he started to speak by this time and the language he chose was German.

Berens was amazingly well educated. In addition to his excellent knowledge of German and world history and a keen understanding of politics, he was a true expert on Franconia. He grew up in a different part of Germany – near the city Westphalian of Soest – and obtained his professorship in Erlangen only in 1973. By the time I met him, he had become an authority in local history, geography, and culture. Were he not an excellent mathematician, he could be an excellent tour guide.

Thus, he was my teacher not only in mathematics, but also in history and culture. I started to understand Germany through him. Moreover, he had an influence on my very personality, and on my way to understand the world. Going through my phases of frustration, I frequently think of his words – some remarks he made in our discussions – and they provide me an orientation and something to hold on. And then all these practical issues! I came to Germany from post-Soviet Russia, and it was pretty much a different world. I was helpless with insurances, banks, taxes, etc. – all things I never had accounted before. To buy a bus ticket at a vending machine already was a challenge. And Berens went with me to all these insurance offices, banks and immigration offices. He spent a huge amount of time processing

my issues and teaching the German reality to me. His wife told me that he always did this for all his visitors.

We started to work on questions related to the so-called Turán problem. By that time, Berens just had finished a series of papers on the harmonic analysis for ℓ_1-radial functions. The key idea appeared in joint investigations with Yuan Xu on ℓ_1 summability of Fourier integrals [59] and was further developed in a joint paper with Wolfgang zu Castell [61]. I came with the Turán problem in my pocket, and it was a welcomed application for their theory. The Turán problem consists of determining the maximal possible value of integrals of positive definite functions vanishing outside a given centrally symmetric body, normalized by one at the origin. Being motivated by results for the ball, where one can restrict the consideration to ℓ_2-radial functions, we decided to consider the Turán problem for ℓ_1-radial functions, a problem which we still have no solution today. We did give a lower bound in terms of n-times monotone functions, and we did introduce a discrete variant of the Turán problem that gives an upper bound. In [68] we solved the discrete problem in the two-dimensional case. In [69] and [70] we considered the case of higher dimensions and obtained some particular results, including the solution of the discrete problem for dimensions 3, 5 and 7.

Paper [66], on what we called the pointwise Turán problem (i. e., instead of the integral we maximize the value of the function at a given point), was a result of a month-long stay of Vitalii Arestov in Erlangen. Berens, again displaying his qualities of a good organizer, managed to obtain a DFG fellowship for Arestov. That month was extremely productive. The common language the two gentlemen found was – imagine my shock – French, and I frequently found them discussing mathematics in French at the blackboard when I was coming to the institute in the morning. How fortunate that the formulae are the same in every language!

I never worked on Bernstein type operators when I was in Erlangen. However, I strongly suspect that "her coming from Berens" was the reason why Kurt Jetter invited me to join his group in Hohenheim. In Berens' publication list, I counted 15 papers on Bernstein type operators and Korovkin type theorems for positive operators. The earlier papers are coauthored with G.G. Lorentz and R. DeVore and are devoted, in particular, to saturation of Bernstein type operators and to Korovkin type theorems in abstract spaces as well as Korovkin type theorems for contractions in L^p. The later papers, coauthored with Yuan Xu and partly with H.J. Schmid, introduce and study the Bernstein-Durrmeyer operator with Jacobi weights. Berens told me that the reason Yuan and he had introduced Jacobi weights into the picture was that they had realized that the (unweighted) Bernstein-Durrmeyer operators are almost the de La Vallée Poussin means, just with a different weight. They also had learned then that the construction already had been introduced by Radu Păltănea a couple of years earlier, but it is [47] that remains the best introduction into the topic.

Berens and I kept seeing each other a couple of times a year, first in Stuttgart or at the Mathematical Institute in Erlangen, later in his house in Barthelmeßstraße. I enjoyed the hospitality provided by Mr and Mrs Berens. Last time I visited Berens was in August 2014, and our last phone conversation was in January 2015. He was already exhausted by his long illness, but I heard the same witty, very definite, very opinionated Berens. Our last joint paper [70] appeared in fall 2014. We made a step forward, but the problem still remains unsolved. I told him that we will write

a continuation together. I still cannot believe I will have to write it without him. I cannot believe I will come to Erlangen and will not meet him. I thank him for everything he did for me. And I thank God for having this person in my life.

Paul Butzer[3]

The RWTH Aachen, founded in 1870, was foremost an Institute of Technology with four well-known engineering faculties and a faculty of general science (essentially mathematics, physics, chemistry and some cultural fields). In 1958, with as many as 10,000 students in the Faculty of Mathematics and General Science, mathematics comprised three institutes (in particular three chairs), the "Institut für Reine und Angewandte Mathematik", "Institut für Mathematik" and "Institut für Geometrie und Praktische Mathematik". These institutes were responsible for all mathematical courses required by the engineering faculties; in 1961/62 there were alone 2100 first year engineering students. Faculties of Philosophy and Medicine were added during the mid-sixties.

When I joined the first named Institute in 1958, it was directed by Claus Müller (1920–2008) who had come from Bonn in 1956 to strengthen mathematics as a whole, together with Heinz König (who came from Würzburg in 1957, but left for Cologne in 1962). New courses in intermediate and advanced mathematics were added to enable students to obtain their masters and doctoral degrees in mathematics (with physics as a minor). Two more Chairs were added to the mathematics group in 1964 and 1965, headed by Heinz Schöneborn (Algebra) and Friedhelm Erwe (Analysis).

After I was appointed associate professor with a new Chair (Analysis) in 1961, and being promoted to full professor (Persönlicher Ordinarius) in 1962, Hubert Berens and Rolf Nessel, as junior assistants, were assigned to the new chair. Both had come to the RWTH in 1956 and were the first junior assistants at Müllers institute. They participated in my advanced courses, and wrote their Diplomarbeit/Staatsexamensarbeit (Masters thesis/teachers state examination) under me (1962). Hubert and I, with Rolfs assistance, had to build up this chair, now called Lehrstuhl A, in its delicate founding years 1958–1968. It was a tough, time consuming job which they managed in every respect.

The advanced (graduate) courses held at Lehrstuhl A from the very beginning included Approximation Theory, Special Functions of Mathematical Physics, Fourier Analysis, Functional Analysis and Semigroups of Operators. On top came the elementary courses, essentially the four semester course Analysis I–IV with the associated exercises (attended by at least 150 students in the first year – and 450–625 from 1968 onwards). All of these had to be assisted in some form or other by Hubert, and also Rolf. But the major work in building up the Lehrstuhl was the research involving the Diploma and Staatsexamen theses (both roughly equivalent to a masters thesis), basically due to the fact that our aim was to offer our students the possibility to work on problems that had so far not been solved, not only ones that were fashionable at the time.

A good portion of those graduating with those degrees continued on with their doctorate, often connected with deeper investigations dealing with their foregoing theses. In the course of Huberts years in Aachen (1958–1968) and shortly thereafter twelve of our students received their doctorates. Apart from Hubert (1964) and Rolf (1965), there were Dieter Ernst (1965), Hermann Schulte (1966), Ernst Görlich

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(1967), Hans-Günther Neuhausen, Walter Köhnen, Karl Scherer, Walter Trebels, and Ursula Westphal (all 1969). Hans Johnen and Eberhard Stark followed in 1970. Ten of them became professors at German universities; Dieter became Director of the Akademie of the Bunderwehr in Bremen. Considering the fact that the students need four to seven years to receive the initial degree, and ca. 2 to 4 further years for their doctorate, Hubert and Rolf always faced a huge amount of work before them.

Hubert assisted me with the lecture course on Semigroups of Operators held 1962, a job for which he was truly predestined. In fact, in the years preceding his doctorate he had studied mathematics and physics, the most important professor being the internationally known (mathematical) physicist Josef Meixner (1908–1994), famous for the Meixner scheme of orthogonal polynomials. As co-advisor of Huberts doctoral thesis Approximationssätze für Halbgruppenoperatoren in intermediären Räumen [Approximation theorems for semigroup operators in intermediate spaces] [4] Meixner revealed how tough he was as an examiner, dreaded by his students. Nevertheless, Hubert passed the exams with honors.

The semigroup course opened the way to Huberts first research field and led to Volume 145 of the "Grundlehren der mathematischen Wissenschaften", namely "Semi-Groups of Operators and Approximation" [B1] (Springer 1967, xi+318 pp.), presenting a systematic treatment of semigroups of bounded linear operators on Banach spaces and their connections with approximation theoretical questions.

The second half of the volume is devoted to the incorporation of the foregoing semigroup theory into the setting of intermediate spaces between the initial Banach space and the domain of (fractional) powers of the semigroups infinitesimal generator. These goals are achieved by applying the famous K-functional method, due to Jaak Peetre (1963). It was the first presentation of the new theory in book-form and it was precisely Hubert who worked it out meticulously.

Apart from Huberts outstanding doctoral thesis [4] he published 15 further papers while in Aachen in the broad areas of approximation and intermediate space theories. Of these he co-authored papers with Ernst Görlich on Fourier series [9] and Rolf Nessel [11, 15]. The joint paper [12] is regarded as the first one ever treating a PDE of fractional order. Köhnen in his doctoral thesis extended Hubert's results to evolution equations in Banach spaces.

One of the chair's early areas of research in approximation theory was the saturation phenomenon, to which I was introduced by Jean Favard (1902–1965) in Paris in September 1955. It concerns the optimal rate of convergence which can be achieved by an approximation process.

One of the basic innovations of Hubert's Habilitation thesis of 1968 [B2], was his incorporation of saturation theory together with an extension of E. Gagliardo's concept of relative completion into the broad area of approximation theory. In order to include the approximation behaviour of semi-group operators, also in the setting of intermediate space theory, Gagliardo's concept had first to be supplied with essentially new results.

In the simple case of a (C_0)-semigroup $\{T(t); t \geq 0\}$ on a Banach space X with norm $|\cdot|$, the saturation order is always $\mathcal{O}(t)$. This order cannot be improved, unless f belongs to some trivial class of functions, indeed $|T(t)f - f| = o(t)$ implies $T(t)f = f$ for all $t \geq 0$ (see [B1; Thm. 2.1.2]). The saturation class $S(\{T(t)\}) = \{f \in X; |T(t)f - f| = \mathcal{O}(t), t \to 0+\}$, can then be characterized by $S(\{T(t)\}) =$

$\widetilde{D(A)}^X$. Here A denotes the infinitesimal generator of the semigroup, $D(A)$ its domain equipped with the graph norm $\|\cdot\|$, and the relative completion of $D(A)$ with respect to X is defined by

$$\widetilde{D(A)}^X =: \big\{ f \in X; \exists\, (f_n)_{n\in\mathbb{N}} \subset D(A) \text{ with } \|f_n\| = \mathcal{O}(1),\ |f_n - f| = o(1),\ n \to \infty \big\}.$$

In case X is a reflexive Banach space, this class is simply $D(A)$ itself.

As an example, the Abel-Poisson singular integral $V(\rho)f$, with $f \in X = X_{2\pi}$ (standing for $C_{2\pi}$ or $L_{2\pi}^p$, $1 \le p < \infty$), which is the solution of Dirichlet's problem for the unit disk $0 \le \rho < 1$, defines a contraction semi-group of class (C_0) for $\rho = e^{-t}$ with infinitesimal generator $Af = -(\widetilde{f})'$, $\widetilde{f}$ being the conjugate function of the Fourier series of $f \in X_{2\pi}$. Its domain is given by $D(A) = \{f \in X_{2\pi}; (\widetilde{f})' \in X_{2\pi}\}$, and its completion turns out to be

$$\widetilde{D(A)}^{X_{2\pi}} = \Big\{ f \in X_{2\pi}; \big| \widetilde{f}(\cdot + h) - \widetilde{f}(\cdot) \big|_{X_{2\pi}} = \mathcal{O}(|h|),\ h \to 0 \Big\}$$

$$= \begin{cases} \big\{ f \in C_{2\pi}; (\widetilde{f})' \in L_{2\pi}^\infty \big\}, & X_{2\pi} = C_{2\pi}, \\ \big\{ f \in L_{2\pi}^p; (\widetilde{f})' \in L_{2\pi}^p \big\}, & X_{2\pi} = L_{2\pi}^p, 1 < p < \infty, \\ \big\{ f \in L_{2\pi}^1; \widetilde{f} \in BV_{2\pi} \big\}, & X_{2\pi} = L_{2\pi}^1. \end{cases}$$

Since $(1 - \rho) \sim t$, the saturation order is given by $\mathcal{O}(1 - \rho)$, i.e.

$$|V(\rho)f - f|_{X_{2\pi}} = \mathcal{O}(1 - \rho),\ \rho \to 1- \iff f \in \widetilde{D(A)}^{X_{2\pi}},$$

$$|V(\rho)f - f|_{X_{2\pi}} = o(1 - \rho),\ \rho \to 1- \iff (\widetilde{f})' = 0 \iff V(\rho)f = f,\ 0 \le \rho < 1.$$

Already in 1968/69, the joint papers [11, 15] by Rolf and Hubert dealt with multivariate saturation.

Hubert received the Habilitation degree at the University of Mainz during the summer semester 1968, a procedure which was fully completed within seven weeks with the support of its Rector, Hans Rohrbach (1903–1993) together with Gottfried Koethe (1905–1989).

Immediately thereafter Hubert left with his family, Uschi and their two sons Christian and Harald (both born in Aachen), for Santa Barbara, to accept an assistant professorship at the University of California there. Hubert stayed for two years and then accepted an associate professorship at the University of Texas at Austin. George Lorentz, my own teacher, was stationed there and they published six seminal joint papers [19, 22, 24–27]. Huberts daughter Eve was born in Austin. During his sixth year in the USA he was invited as chair holder at the University of Erlangen.

Hubert always regarded his years in the USA as the best of his teaching career, especially the time at Santa Barbara with its beautiful location, facing the Pacific. Clemens Market (doctorate and Habilitation under Ernst Görlich (1940–2014)), who also received Huberts personal support, recalls Huberts words one ought to go to Santa Barbara if one longs for heaven.

Eberhard Stark and Rudolf Stens participated in the International Conference on Constructive Theory of Functions, conducted by S. M. Nikolskiĭ (1905–2012), held at Kiev in 1983. They were especially happy to meet their model Hubert. Approximation being the joint basis of the research at the Aachen school, the work

of the Russian approximation school was closest to them. There the three had the unique opportunity in communicating directly (Stark could speak Russian) with the Russian and Eastern Block approximation theorists on their own home grounds. We respected their excellent work, they ours. They were among friends. Living together in the same hotel and relaxing from the strenuous lecture sessions, Hubert recalled his ten years spent in Aachen as especially productive.

At the colloquium held in honor of the retirement of Ernst Görlich at Aachen in 2005, it was Hubert who presented the invited lecture. It was so pleasant for Ernst, Rolf Nessel, Rudolf Stens and myself to have Hubert back with us in Aachen.

Professor Wang Kunyang of Beijing Normal University gave me a copy of his book Harmonic Analysis and Approximation on the Unit Sphere (Science Press, Beijing/New York 2000) and told me that our joint paper [14] of 1968 was the first paper on approximation of functions on the sphere, a topic which became an active research field only in 1980. Hubert had written two papers on this topic together with the co-author Luoqing Li [63, 64].

Hubert was a man of principle, straightforward, modest and slightly reserved in public, but quite open among friends. The friendly atmosphere he had experienced in the USA was different than what he encountered at Erlangen. The institute structure did not foster a truly collegial climate, and which Hubert had to bear. But Hubert had the opportunity to build up a chair with good students, many working on master, doctoral and Habilitation theses. His students and collaborators there have included Lothar Hetzelt (1981), Martina Finzel (1990) and Tomas Sauer (1993) (the three received their Habilitation degree under him, too), and Wolfgang zu Castell (2000). He published many joint papers with most of them.

A colleague with whom he shared many of his views was Dietrich Kölzow (born 1930), who met Hubert at Santa Barbara and later at Austin. Kölzow, who spent the years 1961–65 in Aachen and is stationed at Erlangen since 1968, mentioned that he was instrumental in regard to Huberts return to Germany. In fact, the German government had set up a program at the time enabling German scholars teaching in the USA to return home. A new chair was created for Hubert in Erlangen. The institute-chair structure was quite different to that which existed in Aachen. In Erlangen all chairs were a part of one large Mathematical Institute, although now they are independent.

After Hubert retired as Professor Emeritus in 2001, this chair was devoted to PDEs. The American custom of always leaving ones office door open for students was not appreciated at Erlangen. Kölzow emphasized that Hubert, being a renowned approximation theorist, should indeed be honored with a memorial colloquium.

Hubert, an independent, truly broad mathematical analyst, demanded good, solid mathematics from his students, was strict but always fair, as particular as he was with himself. One could always depend upon his judgement and constructive work in all aspects and many duties of the chair. He was a unique collaborator for me. His position was taken up by Rolf Nessel after he left for Santa Barbara.

In our last two telephone talks of fall and Christmas of 2014 – he died peacefully on February 9, 2015 – Hubert expressed his joy in finally having completed his paper [70], work which he had begun some five years earlier. Due to his serious illness, he was concerned that his collaborator Elena Berdysheva had to polish up the final version almost by herself. In his final call he told me how grateful he and

his spouse Uschi were to be able to enjoy the visits of their eight grandchildren, some of them almost daily.

The author extends his sincere thanks to Carl Butzer, B.A., J.D., LL.M. for his revision of this manuscript, and to Rudolf Stens for sharing his recollections of meetings with Hubert, for his critical study of this manuscript, especially the mathematical part, and for preparing the final version.

Wolfgang zu Castell[4]

The relationship between a candidate and his thesis advisor is often much deeper than just a matter of professional interaction, knowledge transfer and supervision. In German, the notion of *Doktorvater* comprises an appeal to a family type relation between parent and child. As such, the educational period as undergraduate/graduate student towards adulthood within the scientific community, i.e., the PhD degree, is certainly characterized by more than just mathematical education. The candidate is guided through the process of formulating and addressing a proper problem, is escorted along his struggle with the ups and downs of mathematical research, and, last but not least, obtains a living example of a pronounced attitude towards science and –in some, valuable cases– on some aspects of life in general.

Thus, the period of close interaction with the PhD advisor during these years of academic development leaves clear traces on one's own scientific habitualitie and taste, and contributes significantly to the young researcher's *world view on science.*

When I first encountered Hubert Berens as lecturer of an introductory class on numerical analysis at the Mathematical Institute of the University of Erlangen, all these thoughts were far beyond my perspective at that time. Hubert Berens had never been one of these instructors one notices while taking a class and leaving fading impressions, the longer the course moves back into the past. Right from the beginning, it became clear that his standards and expectations reached far beyond an unemotional accumulation of credits. His requests to deal with the subject and to explore ideas behind the material had been highly present from the first lecture on and never faded. It might not have been the large crowds of students being touched by his style of lecturing, but those who did followed him throughout their stay at the Mathematical Institute.

At the time of starting on a project for the German diploma thesis, Hubert Berens gave me three problems to choose from, all related to his recent work with Yuan Xu on ℓ_1-summability of the inverse Fourier integral [59]. I started working on one of them, which turned out to be much more fundamental than what had been anticipated when formulating the problem. At the core of the problem was the question of non-negativity of the integral

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(1) \qquad \int_0^\xi (\xi - t)^\lambda t^{\alpha+1} J_\alpha(t)\, dt, \qquad \xi \geq 0.
$$

This integral is well-known within the context of the Fourier-Bessel transform. For example, it shows that the classes of positive definite, ℓ_2-radial functions are properly nested with respect to the dimension. Berens & Xu used results from Richard Askey and George Gasper dealing with this integral within the proof of the positivity of the Cesàro (C, δ) means of the ℓ_1-partial integral. It was not until much later that I obtained a deeper understanding of the fundamental significance of such Bessel-type integrals. The question Hubert Berens asked me to work on was to find

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a proof of the positivity without referring back to the Bessel function integral (1). From a broader perspective, the aim to provide a proof of the positivity of the (C, δ) means in the ℓ_1-case without referring back to the results for ℓ_2-radial functions reflects the impression already present in the work on ℓ_1-summability of Fourier series [57-58], that changing the norm substantially changes the character of the summability method. We finally obtained a better understanding of the underlying structure and the close relations between ℓ_1 summability and the classical results for ℓ_2-radial functions (see for example (3)), although not within the reach of a single diploma thesis.

Thus, the advice was to work on the second problem, a characterization of the kernels involved in the Fourier- and inverse Fourier transform of ℓ_1-radial functions. This topic worked out much better and lead us into studying the hypergeometric $_1F_2$-function

(2) $$h_d(\xi) = 2^{-d+1} \, _1F_2 \left[\begin{array}{c} d \\ \frac{d}{2}, \frac{d+1}{2} \end{array} \middle| -\frac{\xi^2}{4} \right], \qquad \xi \in \mathbb{R}_+,$$

as a kernel of an integral transform on $\mathbb{R}_+$ (see [61]). Starting out from a natural question in summability, i.e., summing the series with respect to the total degree of the polynomial [58], lead onto a path studying Fourier integrals of radial functions and moving forward into the field of hypergeometric functions and Meijer's G-functions [61].

The first lesson I was taught during this time was that I must never shut my mathematical mind. Being open to other fields, some maybe related, others being further apart, was one of the fundamental characteristics of Berens' scientific attitude. Looking at the comparably short list of students he had during his academic career, one encounters a diversity which is rarely found. Almost none of the theses he supervised have been closely related in mathematical terms, or even showing overlapping contents. The field of expertise he covered as a supervisor reflects his broad scientific interest in approximation and related fields. And his interest did not stop within mathematics. I distinctly recall a talk on *Number Symbolism in Johann Sebastian Bach's Mass in h-Minor* given within the Seminar on Approximation Theory, just to mention an example.

Having encountered an area of potential interest, Hubert Berens further dwelt upon it, exploring its consequences and richness. Work on such projects could well spread over years, coming back to the topic several times. This way of carrying scientific problems along might be a second feature of Hubert Berens' scientific work. It can be read off his list of publications which also shows topics, being picked-up again much later, sometimes with another collaborator. For example, the work with Finzel [43] and Schmid [44] on Chebyshev sets in matrix spaces has been revisited in the late nineties [62], when Alexey Alimov visited the institute in Erlangen.

Regular visits of mathematicians from all over the world complemented the group of people working on approximation theory in Erlangen. One of the inspiring experiences shaping my academic life was an intense discussion with Charles Micchelli early during my work on my PhD thesis. Micchelli's paper on *Interpolation of Scattered Data: Distance Matrices and Conditionally positive Definite Functions* in Constr. Approx. (1986) has marked my entrance into the approximation theory group in Erlangen. My task was to present the paper in a seminar run by Hubert Berens and Tomas Sauer on multivariate interpolation. The seminar gave Berens

the motivation to get me involved into topics surrounding Bochner's Theorem. Our efforts soon showed that the modification to consider ℓ_1-radial functions instead of ℓ_2-radial ones provided a substantial alteration of the problem of characterizing radial positive definite functions, revealing that the classical tools from harmonic analysis could not be applied in a straightforward way. B-splines mingled with Bessel functions, and the Mellin transform turned out to provide a suitable tool. All these, apparently different tesserae grouped together in the work by Berens & Xu on ℓ_1-summability [57-59,61,64-65], mentioned before.

The quest for scientific achievement and recognition only too quickly merges into a struggle for originality and scientific depths. Hubert Berens always took a very clear attitude towards these facets of modern scientific life. *In dubio pro reo*, he withdraw his claim, acknowledging the contribution of the other. The first manuscript of [57] was withdrawn, once the authors found out that one part to some extent had been stated in a paper by Podkorytov (1981), although the context was substantially different. Similarly, Berens never failed to mention Cambanis, Keener & Simons when referring to the inverse kernel of (2) appearing in the Fourier transform of ℓ_1-radial functions, i.e.,

$$(3) \quad \omega_d(\xi) = \frac{2^{\frac{d}{2}} \Gamma^2\left(\frac{d}{2}\right)}{\Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{d-1}{2}\right)} \, \xi^{-\frac{d-2}{2}} \int_1^\infty \left(\tau^2 - 1\right)^{\frac{d-3}{2}} \tau^{-\frac{3d-4}{2}} J_{\frac{d-2}{2}}(\xi\tau) \, d\tau, \quad \xi \in \mathbb{R}_+,$$

as it had been introduced earlier by Cambanis et al. (1983), a paper which Berens & Xu did not get to know until some years after publishing their work. Hubert Berens always tried to trace back the original source of a thought and attribute it to its inventor. Such an attitude might well sound self-evident in academic ethics, but everybody having spent long enough in academic writing also knows that life often turns out to be much more involved, and that wholeheartedly keeping a fair-minded position in such cases deserves being honored.

Being a student of Hubert Berens, his attitude enabled for a young mathematician like me to form my own character as a researcher, a fact which can be readily seen with all of Hubert Berens' PhD students. Berens never failed to follow our further paths, engaging whenever he felt able to contribute. Thus, he joined in when Frank Filbir and I started out to better understand the harmonic analysis background of the dual pair (h_d, ω_d) of kernels related to the Fourier transform of ℓ_1-radial functions. Although this project never made it up to the level of a publication, our discussions with Hubert Berens and Rupert Lasser at the Institute of Biomathematics and Biometry in Munich laid the foundation for our further joint work on positive definite functions.

With all his mathematical work, quality went before quantity – a third lesson, that I have been taught as a graduate student of Hubert Berens. Clearly, publishing constitutes an integral part of scientific achievement, but originality and mathematical substance never have to take a back seat. Thus, *a posteriori* appraisals of his own work such as "*this paper should never have been appeared as we did it much better later*" occurred as a matter of course.

The same holds true for the mathematical style of writing. Every person develops his/her own characteristics of personal writing, although being formed through advice and constructive criticism. Hubert Berens never returned a manuscript without it being carefully worked through and covered with a wealth of suggestions and recommendations.

Experiencing the loss of a *Doktorvater* goes beyond the loss of a well respected colleague. On the other side, the scientific heritage leaves its permanent traces and shows an impact within the line of academic genealogy. Encountering Hubert Berens together with his advisor Paul Butzer at a workshop in Ballenstedt made me feel this line becoming manifest in people. And some years later, a similar gathering took place in Munich with Georg Berschneider, now with roles shifted by one academic generation. In summary, the feeling of loss is mixed with feelings of gratefulness with the latter dominating over time; gratefulness for having had the chance to lay the foundations of my academic life under the supervision of my *Doktorvater* Hubert Berens.

Wolfgang Dahmen[5]

When I studied in P. L. Butzer's group at RWTH Aachen, Hubert had already left and I first knew of him only through his excellent book with P. L. Butzer on semigroups. Among the few times I met him in person the first one has left me with the strongest impression. In fact, he had invited me to give a talk in Erlangen which was my first colloquium talk ever. During these few days I got to know him as a very supportive and thoughtful person with idealistic views and a tendency to sensitively react to any perceived injustice. At that time I wasn't quite sure in which direction to move on but I am sure this visit with Hubert played an important role in this regard.

Ron DeVore[6]

I first met Hubert Berens at one of the Oberwolfach conference organized by Paul Butzer. These were signature events in approximation theory bringing together not only the first rank of approximation theorists but also the leading figures in nearby fields such as harmonic analysis, numerics, semigroup theory, and PDEs. This was the time of the emergence of approximation theory from a rather self contained discipline to a subject vital to understanding many related application domains – a position it holds today. I was most familiar with Hubert through his book with Butzer on semigroups of operators. This classic book brought forward many concepts, such as interpolation of operators, saturation, and such, that remain at the core of understanding approximation processes. It helped to shape a new generation of approximation theorists. The Butzer conferences were not only exciting for the emerging mathematics but also stimulating social events with music and wine parties added to the traditional Oberwolfach setting. Where else could you hear Dolly Schoenberg or Arnold Schönhage on the piano, or the Jack Benny of mathematics, Dick Varga, on the violin? At this time, Hubert was in the midst of his brief excursion to the USA (Santa Barbara and UT Austin). My first impression of him was of a shy, serious, and thoughtful man.

I was to take a Sabbatical in 1974–75 and asked Hubert if he would be so kind as to host me as a Humboldt fellow. I did not realize at the time that this took considerable work on his part to write the supporting documents, arrange housing, offices, etc. During that sabbatical year, we became the closest of friends. We wrote a series of papers, dealing with approximation by positive operators, a hot topic at that time. We worked together well and I was exposed to his functional analytic way of thinking which has benefited me to this day. A stimulating week

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of mathematical discussions usually culminated in a weekend family excursion to the Bavarian countryside. Hubert had a grand plan to explore the nearby circle of towns such as Neuschwanstein, Bamberg, Nürnberg, Würzburg, before gradually expanding out. So I learned not only mathematics but German culture, architecture, and art, and of course the German beers, wurst, white wines. Unfortunately, I probably took for granted all the effort he put into being the perfect host. We had many a Sunday at the Berens' house for Kaffee und Kuchen prepared by Uschi and shared by our families. By the way, Uschi made a special cake (a version of Frankfurter Kranz) I have never seen replicated. A classic Berens story occurred at one of these social events, the annual evening Gala in Erlangen where all the dignitaries met for a wine and dine in the city park. After you learned German, you began to detect there is something slightly different when Hubert speaks his mother tongue. Any way, somehow, Hubert got us an invitation to the event and with newly bought fancy threads we entered the park to flashbulbs and a crowd of onlookers. At the dinner one of the ladies at the table began praising Hubert on his German and asked where he was born and how he had learned to speak German so well. He said "Ich bin Deutscher!"

Hubert and I remained close friends in the succeeding years but with less frequent meetings as our mathematical interests diverged. He always had a serious, socially conscious component which seemed to grow on him as he witnessed the injustices in the mathematical community. Many of us enter mathematics expecting a non-subjective system. When we witness reality, we need to find a way to digest this new truth. With Hubert, this was a difficult experience. But, I think this was an essential part of this complex but wonderful man.

Frank Filbir[7]

I was shocked and sad when Tomas Sauer informed me that Hubert Berens passed away in February 2015. I had spoken with Tomas just at the beginning of that year, and asked him to give my regards to Berens the next time when Tomas would see him.

I already knew Berens by name from his various substantial contributions in approximation theory. I do not remember exactly when I met Hubert Berens personally for the first time. It was probably at one of the meetings called 'Süddeutsches Approximationskolloquium' approximately 20 years ago. Hubert Berens was one of the founders of this colloquium series, and it was a tradition to meet every winter semester at the University of Erlangen-Nuremberg.

I had the chance to come into closer contact with Hubert Berens around 2000. This was initiated by Wolfgang zu Castell who had just finished his Ph.D. thesis under the supervision of Hubert Berens. Shortly after Wolfgang joined the Institute of Biomathematics and Biometry in Munich, Hubert Berens, Wolfgang, and I started to work on the existence of a product formula for the kernels of the Fourier transform of ℓ_1-radial functions. Our work was stimulated by a superb paper of Hubert Berens and Yuan Xu on ℓ_1 summability of multiple Fourier integrals and positivity [59]. We had several workshops on the subject at the institutes in Munich and in Erlangen. It turned out that the problem is hard and, unfortunately, we did not succeed with the project. Nevertheless, I learned a lot during that time and this alone made these meetings so valuable.

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We went to several conferences together since then. I remember especially our joint participation in one of the Approximation Conferences at Maratea in Italy. We had several discussions about mathematics, and Hubert Berens shared his view on our research field with me. The last time I met Hubert Berens personally was in March 2009 at the conference on Approximation Theory and Signal Analysis in Lindau, Lake Constance. Some years later I met Paul Butzer at a visit to Aachen and I was sad to hear about the serious health problem of Hubert Berens.

Hubert Berens was not only a excellent mathematician but also a polite, friendly and modest gentleman. I will never forget his pronounced way of talking. He always opposed strongly any kind of celebration in his honor. This might be the reason why there were no birthday conferences organized in honor of him, although he certainly well-deserved such honors. Hubert Berens was a brilliant character, and all of his friends and colleagues will terribly miss him.

Kurt Jetter[8]

It was in 1976, during the second Austin conference on Approximation Theory, when I met Hubert Berens for the first time. One evening, in George and Tanny Lorentz' house, as a young postdoc I was impressed by being among this fine group of famous and already well–established scientists whom I had known so far only through their scientific work. At that time, for me, Berens' name was connected with work on Bernstein polynomials and on Korovkin type theorems which I had to study as a student and as a young scientist, at the University of Tübingen under the instruction of Karl Zeller and Franz Locher. And now, through the help of George's arrangement I got to know Hubert Berens personally, in his fine and reserved manner, in his open and friendly nature, and in his straight and critical, yet supporting, attitude in scientific discussions. During all the years to follow, I have always considered him as a senior advisor, although we met mainly on conferences.

One of these rare encounters was at the Oberwolfach Conference on Approximation, which he co-organized with Ronald DeVore, in 1981. There I got the impression that Hubert Berens showed empathy with other people and really liked sharing his enthusiasm for mathematics. I also received support and, in particular, advice from him. This he offered frequently and voluntarily to young scientists. When I told him during the conference that I was preparing a course on Nonlinear Approximation which I intended to give at my home institution, at that time the University of Duisburg, he immediately provided his notes on the topic. This was quite helpful as a start, where I could build on in the sequel by adding my personal views and experiences.

Our contacts continued this way until 1996, when I accepted my position at the University of Hohenheim. Living in the south of Germany since then, I had the chance to regularly attend the meetings of the approximation theorists from the universities in South Germany which were biannually organized by Hubert Berens and his colleagues from Erlangen and, later, as well from other colleagues at various universities nearby. In this way, all of us were maintaining a good tradition which had been initiated by Günther Meinardus when he was a professor at Erlangen. Our encounters became even more frequent after Elena Berdysheva moved from Erlangen to Stuttgart in order to join our Department of Applied Mathematics and Statistics at the University of Hohenheim. The continuation of her joint research

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with Hubert Berens gave us several nice opportunities to meet, and to deepen our conversation.

Hubert Berens and I never collaborated directly on a mathematical project, nor did we coauthor a paper. However, I am aware of having benefited from his advice and his support in many ways. Of course, while working on Bernstein-Durrmeyer operators on simplices, with my coauthors Elena Berdysheva and Joachim Stöckler, the papers by Hubert Berens and his coauthors on the subject were most inspiring and indispensable. He generously offered helpful comments and suggestions originating from his profound and substantial mastery of the subject. So, I feel and I do believe that Hubert Berens has helped me a lot, in his ways and in his style, during several stages of my academic life.

I am grateful for all this. Thank you, Hubert. May your spirit prevail in all persons and students that you have influenced, as an excellent mathematician and as a fine person.

Hrushikesh Mhaskar[9]

I think I first met Hubert Berens in an international conference honoring Professor Ambikeshwar Sharma, but came closer to him around late 1980s/early 1990s. I am told that he was one of the referees for my Humboldt fellowship, and obviously wrote kind words about me. After I came with my family to Eichstaett, we had many interactions, in which I was privileged to get a first hand experience of his kindness. In particular, he invited me a couple of times for a colloquium in Erlangen. I remember with some amusement his description of Eichstaett as a hill in the negative z direction. Although we never worked together, he freely discuss mathematics with me, and share his life experiences. He was generous to invite us to stay with him in his house, and give us a tour of many churches in Franconia and Bavaria, explaining in detail the differences in architecture and religious symbols. At the same time, he was candid enough to acknowledge that religion was not always spread by peaceful, nonviolent means, and the many saints whose statues are scattered almost everywhere were anything but saints in the true sense of that word. Similarly, he did not mince words in acknowledging that most of the so called great kings in Europe did absolutely no sacrifice for the people whom they were meant to serve, but were considered great only on account of the territory and wealth they could acquire. He probably loved the German language, but was willing to waive the German training required of Humboldt fellows visiting him. Surely, the approximation theory community has lost a great leader and, even more importantly, a very kind and good–natured human being.

Paul Nevai[10]

I am sure I met Hubert several times before 1985 but my first recollection of talking to him goes back to May of 1985 when Hubert and I visited Allan Pinkus in Haifa and Allan arranged for us to stay as roommates in a Technion administered apartment. It was a rather awkward set up since we shared a bathroom and a kitchen but, otherwise, I don't remember what we did and how we got along. Probably Allan kept us busy showing us around the city and the neighborhood such as Acre (Akko).

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After Haifa, we both went to the Toeplitz Conference at Tel-Aviv University where we stayed at a beach-side hotel (in different rooms) and in between the sessions the two of us took long walks all the way to the old city of Jaffa, where, as I recall, Zeev Ditzian grew up. During these walks, Hubert talked constantly about his not so happy experiences with the German academic system. I was young and inexperienced and it took decades for me to grasp the meaning of what Hubert tried to explain to me. After the conference, we parted as friends.

My next encounter with Hubert was while working with Rudy Lorentz on his dad's *Selected Works*. One could even possibly claim that Hubert and I became co-authors since the book is called "... With contributions by Tamás Erdélyi, Paul Nevai, Colin Bennett, and Hubert Berens."

Then, in 2006, came George Lorentz's obituary where Carl de Boor and I had a funny experience with Hubert. After we prepared the obituary with the contributions of some of George's friends, including Hubert, suddenly Hubert recommended that we scrap the project because it was too "'deutschlastig'; it doesn't do justice to Lorentz and his work", and that we replace it by a barebone short CV and by Lorentz's own autobiography. It took heavy duty lobbying by Carl and me to convince Hubert that the obituary should remain as it was originally proposed. First Hubert withdrew his own contribution, then relented, and, in the end, Hubert agreed to keep the obituary as it was. In retrospect, Hubert was right; every contributor had a German name (well, all my grandparents were born with German names) although I don't recall what the reason was for the Deutschlastigkeit. However, Hubert was absolutely correct when criticizing us that the obituary was too one–sided and it didn't highlight George's work in function spaces.

In 2007, when Hubert submitted his paper "Über ein Turánsches Problem für ℓ-1 radiale, positiv definite Funktionen, II" with Elena Berdysheva to JAT, and I wrote him, semi-jokingly, that it appears it was written in a foreign language, he responded "To me, German is an accepted scientific language. And I like to write a paper in my native tongue every now and then." So we decided not to request an English translation. I believe this may have been the only non-English paper published in JAT since 1990, see JAT 16:1-2, 2009, pp. 71–88.

When I congratulated Hubert on his birthday in 2009 and I wrote that he "was born in May 6, 1956, in Suttrop, Germany" by deliberately making him look 20 years younger, he corrected me (1936 and not 1956) and added "My birthday is of hardly any interest, not even to me. Not that I mind being born, but...".

Hubert and I exchanged emails on a reasonably regular basis and sometimes I called him jokingly "Herr Professor Doktor Berens" so when I met Hubert in Eichstaett on July 24, 2009, he started talking to me quickly in German that I barely understood except that it contained the clear statement that he considers us to be on a "du" basis. Considering that even close colleagues call each other "Sie" in Germany, I thought this was the biggest honor I could have ever received from a German Professor.

My most memorable experience with Hubert was on August 20, 2009, when Elena Berdysheva and I visited Hubert who was already in ill health. It was probably the hottest day that summer in Germany and I was fully dehydrated. When we had lunch, I ordered the very first (and very last) time in my life a full one liter dosage of beer that came in the biggest mug I've ever seen. Hubert warned me not to drink it because, in view of the hot weather, I will get even more dehydrated.

He was wrong. It was a very pleasant experience. As it turned out, it was the last time I saw Hubert.

When, in 2010, Hubert's wife had a serious bike accident when she was hit by a car, I wrote him that, in my experience, German drivers are aggressive, impolite, and insist on their rights even if they create dangerous situations. He never reacted to my harsh judgment so I don't know if he agreed or disagreed.

My last communication from Hubert was in August 2011 when he wrote me "I hope I'll be on the road to recovery (from his second operation), but it's not over yet." At the end of the email, with his typical German fastidiousness, he noted "(31-07-11; 10:30h)".

Finally, while preparing these reminiscences, I remembered that I myself used two of Hubert's papers, both from 1974, both related to positive operators, and both with George Lorentz. In one of them, titled "Korovkin theorems for sequences of contractions on L^p–spaces", they claim that "in 1966 Dzjadyk [7] formulated and proved a corresponding result for the Lebesgue spaces $L^p[0,1]$, $1 \le p < \infty$, namely: For equi-continuous sequences ... the set $\{1, x, x^2\}$ is a test-set with respect to convergence in norm", that is, the Korovkin theorem remains true in L^p too. Well, my dear Hubert and George, googling "mathnet.ru sm4239.pdf", one can download and read the Dzjadyk paper [7] you refer to, and, surprise surprise, there is no such concrete statement nor proof. More precisely, in the very last paragraph, Dzjadyk casually mentions that such a result can be proved; the paper itself is about trigonometric approximation.

It's nice to realize that I had (at least) two German friends who were not perfect.

Rest in peace my beloved Freund Herr Professor Doktor Hubert Anton Berens.

Tomas Sauer[11]

Though a surprisingly wide range of mathematics is covered by the PhD theses he supervised, Hubert Berens only had had seven doctoral students (see the list at the end of this section). The award for being a member of such a small and exclusive club was the excellent mentoring we experienced: his door was always open for his students and he spent hours discussing our work with us, critically checking our ideas and approaches and suggesting methods and techniques to attack the problems. He had a rich mathematical knowledge that he always was willing to share as generously as he spent his time with and on his students, if necessary even on the weekend. The byproduct of these many sessions was that we not only were able to accumulate the results for our theses, we learned how to do mathematics.

Hubert Berens was an "old style" mathematician in the most positive meaning of this word: he cared for mathematical quality, independently of the actually fashionable topic and he asked for mathematical substance. A proof that was worth it always had to contain a point where hard work and new ideas were necessary, straightforward arguments or "soft functional analysis" were not enough. Nevertheless he was open to and interested in all fields of mathematics and never stopped learning new things: often we spent our weekly seminar by jointly reading papers on a new topic that he found interesting.

There are two episodes that I particular remember and that nicely characterize Hubert Berens and they accidentally mark the beginning and the end of our direct mathematical collaboration. When I approached him as young student asking for a topic for a diploma thesis, he answered "Bernstein polynomials in several variables

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do not preserve convexity. Make something out of it." Of course, I was not left alone with this challenge, he gave me his unpublished notes with H. J. Schmid and patiently explained to me all that had to be known there. As afterwards the results were to be published, he insisted not to be coauthor, even if had significantly contributed. The second episode was much later when I had started to collaborate with Yuan Xu on multivariate polynomial interpolation. He took me aside and simply told me that he was glad that I went my own ways, that he would continue to support me but that mathematically here he could not help me any more.

Hubert Berens was a wonderful teacher and mentor, responsible, honest, direct, critical and motivating, supportive and generous. He had his own opinion on things, he could be stubborn, but he was never unfair and always approached things with good humor. He was interested in many things besides mathematics and also loved to discuss them, especially during the daily coffee breaks of our group. He also organized regular hiking tours and his shortcuts were legendary.

Being a student of Hubert Berens was a gift and privilege. There are only seven, but what they received in in terms of mentoring, support and humanity would have been sufficient for a much larger number.

Larry Schumaker[12]

Hubert and I crossed paths fairly early in our careers. In 1968 he went to the University of California in Santa Barbara, while George Lorentz and I both joined the approximation group that Ward Cheney was assembling at the University of Texas in Austin. I first met Hubert when he came to spend a year at UT in the fall of 1970. By then Jörg Blatter had also joined the group, and we all got to know each other quite well at many departmental parties (we were all still young back then). Our contact was enhanced by the fact that my wife is from Berlin, and we all had children of more or less the same age.

Hubert went back to Santa Barbara the following year, but in 1972 he returned for a second year in Austin. Although we did not write any papers together, we did share a unique experience – team teaching a graduate course in Approximation Theory. He would teach one week, and then I would do the next week. We also collaborated with the rest of the group on organizing the first of the so-called "Texas Approximation Conferences". It was held in January of 1973 in Austin. After more than forty years, these meetings still take place every three years.

Even after he returned to Germany, we still saw each other fairly often, either at later Texas meetings, at Oberwolfach, or in Erlangen. My wife and I both remember fondly one visit to Erlangen when, on a beautiful warm and sunny afternoon, he took us to a wonderful beer garden near Erlangen where we enjoyed more than a few glasses of the local brew. Hubert was always a perfect gentleman, and a kind and gracious host. He will be missed.

Manfred Sommer[13]

My first contact with Professor Berens was in October 1974 when he started a two-semester lecture for the engineering students at the university of Erlangen. I was responsible for the Übungen (course exercises). After having finished my dissertation in 1975, a long and intensive period of scientific communication with Berens began. He formed a very productive research group in Erlangen, including Klaus

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Bartke, Lothar Hetzelt, Günther Nürnberger, Ullrich Sattes, Gerhard Schmeisser, Hans Joachim Schmid, Hans Strauss, myself and some other guys. Starting in 1975, in a weekly seminar we presented and discussed our newest results. Of course, Hubert Berens gave a series of talks on his research on best nonlinear approximation in Banach and Hilbert spaces. Communication with Berens was always a great pleasure. He was very interested in my results on continuous selections for metric projections and encouraged me to continue my research on splines, generalized splines, weak Chebyshev subspaces and best L_1-approximation. In November 1975, he asked Professor G.G. Lorentz to invite me to the second Texas Conference on Approximation Theory at Austin, 1976. The contacts which I got at this conference were very important for my later research. In 1978, Berens was one of the referees for my Habilitation at Erlangen.

It was also a great deal for our research group in Erlangen that Hubert Berens could get Ron DeVore as a guest professor for one year in 1975/76. Also due to his good contacts, he invited many international experts in Approximation Theory for shorter periods, e.g., George G. Lorentz, Larry L. Schumaker, Charles Micchelli and many others. After their talks in the "Mathematisches Kolloquium" he and his wife were perfect hosts in their home. I enjoyed these evenings very much.

In 1982, I went to the Catholic University of Eichstätt. To continue the close contact to Berens and the group in Erlangen, a workshop on Approximation Theory held twice a year at Erlangen resp. Eichstätt was initiated. Some years later the workshop expanded to some other universities in South Germany, such as Mannheim, Würzburg, TU München, Stuttgart-Hohenheim, Jena and Passau. Berens gave many interesting talks at this workshop, the last one in 2008. For the last time I saw him in March 2010, during a workshop in München.

Hans Strauss[14]

I first met Hubert Berens in the year 1973 when he was appointed Full Professor for Applied Mathematics at the University of Erlangen. He followed Professor Dr. Wilhelm Specht at this Chair. I had obtained my doctoral degree some years ago under the supervision of W. Specht.

When Professor Berens came to Erlangen I was a member at the Institute of Günter Meinardus who also held a Chair for Applied Mathematics but in another department. Meinardus left Erlangen in the year 1974.

Berens was very active in Erlangen. He was an important academic researcher who made outstanding contributions to several topics in approximation theory and wrote several joint papers with famous co-authors. Guests from foreign countries often visited him.

He soon had many students in Erlangen and supported the younger generation of mathematicians very much. He was organizing a workshop on Approximation Theory every week. I always attended this workshop and also gave several talks. In the year 1976 I finished my habilitation and was very much supported by Hubert Berens.

Since 1983 another workshop was organized by him twice a year in Erlangen and in Eichstätt. Some years later it expanded to several universities in South-Germany. These workshops always were a very good opportunity to exchange our newest research results.

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Berens stayed in Erlangen until his retirement in 2001. I had a very good contact to him during this long time although we were not working at the same institute. We often traveled together to international conferences, e.g., to the Texas conferences on approximation theory and many other meetings. We made excursions, visited art exhibitions and had long conversations. He was a very nice colleague.

Berens remained active mathematically after his retirement for many years but he fell ill several years ago.

I will miss him very much.

Yuan Xu[15]

Hubert Berens was my mentor, collaborator and, above all, a dear friend.

I had the good fortune to meet him at the beginning of my career, the second year, 1989, of my postdoc at UT Austin, when Hubert spent a semester visiting George Lorentz. Soon after I gave a talk at our Approximation Theory seminar at the beginning of the term, Hubert came to talk to me and introduced me to the subject he was interested in at the time, Bernstein-Durrmeyer operators. He was open and generous from the beginning, we soon started working together and showed that the B-D operators with respect to the Jacobi weight functions are exactly the de la Vallée Poussin means of the Fourier-Jacobi series. A few months after Hubert returned to Germany, he sent me a draft, beautifully written and composed [47]. I still remembered the wonder and thrill when first seeing it. We then generalized, together with Hans Schmid, our results from $[-1, 1]$ to the simplex in $\mathbb{R}^d$ [50], which was my first encounter with orthogonal polynomials of several variables, a subject not well documented at the time and much of what we needed was in the 1926 book by Appell and de Fériet (Hubert sent me an xerox copy of the book from Germany). The sparsity of information on orthogonal polynomials of several variables provided me an opportunity that shaped my career – I have been working on the topic ever since.

Hubert helped me secure a Humboldt fellowship and I spent the academic year 1993-94 working with him in Erlangen. It was a very productive year for me. Hubert and I wrote two papers together and I worked on my own and with his student Tomas Sauer on several others. Our first paper [58] is about the Fejér means for multivariate Fourier series when the partial sums are taken over multi-indexes in ℓ_1 balls, and our main result shows that the Cesàro (C, δ) means of such partial sums are positive if and only if $\delta \geq 2d - 1$. After we finished [58], Hubert told me that it should be more natural to consider the Riesz means for the partial integrals of the Fourier transform taken over ℓ_1 ball. His hunch was right on, we proved, in [59], not only the analogous result on the positivity but also analogues of Schoenberg's modification of Bochner's theorem on positive definite functions on $\mathbb{R}^d$ and some other results in the ℓ_1 ball setting. There has been further research in this direction, as shown by the work of Wolfgang zu Castell, Elena Berdysheva, some joint with Hubert, and others. Hubert was a mathematician with high standards and a superb taste.

For my stay in Erlangen I could not have asked for a better host. Hubert took good care of me and helped me settle in right from the beginning, from buying a bicycle to getting a BahnCard, nothing was too small. I was a frequent guest in his house: coffee and Ursula's delicious cakes and blueberry pies, backyard under the apple tree, watching the American Super Bowl (a spectacle on German TV

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back then), Christmas day with his entire family. I have many fond memories of our time together: short outings and hiking in Franconia countryside, beer garden, bicycling to a nearby village for new wine. Hubert made sure that I had a good exposure to German culture, took me to quite a few nearby cities and towns (Bamberg, Bayreuth, Nürnberg, Regensburg, Würzburg ...), where we visited churches (Gothic style was his favorite) and museums (Dürer, Riemenschneider, ...), and to the Kloster Weltenburg (the oldest operating monastery brewery for the dark beer). At the end of my stay in Erlangen, he took me on a three day trip and we drove through the wine region all the way to Trier and back.

During my year in Erlangen, I got to know Hubert well. He was gentle, kind and compassionate. He was a German through and through, in the best possible way, but he also liked the American life and was a long–term subscriber of *The Times* magazine (his English was impeccable). He loved American casualness, insisted, from the very beginning, that we should address each other by the first name. I have a fond memory of my first coffee break with his group in Erlangen. While we talked, I could sense an uneasiness in the air but didn't know its source. I was told later that it was shocking to hear that he was addressed by his first name instead of the customary, in Germany, Herr Professor Berens. When asked if I should do as Germans do, Hubert wanted none of it. He always made me feel comfortable and never pressured me to do anything, although he probably wanted to, and should, make me learn more German. In his autobiography, *I want to be a Mathematician*, Paul Halmos mentioned, on page 387, that Hubert gave his talk in Oberwolfach in German in 1980 since "he was not willing to relinquish 100 percent linguistic control." Two of Hubert's last papers [67, 68], joint with Berdysheva, were in German.

Hubert had a strong sense of what is right, or appropriate, in our profession and in the society at large. He was passionate, forthright and totally honest, sometimes too much so. He did not always get along with his colleagues but he was always consistent, genuine and truthful. Hubert provided strong support for colleagues in need, especially for younger mathematicians. He was cordial, gracious, completely unpretentious, a great friend for those who were fortunate to know him.

After my stay in Erlangen, Hubert became my trusted adviser and friend. We did not merely keep in touch but wrote to each other frequently for years. A search of my computer reminds me how many messages we wrote to each other – literarily hundreds. We wrote about mathematics, papers we found interesting, new topics and questions, conferences, but also on many other things, ups and downs in work, good books, politics, messages between friends. We exchanged season's greetings on Christmas and New Years, and Hubert sent me greetings on Chinese New Years. We arranged to meet whenever possible. Hubert visited me in Eugene when I first moved to Oregon in 1993, stayed with my family for several days in 1998 (he enjoyed our trip into the Cascades, water falls and old growth forest), but didn't want to come to America during George W. Bush's presidency. I made stops in Erlangen in 1997 and 2001, and in 2004 with my family. When I visited GSF in Munich in the summer of 2005, we met in Augsburg, a short train ride for each of us, for sightseeing on a Saturday, and we did the same and met in Regensburg on 2007, and I visited him in Erlangen in 2009.

As time went by, our frequent communications dwindled. We exchanged holiday greetings as usual at the end of 2011, all seemed normal enough given the

circumstances. My first inkling that his health had seriously worsened came when Hubert did not answer my letter in early September, 2012, before my departure to a conference in Munich. When he did answer, a week later, he wrote that he was in hospital for a week and it would be nice if we could see each other and "have a talk about 'Gott und die Welt' ". I called immediately and went to Erlangen. It was shocking to see how thin and fragile he had become, but he was in good spirits and, after an initial awkwardness, we talked and he even showed me the manuscript that he had been working on with Elena. For a moment, it felt like the good old days. That was the last time I saw him. I called him right after Christmas, 2014, over the muffled sound of his grandkids in the background, he said "I'm not doing well, we shall see" but his voice was strong and reassuring as usual. That was the last time I spoke to him.

Even though I knew it was coming, when the news of his departure reached me, I was deeply affected and saddened. Germany suddenly felt that much farther and colder. Hubert, I miss you. Thank you for being there for me for all these years. Tschüss, till we meet again wherever it may be.

List of doctoral Students of Hubert Berens

1. Klaus Bartke (1979)
2. Martina Finzel (1990)
3. Lothar Hetzelt (1981)
4. Gert Renner (1985)
5. Ullrich Sattes (1980)
6. Tomas Sauer (1993)
7. Wolfgang zu Castell (2000)

List of Publications of Hubert Berens

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Communicated by Yuan Xu[16]

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