Orthogonal Polynomials and Summability in Fourier Orthogonal Series on Spheres and on Balls

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(*Received* 13 *October* 1999)

Abstract

For a general family of weight functions, a relation between orthogonal polynomials on the unit sphere in $\mathbb{R}^{d+m}$ and orthogonal polynomials on the unit ball in $\mathbb{R}^d$ is discussed and used to show that the summability of the Fourier orthogonal series on the unit ball follows from the summability on the unit sphere. Convergence of the Cesàro summability is established for a number of weight functions on the sphere and on the ball.

1. *Introduction*

There is a close relation between orthogonal structure on the unit sphere and on the unit ball. Let $B^d = \{\mathbf{x} : |\mathbf{x}| \le 1\}$, where $|\mathbf{x}|$ is the usual Euclidean norm, and let $S^{d-1} = \{\mathbf{x} : |\mathbf{x}| = 1\}$ be the boundary of B^d. It was proved in [**27**] that a basis of homogeneous orthogonal polynomials with respect to a weight function H on S^d can be given in terms of orthogonal polynomials with respect to two weight functions on B^d (see below). The result allows us to describe the space of orthogonal polynomials on S^d without the benefit of differential or differential-difference operators as in the theory of classical spherical harmonics or in the theory of Dunkl's h-harmonics associated with reflection groups. In the present paper, we study the relation between orthogonal polynomials on S^{d+m-1} and those on B^d and we show how to use the relation to study the summability of the Fourier orthogonal expansion on B^d.

Throughout this paper we adopt the following multi-index notation. Let $\mathbb{N}_0$ be the set of nonnegative integers. For $\alpha \in \mathbb{N}^d$ and $\mathbf{x} \in \mathbb{R}^d$, we let $\mathbf{x}^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$, which are the monomials. The number $|\alpha|_1 = \alpha_1 + \cdots + \alpha_d$ is the (total) degree of $\mathbf{x}^\alpha$. We denote by Π^d the set of polynomials in d variables on $\mathbb{R}^d$ and by Π_n^d the subset of polynomials of total degree at most n. We also denote by $\mathcal{P}_n^d$ the space of homogeneous polynomials of degree n on $\mathbb{R}^d$ and we let $r_n^d = \dim \mathcal{P}_n^d$. It is well-known that $\dim \Pi_n^d = \binom{n+d}{n}$ and $r_n^d = \binom{n+d-1}{n}$.

The structure of orthogonal polynomials on the unit ball B^d is well understood. Let W be a nonnegative weight function with finite moments on B^d. The set of polynomials of degree n that are orthogonal to all polynomials of lower degree forms a vector space, which we denote by $\mathcal{V}_n^d(W)$. By the Gram-Schmidt process, $\dim \mathcal{V}_n^d(W) = r_n^d$. We denote an orthonormal basis of $\mathcal{V}_n^d(W)$ by $\{P_\alpha^n\}_{|\alpha|_1=n}$, where the superscript n means that $P_\alpha^n \in \Pi_n^d$ and the subscript α can be ordered according to the lexicographical order. We note that

$\mathcal{V}_n^d(W)$ can have a basis whose elements are not orthogonal amongst each other. A typical example is the classical weight function $(1-|\mathbf{x}|^2)^{(m-1)/2}$ on B^d, for which Hermite and Didon introduced bi-orthogonal polynomial bases (cf. [**1**] and [**9**, Chapt. 13]), while an orthonormal basis can be easily constructed (cf. [**11**, **28**]).

For orthogonal polynomials on the unit sphere S^{d-1}, the most well-known example is the spherical harmonics (cf. [**3**, **15**, **20**, **23**]), which are the restriction of harmonic polynomials to S^{d-1}. Recently, Dunkl ([**5**, **6**, **7**]) has developed an important theory of h-harmonics for weight functions invariant under finite reflection groups (see Section 2). The structure of orthogonal polynomials on the sphere S^{d-1} with respect to a family of general weight functions has been studied in [**27**]. For a weight function H defined on $\mathbb{R}^d$, we denote by $\mathcal{H}_n^d(H)$ the space of homogeneous polynomials that are orthogonal to lower degree polynomials with respect to $H d\omega_d$ on S^{d-1}. The weight functions considered in [**27**] include those that are even in each of their variables. If H is such a weight function, then it is proved that one orthonormal basis of $\mathcal{H}_n^d(H)$ can be derived from the orthonormal polynomials of degree n with respect to $H(\mathbf{x}, \sqrt{1-|\mathbf{x}|^2})/\sqrt{1-|\mathbf{x}|^2}$ and the orthonormal polynomials of degree $n-1$ with respect to $H(\mathbf{x}, \sqrt{1-|\mathbf{x}|^2})\sqrt{1-|\mathbf{x}|^2}$ on the unit ball B^{d-1}.

The fact that the orthogonal polynomials with respect to $(1-|\mathbf{x}|^2)^{(m-2)/2}$ on B^d and the spherical harmonics on S^{d+m-1} are related was observed, based on the explicit formulae of the bi-orthogonal bases, already in the work of Hermite, Didon, Appell and Kampé de Fériet, (cf. [**1**, Part 2] and [**9**, Chapt. 11 and Chapt. 12]). This connection was used in the work of Koschmieder ([**12**, **13**]) to study the convergence of the Fourier orthogonal expansion with respect to $(1-|\mathbf{x}|^2)^{(m-2)/2}$ on B^d. In the Section 2 of the present paper, we show that the relation between orthogonal polynomials on S^{d+m-1} and orthogonal polynomials on B^d holds for a large family of weight functions and that a basis for $\mathcal{H}_n^{d+m}(H)$ can be given in terms of orthogonal polynomials with respect to a family of weight functions on B^d. As a consequence, we will derive relations between reproducing kernels on B^d and on S^{d+m-1}. In Section 3, we show that the summability of the Fourier orthogonal series follows from the summability on the sphere. We then derive a number of results on Cesàro summability for several classes of weight functions on B^d.

2. *Orthogonal polynomials on spheres and on balls*

We denote by $d\omega_d$ the surface measure on the unit sphere S^{d-1}, the surface area of S^{d-1} is $\omega(S^{d-1}) = \int_{S^{d-1}} d\omega_d = 2\pi^{d/2}/\Gamma(d/2)$. The following formula connects integrals on S^{d+m-1} and on B^d (cf. [**3**, p. 216]); it plays an important role in our investigation.

LEMMA 2·1. *Let d and m be positive integers. Then*

$$\int_{S^{d+m-1}} f(\mathbf{y}) d\omega_{d+m} = \int_{B^d} (1-|\mathbf{x}|^2)^{\frac{m-2}{2}} \Big[\int_{S^{m-1}} f(\mathbf{x}, \sqrt{1-|\mathbf{x}|^2}\eta) d\omega_m(\eta) \Big] d\mathbf{x}.$$

We now define our weight functions. A function f defined on a set $\Omega \in \mathbb{R}^d$ is called *centrally symmetric* if $\mathbf{x} \in \Omega$ implies that $-\mathbf{x} \in \Omega$ and $f(\mathbf{x}) = f(-\mathbf{x})$. A function f on $\mathbb{R}^m$ is positive homogeneous of degree σ if $f(t\mathbf{x}) = t^\sigma f(\mathbf{x})$ for $t > 0$.

DEFINITION 2·2. *The weight function H defined on $\mathbb{R}^{d+m}$ is called admissible if*

$$H(\mathbf{x}) = H_1(\mathbf{x}_1) H_2(\mathbf{x}_2), \qquad \mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}^{d+m}, \quad \mathbf{x}_1 \in \mathbb{R}^d, \quad \mathbf{x}_2 \in \mathbb{R}^m,$$

where we assume that H_1 is a centrally symmetric function, H_2 is homogeneous of order 2τ and even in each of its variables; moreover, we assume that $\int_{S^{d+m-1}} H(\mathbf{x}) d\omega_{d+m} = 1$.

Examples of admissible weight functions include $H(\mathbf{x}) = c\prod_{i=1}^{d+m} |x_i|^{\kappa_i}$ for $\kappa_i \geq 0$, in which H_1 and H_2 are of the same form but with fewer variables, and $H(\mathbf{x}) = cH_1(\mathbf{x}_1)\prod |x_i^2 - x_j^2|^{\beta_{i,j}}$, where $d+1 \leq i < j \leq d+m$ and $2\tau = \sum_{i,j} \beta_{ij}$.

For convenience we also assume that H_1 and H_2 are normalized so that

$$\int_{S^{m-1}} H_2(\mathbf{y}) d\omega_m = 1 \quad \text{and} \quad \int_{B^d} H_1(\mathbf{x})(1-|\mathbf{x}|^2)^{\tau+(m-2)/2} d\mathbf{x} = 1. \tag{2·1}$$

It follows from the formula in Lemma 2.1 that these normalizations are consistent with the assumption that H has unit integral over S^{d+m-1}. We note that the normalization of H_1 depends on the homogeneous degree of H_2. We shall denote by W_m^H the weight function

$$W_m^H(\mathbf{x}) = H_1(\mathbf{x})(1-|\mathbf{x}|^2)^{\tau+(m-2)/2}, \qquad \mathbf{x} \in B^d. \tag{2·2}$$

Moreover, when $m = 1$, we write $W^H = W_1^H$. We note that W_m^H has unit integral over B^d. If H, H_1 and H_2 are all equal to constants, we denote the weight function W_m^H by $W_{(m-1)/2}$, which is the case of $\mu = (m-1)/2$ of the following weight function,

$$W_\mu(\mathbf{x}) = w_\mu(1-|\mathbf{x}|^2)^{\mu-1/2}, \qquad w_\mu = \frac{\Gamma(\mu + (d+1)/2)}{\pi^{d/2}\Gamma(\mu+1/2)}, \tag{2·3}$$

normalized to have unit integral over B^d. The weight function W_μ on B^d is the classical weight function; most work about orthogonal polynomials on the ball in the literature deals with $W_{(m-1)/2}$ (cf. [**1**] and [**9**, Chapt. 12]).

For a function f defined on B^d, we can extend it to a function g defined on S^{d+m-1}; that is, we define $g(\mathbf{y}) = f(\mathbf{y}_1)$, where $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2) \in S^{d+m-1}$, $\mathbf{y}_1 \in B^d$ and $\mathbf{y}_2 \in B^d$. The formula in Lemma 2.1 then shows that, using the normalization of H and W_m^H,

$$\int_{S^{d+m-1}} g(\mathbf{y}) H(\mathbf{y}) d\omega_{d+m}(\mathbf{y}) = \int_{B^d} f(\mathbf{y}_1) W_m^H(\mathbf{y}_1) d\mathbf{y}_1.$$

In the following we denote by $\{P_\alpha^n(W_m^H)\}$ a sequence of orthonormal polynomials with respect to W_m^H on B^d, where the superscript n means that $P_\alpha^n(W_m^H)$ are of degree n as in Section 1. Since H_1, thus W_m^H, is centrally symmetric, it follows that $P_\alpha^n(W_m^H)$ is a sum of monomials of even degree if n is even and a sum of monomials of odd degree if n is odd. For a polynomial satisfying such a property, we say that it has the same parity as n.

Our first result shows that $P_\alpha^n(W_m^H)$ is in fact a homogeneous orthogonal polynomial with respect to H on S^{d+m-1}.

THEOREM 2·3. *Let H and P_α^n be defined as above. Then the functions*

$$Y_\alpha^n(\mathbf{y}) = |\mathbf{y}|^n P_\alpha^n(W_m^H; \mathbf{y}_1'), \qquad \mathbf{y} = r(\mathbf{y}_1', \mathbf{y}_2') \in \mathbb{R}^{d+m}, \quad r = |\mathbf{y}|, \quad \mathbf{y}_1' \in B^d$$

are homogeneous polynomials in $\mathbf{y}$ and they are orthonormal with respect to $H(\mathbf{y})d\omega_{d+m}$ on S^{d+m-1}.

Proof. We first prove that Y_α^n is orthogonal to polynomials of lower degree. It is sufficient to prove that Y_α^n is orthogonal to $g_\gamma(\mathbf{y}) = \mathbf{y}^\gamma$ for $\gamma \in \mathbb{N}^d$ and $|\gamma|_1 \leq n-1$. We have from Lemma 2.1 that

$$\int_{S^{d+m-1}} Y_\alpha^n(\mathbf{y}) g_\gamma(\mathbf{y}) H(\mathbf{y}) d\omega_{d+m}$$
$$= \int_{B^d} P_\alpha^n(W_m^H; \mathbf{y}_1') \left[\int_{S^{m-1}} g_\gamma(\mathbf{y}_1', \sqrt{1-|\mathbf{y}_1'|^2}\,\eta) H_2(\eta) d\omega_d(\eta) \right] W_m^H(\mathbf{y}_1') d\mathbf{y}_1'.$$

If g_γ is odd with respect to at least one of its variables $y_{d+1}, ..., y_{d+m}$, then we can conclude, using the fact that H_2 is even in each of its variables, that the integral inside the square bracket is zero. Hence, Y_α^n is orthogonal to g_γ in this case. If g_γ is even in every variable of $y_{d+1}, ..., y_{d+m}$, then the function inside the square bracket will be a polynomial in $\mathbf{y}_1'$ of degree at most $n-1$, from which we conclude that Y_α^n is orthogonal to g_γ by the orthogonality of P_α^n with respect to W_m^H on B^d. Moreover, we also have

$$\int_{S^{d+m-1}} Y_\alpha^n(\mathbf{y}) Y_\beta^n(\mathbf{y}) H(\mathbf{y}) d\omega_{d+m} = \int_{B^d} P_\alpha^n(W_m^H; \mathbf{y}_1') P_\beta^n(W_m^H; \mathbf{y}_1') W_m^H(\mathbf{y}_1') d\mathbf{y}_1' = \delta_{\alpha,\beta},$$

which shows that $\{Y_\alpha^n\}$ is an orthonormal set.

Finally, since P_α^n has the same parity as n, we can write $P_\alpha^n(W_m^H; \mathbf{y}_1')$ as a linear combination of monomials of the form ${\mathbf{y}'}_1^\beta$, $|\beta| = n - 2k$, $0 \le k \le n$. Hence, upon using the fact that $y_i = |\mathbf{y}| y_i'$ for $1 \le i \le d$, it follows that Y_α^n is a sum of the terms $c_\beta |\mathbf{y}|^{2k} y_1^{\beta_1} \cdots y_d^{\beta_d}$, $|\beta|_1 = n - 2k$. Hence, the polynomial $Y_\alpha^n(\mathbf{y})$ is homogeneous of degree n in $\mathbf{y}$. □

In the case that H, H_1 and H_2 are all constants, the theorem states that the polynomials $P_\alpha^n(W_{(m-1)/2})$ of d variables correspond to ordinary spherical harmonics of $d+m$ variables. This fact was known for the bi-orthogonal polynomials defined by Hermite and Didon for the space $\mathcal{V}_n^d(W_{(m-1)/2})$ (cf. [**9**, Chapt. 12]). In the case $m = 1$, the above result has been proved in [**27**, p. 784]. In that case, the integral over S^{m-1} degenerates to point evaluations at 1 and -1 with weights 1/2 for each point; we no longer need to split H as a product of H_1 and H_2. In fact, the result in [**27**] is established for the weight function $H(\mathbf{y})$ which is centrally symmetric with respect to the first d variables and is even in its last variable (called S-symmetric there). In that case, the weight function W^H in (2.2) becomes $W^H(\mathbf{x}) = H(\mathbf{x}, \sqrt{1-|\mathbf{x}|^2})/\sqrt{1-|\mathbf{x}|^2}$.

The set $\{Y_\alpha^n\}$ in Theorem 2.3 forms a basis for a subspace of $\mathcal{H}_n^{d+m}(H)$. In the case $m = 1$, we have shown in [**27**, Theorem 3.3] that the orthogonal polynomials of degree n with respect to $H(\mathbf{x}, \sqrt{1-|\mathbf{x}|^2})\sqrt{1-|\mathbf{x}|^2}$ provide the basis for the orthogonal complement of $\{Y_\alpha^n\}$ in $\mathcal{H}^{d+1}(H)$. We shall extend the result in the following. First we need

LEMMA 2·4. *Let H_2 satisfy the condition in Definition 2.2. Then there is an orthonormal basis $\{S_\beta^n\}$ of $\mathcal{H}_n^m(H_2)$ such that each S_β^n is homogeneous, and S_β^n is even in each of its variables if n is even and odd in each of its variables if n is odd.*

This lemma follows from the fact that a basis of $\mathcal{H}_n^m(H_2)$ is given in terms of the orthogonal polynomials with respect to the weight functions $H_2(\mathbf{x}, \sqrt{1-|\mathbf{x}|^2})/\sqrt{1-|\mathbf{x}|^2}$ and $H_2(\mathbf{x}, \sqrt{1-|\mathbf{x}|^2})\sqrt{1-|\mathbf{x}|^2}$, where $\mathbf{x} \in B^{m-1}$, and these polynomials are even in each of their variables if n is even and odd in each of their variables if n is odd, owing to the fact that both weight functions are even in their variables. Let us recall that $\mathcal{V}_n^d(W)$ is the space of orthogonal polynomials of degree exactly n with respect to W on B^d.

THEOREM 2·5. *Let $\{S_\beta^k\}$ be an orthonormal basis of $\mathcal{H}_k^m(H_2)$ as in Lemma 2.4, and* *let* $\{P_\alpha^{n-k}(W_{m+2k}^H)\}$ *be an orthonormal basis for* $\mathcal{V}_{n-k}^d(W_{m+2k}^H)$ *with respect to the weight function* W_{m+2k}^H *on* B^d*, where* $0 \le k \le n$*. Then the polynomials*

$$Y_{\alpha,\beta,k}^n(\mathbf{y}) = r^n A_{m,k} P_\alpha^{n-k}(W_{m+2k}^H; \mathbf{y}_1') S_\beta^k(\mathbf{y}_2'), \quad \mathbf{y} = r(\mathbf{y}_1', \mathbf{y}_2'), \ \ r = |\mathbf{y}|,$$

where $\mathbf{y}_1' \in B^d$*,* $\mathbf{y}_2' \in B^m$ *and* $[A_{m,k}]^{-2} = \int_{B^d} W_m^H(\mathbf{x})(1-|\mathbf{x}|^2)^k d\mathbf{x}$*, are homogeneous of degree* n *in* $\mathbf{y}$ *and they form an orthonormal basis for* $\mathcal{H}_n^{d+m}(H)$*.*

Proof. Since the last property of S_β^k in Lemma 2.4 implies that it has the same parity as n, and $P_\alpha^{n-k}(W_{m+2k}^H)$ has the same parity as $n-k$, the fact that $Y_{\alpha,\beta,k}^n$ is homogeneous of degree n in $\mathbf{y}$ follows as in the proof of Theorem 2.3.

We prove that $Y_{\alpha,\beta,k}^n$ is orthogonal to all polynomials of lower degree. Again, it is sufficient to show that $Y_{\alpha,\beta,k}^n$ is orthogonal to $g_\gamma(\mathbf{y}) = \mathbf{y}^\gamma$, $|\gamma|_1 \le n-1$. Using the notation $\mathbf{y} = r(\mathbf{y}_1', \mathbf{y}_2')$, we write g_γ as

$$g_\gamma(\mathbf{y}) = r^{|\gamma|_1} \mathbf{y}_1'^{\gamma_1} \mathbf{y}_2'^{\gamma_2}, \qquad |\gamma_1|_1 + |\gamma_2|_1 = |\gamma|_1 \le n-1.$$

Using the fact $|\mathbf{y}_2'|^2 = 1 - |\mathbf{y}_1'|^2$ and the basic formula in Lemma 2.1 we conclude that

$$\begin{aligned}
\int_{S^{d+m-1}} & Y_{\alpha,\beta,k}^n(\mathbf{y}) g_\gamma(\mathbf{y}) H(\mathbf{y}) d\omega_{d+m} = A_{m,k} \int_{B^d} P_\alpha^{n-k}(W_{m+2k}^H; \mathbf{y}_1') \\
& \times \left[\int_{S^{m-1}} S_\beta^k(\eta) g_\gamma(\mathbf{y}_1', \sqrt{1-|\mathbf{y}_1'|^2}\eta) H_2(\eta) d\omega_m(\eta) \right] (1-|\mathbf{y}_1'|^2)^{k/2} W_m^H(\mathbf{y}_1') d\mathbf{y}_1' \\
= & [A_{m,k}]^{-1} \int_{B^d} P_\alpha^{n-k}(W_{m+2k}^H; \mathbf{y}_1') \mathbf{y}_1'^{\gamma_1} (1-|\mathbf{y}_1'|^2)^{(|\gamma_2|_1 - k)/2} W_{m+2k}^H(\mathbf{y}_1') d\mathbf{y}_1' \\
& \qquad \times \left[\int_{S^{m-1}} S_\beta^k(\eta) \eta^{\gamma_2} H_2(\eta) d\omega_m(\eta) \right],
\end{aligned}$$

where we have used the fact that $W_{m+2k}^H(\mathbf{x}) = [A_{m,k}]^2 W_m^H(\mathbf{x})(1-|\mathbf{x}|^2)^k$. We show this integral is zero by considering the following cases: If $|\gamma_2|_1 < k$, then the integral in the square bracket is zero because of the orthogonality of S_β^k. If $|\gamma_2|_1 \ge k$ and $|\gamma_2|_1 - k$ is an odd integer, then $|\gamma_2|_1$ and k have different parity. Hence, since S_β^k is homogeneous, a change of variable $\eta \mapsto -\eta$ leads to the conclusion that the integral in the square bracket is zero. If $|\gamma_2|_1 \ge k$ and $|\gamma_2|_1 - k$ is an even integer, then $\mathbf{y}_1'^{\gamma_1}(1-|\mathbf{y}_1'|^2)^{(|\gamma_2|_1 - k)/2}$ is a polynomial of degree $|\gamma_1|_1 + |\gamma_2|_1 - k \le n-1-k$; hence, the integral is zero by the orthogonality of $Q_\alpha^{n-k}(W_{m+2k}^H)$.

The same consideration also shows that the polynomial $Y_{\alpha,\beta,k}^n$ is normalized, that is,

$$\int_{S^{m+d-1}} Y_{\alpha,\beta,k}^n(\mathbf{y}) Y_{\alpha',\beta',k'}^n(\mathbf{y}) H(\mathbf{y}) d\omega_{d+m} = \delta_{\alpha,\alpha'} \delta_{\beta,\beta'} \delta_{k,k'},$$

using the normalization of $P_\alpha^{n-k}(W_{m+2k}^H)$ and S_β^n.

Finally, we show that $\{Y_{\alpha,\beta,k}^n\}$ forms a basis of $\mathcal{H}_n^{m+d}(H)$. By orthogonality, the elements of $\{Y_{\alpha,\beta,k}^n\}$ are linearly independent; since $\dim \mathcal{H}_k^m(H) = r_k^{m-1} + r_{k-1}^{m-1}$, their number is equal to

$$\sum_{k=0}^n r_{n-k}^d (r_k^{m-1} + r_{k-1}^{m-1}) = \sum_{k=0}^n r_{n-k}^d r_k^{m-1} + \sum_{k=0}^{n-1} r_{n-1-k}^d r_k^{m-1},$$

which is the same as the dimension of $\mathcal{H}_n^{m+d-1}(H)$, as seen by using the combinatorial identity

$$\sum_{k=0}^{n} r_{n-k}^{d} r_{k}^{m-1} = \sum_{k=0}^{n} \binom{n-k+d-1}{n-k}\binom{k+m-2}{k} = \binom{n+d+m-2}{n} = r_{n}^{d+m-1}$$

(cf. [**16**, p. 618, Formula 36]). This completes the proof. □

For $m = 1$, the integral on S^{m-1} becomes point evaluations and H_2 is constant; the space $\mathcal{H}_k^1(H_2)$ reduces to the space of linear polynomials for any k and a basis $\{S_\beta^k\}$ can be taken as $\{1, y_{d+1}\}$. In this case, the theorem reduces to the result proved in [**27**, Theorem 3.3].

As an application of the above results, we can derive a relation between reproducing kernels on the sphere and on the ball. For orthonormal polynomials on the ball, we denote the reproducing kernel of the space $\mathcal{V}_n^d(W)$ with respect to a weight function W as $\mathbf{P}_n(W;\cdot,\cdot)$, which is defined by the property that

$$\int_{B^d} P_n(W;\mathbf{x},\mathbf{y}) f(\mathbf{y}) W(\mathbf{y}) d\mathbf{y} = f(\mathbf{x}), \qquad \mathbf{x} \in B^d, \tag{2·4}$$

for any $f \in \mathcal{V}_n^d(W)$. In terms of an orthonormal basis $\{P_\alpha^n(W)\}$ of $\mathcal{V}_n^d(W)$, the kernel can be written as

$$\mathbf{P}_n(W;\mathbf{x},\mathbf{y}) = \sum_{|\alpha|_1=n} P_\alpha^n(W;\mathbf{x}) P_\alpha^n(W;\mathbf{y}), \qquad \mathbf{x},\mathbf{y} \in B^d. \tag{2·5}$$

For orthogonal polynomials on the sphere, we denote the reproducing kernel of the space $\mathcal{H}_n^{m+d}(H)$ by $P_n(H;\cdot,\cdot)$. It is defined by a reproducing property similar to (2.4), and likewise it can be written in terms of an orthonormal basis of $\mathcal{H}_n^{m+d}(H)$. In particular, in terms of the orthonormal basis in Theorem 2.5, we can write

$$P_n(H;\mathbf{x},\mathbf{y}) = \sum_{k=0}^{n} \sum_{\alpha} \sum_{\beta} Y_{\alpha,\beta,k}(\mathbf{x}) Y_{\alpha,\beta,k}(\mathbf{y}), \qquad \mathbf{x},\mathbf{y} \in \mathbb{R}^{d+m}. \tag{2·6}$$

For ordinary spherical harmonics, that is, $H(\mathbf{x}) = 1/\omega(S^{d+m-1})$, the reproducing kernel P_n is the so-called zonal polynomial.

In the following, for $\mathbf{y} \in S^{d+m-1}$ we write $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2)$. One consequence of Theorem 2.5 is the following result on the relation between the reproducing kernels on B^d and on S^{d+m-1}.

THEOREM 2·6. *Let H be an admissible weight function defined on $\mathbb{R}^{n+m}$ and W_m^H be the associated weight function on B^d defined in (2.2). Then*

$$\mathbf{P}_n(W_m^H;\mathbf{x}_1,\mathbf{y}_1) = \int_{S^{m-1}} P_n(H;\mathbf{x},(\mathbf{y}_1,\sqrt{1-|\mathbf{y}_1|^2}\eta)) H_2(\eta) d\omega_m(\eta)$$

where $\mathbf{x} = (\mathbf{x}_1,\mathbf{x}_2) \in S^{d+m-1}$ with $\mathbf{x}_1 \in B^d$, $\mathbf{y} = (\mathbf{y}_1,\mathbf{y}_2) \in S^{d+m-1}$ with $\mathbf{y}_1 \in B^d$, $\mathbf{y}_2 = |\mathbf{y}|_2\eta \in B^m$, and $\eta \in S^{m-1}$. In particular, when $m = 1$, we have

$$\mathbf{P}_n(W_1^H;\mathbf{x}_1,\mathbf{y}_1) = \left[P_n(H;\mathbf{x},(\mathbf{y}_1,\sqrt{1-|\mathbf{y}_1|^2})) + P_n(H;\mathbf{x},(\mathbf{y}_1,-\sqrt{1-|\mathbf{y}_1|^2}))\right]/2.$$

Proof. We use the formula (2.6) of $P_n(H)$. When we integrate $P_n(H;\mathbf{x},\mathbf{y})$ with respect to $H_2(\mathbf{y}_2)$ over S^{m-1}, we write $|\mathbf{y}_2|^2 = 1 - |\mathbf{y}_1|^2$ and use the fact that S_β^k is homogeneous and orthogonal with respect to $H_2(\mathbf{y}_2)d\omega_m$ to conclude that the integral of $Y_{\alpha,\beta,k}^n$ on S^{m-1} equals to zero for all $k \neq 0$, while for $k = 0$ we have $\beta = 0$ and $Y_{\alpha,0,0}^n(\mathbf{y}) =$

$A_{m,0}P^n_\alpha(W^H_m;\mathbf{y}_1)$. Since $A_{m,0}=1$ by its definition, the desired result follows from the formula of $\mathbf{P}_n(W^H_m)$ in (2.5). When $m=1$, the integral over S^{m-1} degenerates to the point evaluation at $\eta=1$ and -1, we then use $\mathbf{y}_2=y_{d+1}=\sqrt{1-|\mathbf{y}_1|^2}$. □

For one important family of weight functions, those invariant under the reflection groups studied by Dunkl recently, it is possible to give a compact formula for the kernel $P_n(H;\cdot,\cdot)$. First let us recall the basics of Dunkl's theory of h-harmonics (see [**5**, **6**, **7**] and the references therein). Let G be a finite reflection group on $\mathbb{R}^d$ with positive roots $R_+=\{\mathbf{v}_1\ldots,\mathbf{v}_m\}$; assume that $|\mathbf{v}_i|=|\mathbf{v}_j|$ whenever σ_i is conjugate to σ_j in G, where $\sigma_i=\sigma_{\mathbf{v}_i}$, $1\le i\le m$, are reflections with respect to $\mathbf{v}_i$. For a nonzero vector $\mathbf{v}\in\mathbb{R}^d$, the reflection $\sigma_\mathbf{v}$ is defined by $\mathbf{x}\sigma_\mathbf{v}:=\mathbf{x}-2(\langle\mathbf{x},\mathbf{v}\rangle/|\mathbf{v}|^2)\mathbf{v}$, $\mathbf{x}\in\mathbb{R}^d$. Let κ be a nonnegative multiplicity function defined on R_+ such that $\kappa_i=\kappa_j$ whenever σ_i is conjugate to σ_j in G. The key ingredient of the theory is a family of commuting operators, $\mathcal{D}_i$ (Dunkl's operators), associated to G and κ, which are defined by

$$\mathcal{D}_i f(\mathbf{x}):=\partial_i f(\mathbf{x})+\sum_{j=1}^m\kappa_j\frac{f(\mathbf{x})-f(\mathbf{x}\sigma_j)}{\langle\mathbf{x},\mathbf{v}_j\rangle}\langle\mathbf{v}_j,\mathbf{e}_i\rangle,\qquad 1\le i\le d,$$

where ∂_i is ordinary partial derivative with respect to x_i and $\mathbf{e}_1,\ldots,\mathbf{e}_d$ are the standard unit vectors of $\mathbb{R}^d$. The h-harmonics are homogeneous polynomials satisfying the equation $\Delta_h p=0$, where $\Delta_h=\mathcal{D}_1^2+\ldots+\mathcal{D}_d^2$. The h-harmonic polynomials are orthogonal with respect to the weight function $h_\kappa^2(\mathbf{y})d\omega_d$ on S^{d-1}, where h_κ is a reflection invariant function defined by

$$h_\kappa(\mathbf{y})=c\prod_{i=1}^m|\langle\mathbf{y},\mathbf{v}_i\rangle|^{\kappa_i},\qquad \kappa_i\ge 0,\quad \mathbf{y}\in\mathbb{R}^d, \tag{2·7}$$

normalized to have unit integral on S^{d-1}. When $\kappa=0$, the h-harmonics are just ordinary harmonics. The intertwining operator V between the algebra of differential operators and the commuting algebra of Dunkl's operators helps us to draw a parallel between the theory of h-harmonics and that of ordinary harmonics. The intertwining operator V is the unique linear operator defined by

$$V\mathcal{P}_n\subset\mathcal{P}_n,\qquad V1=1,\qquad \mathcal{D}_iV=V\partial_i,\qquad 1\le i\le d.$$

For h-harmonics, the reproducing kernel $P_n(h_\kappa^2)$ enjoys a compact formula ([**7**, **26**])

$$P_n(h_\kappa^2;\mathbf{x},\mathbf{y})=\frac{n+\gamma_\kappa+(m-2)/2}{\gamma_\kappa+(m-2)/2}V[C_n^{(\gamma_\kappa+(m-2)/2)}(\langle\mathbf{x},\cdot\rangle)](\mathbf{y}),\qquad \mathbf{x},\mathbf{y}\in S^{m-1}, \tag{2·8}$$

where $\gamma_\kappa=\sum_{i=1}^m\kappa_i$ and $C_n^{(\lambda)}$ is the Gegenbauer polynomial of degree n (see [**22**, p. 80], where the notation $P_n^{(\lambda)}$ is used). If $\kappa=0$, h-harmonics become ordinary harmonics, we have $V=id$ and (2.8) is just the compact formula for ordinary zonal polynomials (cf. [**15**, p. 19] or [**20**, p. 149]).

We are interested in those weight functions h_κ^2 that are even in their variables, so that they satisfy the condition for H_2 in Definition 2.2. Examples of such weight functions include

$$h_{\kappa,\mu}^2(\mathbf{y})=C_{\kappa,\mu}|y_1|^{2\kappa_1}\cdots|y_d|^{2\kappa_d}|y_{d+1}|^{2\mu},\quad \kappa_i\ge 0,\quad \mu\ge 0,\quad \mathbf{y}\in\mathbb{R}^{d+1}, \tag{2·9}$$

where $C_{\kappa,\mu}^2 = \Gamma(|\kappa|_1 + d/2)/(2\Gamma(\kappa_1+1/2)\cdots\Gamma(\kappa_d+1/2)\Gamma(\mu+1/2))$, which is invariant under the group $\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$, as well as

$$h_\kappa^2(\mathbf{y}) = b_{\kappa_0,\kappa_1} \prod_{i=1}^d |y_i|^{2\kappa_0} \prod_{1\le i<j\le d} |y_i^2 - y_j^2|^{2\kappa_1}, \tag{2·10}$$

which is invariant under the octahedral group. The reason we choose the parameters of $h_{\kappa,\mu}^2$ as they are will become clear in Corollary 2.7 below, so is the reason that $h_{\kappa,\mu}$ is defined on $\mathbb{R}^{d+1}$ instead of $\mathbb{R}^d$. The explicit formula of the intertwining operator V is not known for the weight function in (2.10). For the weight function (2.9), however, such a formula is given by [**7**, **25**],

$$Vf(\mathbf{x}) = \int_{[-1,1]^{d+1}} f(t_1x_1,\ldots,t_{d+1}x_{d+1}) \prod_{i=1}^{d+1} c_{\kappa_i}(1+t_i)(1-t_i^2)^{\kappa_i-1} d\mathbf{t}, \tag{2·11}$$

where $c_\lambda = 1/\int_{-1}^1 (1-t^2)^{\lambda-1}dt$ for $\lambda > 0$ and we let $\kappa_{d+1} = \mu$. This compact formula allows us to state the following identity:

COROLLARY 2·7. *For the weight function $W_{\kappa,\mu}(\mathbf{x}) = C_{\kappa,\mu}\prod_{i=1}^d |x_i|^{\kappa_i}(1-|\mathbf{x}|^2)^{\mu-1/2}$, where $\kappa_i \ge 0$ and $\mu \ge 0$, defined on B^d, there is a compact formula*

$$\mathbf{P}_n(W_{\kappa,\mu};\mathbf{x},\mathbf{y}) = \frac{n+|\kappa|_1+\mu+(d-1)/2}{|\kappa|_1+\mu+(d-1)/2} \int_{-1}^1 \int_{[-1,1]^d} C_n^{(|\kappa|_1+\mu+\frac{d-1}{2})}(t_1x_1y_1+\cdots$$
$$+t_dx_dy_d + s\sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2})c_s \prod_{i=1}^d c_{\kappa_i}(1+t_i)(1-t_i^2)^{\kappa_i-1}d\mathbf{t}(1-s^2)^{\mu-1}ds.$$

Let $m = 1$ and $H(\mathbf{y}) = h_{\kappa,\mu}^2$ in Theorem 2.6; then W_m^H in Theorem 2.6 becomes $W_{\kappa,\mu}$. Hence, the above formula follows from (2.8), (2.11) and Theorem 2.6. Recall that $\mathbf{P}_n$ is a multiple sums by its definition in (2.5), we see that the compact formula in the Corollary 2.7 is rather remarkable.

If some $\kappa_i = 0$, then the above formula holds under the limit relation

$$\lim_{\lambda\to 0} c_\lambda \int_{-1}^1 f(t)(1-t^2)^{\lambda-1}dt = [f(1)+f(-1)]/2. \tag{2·12}$$

In particular, if we define $W_\mu = W_{0,\mu}$, which agrees with the definition at (2.3), then we obtain the compact formula of $\mathbf{P}_n(W_\mu)$,

$$\mathbf{P}_n(W_\mu;\mathbf{x},\mathbf{y}) = c_\mu \frac{n+\mu+(d-1)/2}{\mu+(d-1)/2}$$
$$\times \int_{-1}^1 C_n^{(\mu+\frac{d-1}{2})}(\langle \mathbf{x},\mathbf{y}\rangle + t\sqrt{1-|\mathbf{x}|^2}\sqrt{1-|\mathbf{y}|^2})(1-t^2)^{\mu-1}dt, \tag{2·13}$$

by using (2.12) d times. This formula is derived for the first time in [**28**], in which the formula (2.5) is summed up over an explicit basis of orthonormal polynomials using the product formula of the Gegenbauer polynomials. For $\mu = (m-1)/2$ and m being an integer, we can also derive this formula as follows. Let us set H, H_1 and H_2 as constants in Theorem 2.5. Then by the definition of $Y_{\alpha,\beta,k}^n$ and the reproducing kernels, as well as (2.2), we have

$$P_n(H;\mathbf{x},\mathbf{y}) = \sum_{k=0}^n A_{m,k}^2 \mathbf{P}_{n-k}(W_{k+(m-1)/2};\mathbf{x}_1,\mathbf{y}_1)|\mathbf{x}_2|^k|\mathbf{y}_2|^k P_k(H_2;\mathbf{x}_2',\mathbf{y}_2'),$$

where $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in S^{d+m-1}$, $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2) \in S^{d+m-1}$ and $|\mathbf{x}_2'| = |\mathbf{y}_2'| = 1$. Since H_2 is a constant, it follows that $P_n(H)$ and $P_k(H_2)$ are ordinary zonal harmonics on S^{d+m-1} and on S^{m-1}, respectively. They can be written in terms of the Gegenbauer polynomial as in (2.8) with $V = id$ (see [**15**, **20**]). Hence, we conclude that

$$\frac{n + (d+m-2)/2}{(d+m-2)/2} C_n^{((d+m-2)/2)}(\langle \mathbf{x}, \mathbf{y} \rangle)$$
$$= \sum_{k=0}^{n} A_{m,k}^2 \mathbf{P}_{n-k}(W_{k+(m-1)/2}; \mathbf{x}_1, \mathbf{y}_1) |\mathbf{x}_2|^k |\mathbf{y}_2|^k \frac{k + (m-2)/2}{(m-2)/2} C_k^{((m-2)/2)}(\langle \mathbf{x}_2', \mathbf{y}_2' \rangle).$$

If we set $\langle \mathbf{x}_2', \mathbf{y}_2' \rangle = t$, then by the fact that $|\mathbf{x}_1|^2 + |\mathbf{x}_2|^2 = 1$, we can write $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}_1, \mathbf{y}_1 \rangle + \sqrt{1 - |\mathbf{x}_1|^2}\sqrt{1 - |\mathbf{y}_1|^2}\, t$. If we then integrate the above identity with respect to $(1-t^2)^{(m-3)/2} dt$ on $[-1, 1]$, we see that the formula (2.13) with $\mu = (m-1)/2$ follows from the orthogonality of the Gegenbauer polynomial $C_k^{((m-2)/2)}(t)$ on $[-1, 1]$.

3. *Summability of Fourier orthogonal series*

The results on the reproducing kernels provide us a way to study summability of the Fourier orthogonal series. Let W be a weight function defined on B^d. We denote by $L_W^p(B^d)$ [resp. $C(B^d)$] for $1 \le p < \infty$ [resp. $p = \infty$] the space of Lebesgue measurable functions f defined on B^d for which the norm $\int_{B^d} |f(\mathbf{x})|^p W(\mathbf{x}) d\mathbf{x}$ [resp. $\sup_{\mathbf{x} \in B^d} |f(\mathbf{x})|$] is finite. With respect to a family of orthonormal polynomials $\{P_\alpha^n\}$ with respect to W, where $P_\alpha^n \in \Pi_n^d$ as before, the Fourier orthogonal expansion of $f \in L_W^2(B^d)$ is given by

$$f \sim \sum_{n=0}^{\infty} \sum_{|\alpha|_1 = n} a_\alpha^n(f) P_\alpha^n(\mathbf{x}), \qquad \text{where} \qquad a_\alpha^n(f) = \int P_\alpha^n(\mathbf{x}) f(\mathbf{x}) W(\mathbf{x}) d\mathbf{x}. \tag{3·1}$$

We denote the partial sum of such an expansion with respect to the subspace $\mathcal{V}_n^d(W)$ by $\mathbf{P}_n(f, W)$, which is given in terms of the reproducing kernel as

$$\mathbf{P}_n(f, W; \mathbf{x}) = \sum_{|\alpha|_1 = n} a_\alpha^n(f) P_\alpha^n(\mathbf{x}) = \int_{B^d} f(\mathbf{y}) \mathbf{P}_n(W; \mathbf{x}, \mathbf{y}) W(\mathbf{y}) d\mathbf{y}. \tag{3·2}$$

We note that the usual n-th partial sum of the expansion is defined by

$$\mathbf{K}_n(f, W; \mathbf{x}) = \sum_{k=0}^{n} \mathbf{P}_k(f, W; \mathbf{x}).$$

We remark that these sums are independent of the particular choice of orthogonal bases. In fact, we can define $\mathbf{P}_n(f, W)$ as the orthogonal projection of f onto the space $\mathcal{V}_n^d(W)$, which is independent of the bases of $\mathcal{V}_n^d(W)$ (see, for example, [**24**]).

The spaces $L_H^p(S^{d-1})$, $1 \le p < \infty$ of functions defined on S^{d-1} with respect to the weight function H are defined similarly. For an admissible weight function H on S^{d+m-1} and $f \in L_H^2(S^{d+m-1})$, we can consider the Fourier expansion of f in terms of an orthonormal basis, say $Y_{\alpha,\beta,k}^n$, and the partial sum of such an expansion with respect to $\mathcal{H}_n^{d+m}(H)$ is given by

$$P_n(f, H; \mathbf{x}) = \int_{S^{d+m-1}} f(\mathbf{y}) P_n(H; \mathbf{x}, \mathbf{y}) H(\mathbf{y}) d\omega_{m+d}(\mathbf{y}). \tag{3·3}$$

Again, this kernel is independent of the choice of the particular bases. The relation between the reproducing kernel with respect to H on S^{d+m-1} and the kernel with respect to W_m^H on B^d in Theorem 2.6 also yields a connection between the summability of Fourier orthogonal series, which we explore below.

The partial sums $P_n(f,H)$ or $\mathbf{P}_n(f,W_m^H)$ do not converge to the function f that is only continuous. We turn to summability methods in the form of

$$\phi_n(f,H)=\sum_{k=0}^{n} c_{k,n}P_k(f,H) \quad \text{or} \quad \Phi_n(f,W_m^H)=\sum_{k=0}^{n} c_{k,n}\mathbf{P}_k(f,W_m^H),$$

where $c_{k,n}$ are real numbers, often positive, and $\sum_{k=0}^{n} c_{k,n}=1$. If $\phi_n(f,H)$ converges to f in $L_H^p(S^{d+m-1})$ norm, then we say that the Fourier expansion with respect to H is ϕ summable to f in $L_H^p(S^{d+m-1})$. Similar definition is given to $\mathbf{P}_n(f,W_m^H)$ and the expansion with respect to W_m^H. A typical example is the Cesàro (C,δ) summability, which we now describe (cf. [**29**, Chapt. 3]). For $\delta>0$, the Cesàro (C,δ) means, s_n^δ, of a sequence $\{s_n\}$ are defined by

$$s_n^\delta=\frac{1}{\binom{n+\delta}{n}}\sum_{k=0}^{n}\binom{n-k+\delta-1}{n-k}s_k=\frac{1}{\binom{n+\delta}{n}}\sum_{k=0}^{n}\binom{n-k+\delta}{n-k}c_k, \tag{3·4}$$

where the second equality holds if s_n is the n-th partial sum of the series $\sum_{k=0}^{\infty} c_k$. We say that $\{s_n\}$ is Cesàro (C,δ) summable to s if s_n^δ converges to s as $n\to\infty$.

In the following, whenever the L^p norm is used for $1\le p\le\infty$, the norm $\|p\|_\infty$ is understood as uniform norm, since for $p=\infty$ we are working with the space of continuous functions.

THEOREM 3·1. *Let H be an admissible weight function and W_m^H be defined as in (2.2). If the means $\phi_n(\cdot,H)=\sum_{k=0}^{n} c_{k,n}P_k(\cdot,H)$ define a uniformly bounded operator on $L_H^p(S^{d+m-1})$ [resp. $C(S^{d+m-1})$] for $1\le p<\infty$ [resp. $p=\infty$], then the means $\Phi_n(\cdot,W_m^H)=\sum_{k=0}^{n} c_{k,n}\mathbf{P}_k(\cdot,W_m^H)$ define a uniformly bounded operator on $L_{W_m^H}^p(B^d)$. More precisely, if*

$$\Big(\int_{S^{d+m-1}}\Big|\sum_{k=0}^{n} c_{k,n}P_k(F,H,\mathbf{x})\Big|^p H(\mathbf{x})d\omega\Big)^{1/p}\le C\Big(\int_{S^{d+m-1}}|F(\mathbf{x})|^pH(\mathbf{x})d\omega\Big)^{1/p}$$

for $F\in L_H^p(S^{d+m-1})$ [resp. $C(S^d+m-1)$] with $1\le p<\infty$ [resp. $p=\infty$], where C is a constant independent of F and n, then

$$\Big(\int_{B^d}\Big|\sum_{k=0}^{n} c_{k,n}\mathbf{P}_k(f,W_m^H,\mathbf{x})\Big|^p W_m^H(\mathbf{x})d\mathbf{x}\Big)^{1/p}\le C\Big(\int_{B^d}|f(\mathbf{x})|^pH(\mathbf{x})d\mathbf{x}\Big)^{1/p}$$

for $f\in L_{W_m^H}^p(B^d)$ [resp. $C(B^d)$] with $1\le p<\infty$ [resp. $p=\infty$]. In particular, if $\phi_n(F,H)$ converges to F in the $L_H^p(S^{d+m-1})$ [resp. $C(S^{d+m-1})$] norm for $1\le p<\infty$ [resp. $p=\infty$], then the means $\Phi_n(f,W_m^H)$ converges to f in the $L_{W_m^H}^p(B^d)$ norm.

Proof. For $f\in L_{W_m^H}^p(B^d)$, we define a function F on S^{d+m-1} by $F(\mathbf{x})=f(\mathbf{x}_1)$, where $\mathbf{x}=(\mathbf{x}_1,\mathbf{x}_2)\in S^{d+m-1}$, $\mathbf{x}_1\in B^d$. By Lemma 2.1, it follows that

$$\int_{S^{d+m-1}}|F(\mathbf{x})|^pH(\mathbf{x})d\omega=\int_{B^d}|f(\mathbf{x})|^pW_m^H(\mathbf{x})d\mathbf{x};$$

hence, $F\in L_H^p(S^{d+m-1})$. Using Theorem 2.6 and the notation $\mathbf{x}=(\mathbf{x}_1,\mathbf{x}_2)$, $\mathbf{y}=(\mathbf{y}_1,\mathbf{y}_2)\in S^{d+m-1}$ and $\mathbf{y}_2=|\mathbf{y}_2|\eta$, $\eta\in S^{m-1}$, we see that the partial sums $P_n(F,H)$

and $\mathbf{P}_n(f, W_m^H)$ are related as follows,

$$\begin{aligned}\mathbf{P}_n(f, W_m^H; \mathbf{x}_1) &= \int_{B^d} \mathbf{P}_n(W_m^H; \mathbf{x}_1, \mathbf{y}_1) f(\mathbf{y}_1) W_m^H(\mathbf{y}_1) d\mathbf{y}_1 \\ &= \int_{B^d} \Big[\int_{S^{m-1}} P_n(H; \mathbf{x}), (\mathbf{y}_1, \sqrt{1-|\mathbf{y}_1|^2}\eta)) H_2(\eta) d\omega_m(\eta) \Big] f(\mathbf{y}_1) W_m^H(\mathbf{y}_1) d\mathbf{y}_1 \\ &= \int_{S^{d+m-1}} P_n(H; \mathbf{x}), \mathbf{y}) F(\mathbf{y}) H(\mathbf{y}) d\mathbf{y} \\ &= P_n(F, H; \mathbf{x}),\end{aligned}$$

where we have used Lemma 2.1.It then follows from Lemma 2.1 that

$$\begin{aligned}\int_{B^d} \Big|\sum_{k=0}^{n} c_{k,n} \mathbf{P}_k(f, W_m^H, \mathbf{x}_1)\Big|^p W_m^H(\mathbf{x}_1) d\mathbf{x}_1 &= \int_{S^{d+m-1}} \Big|\sum_{k=0}^{n} c_{k,n} \mathbf{P}_k(f, H, \mathbf{x}_1)\Big|^p H(\mathbf{x}) d\omega_{d+m} \\ &= \int_{S^{d+m-1}} \Big|\sum_{k=0}^{n} c_{k,n} P_k(F, H, \mathbf{x})\Big|^p H(\mathbf{x}) d\omega_{d+m}.\end{aligned}$$

Hence, the boundedness of the last integral can be used to conclude that

$$\begin{aligned}\int_{B^d} \Big|\sum_{k=0}^{n} c_{k,n} \mathbf{P}_k(f, W_m^H, \mathbf{x}_1)\Big|^p W_m^H(\mathbf{x}_1) d\mathbf{x}_1 \le& C^p \int_{S^{d+m-1}} |F(\mathbf{x})|^p H(\mathbf{x}) d\omega \\ =& C^p \int_{B^d} |f(\mathbf{x})|^p W_m^H(\mathbf{x}).\end{aligned}$$

This completes the proof for $1 \le p < \infty$. In the case $p = \infty$, the norm becomes the uniform norm and the result follows readily from the fact that $\mathbf{P}_n(f, W_m^H; \mathbf{x}_1) = P_n(F, H; \mathbf{x})$. □

This theorem shows that in order to study the summability of the Fourier orthogonal expansion on the unit ball, we need only to study the summability on the unit sphere. We will look at the application of this theorem for various specified weight functions. First we state the following Lemma that follows from the proof of the theorem.

LEMMA 3·2. *Let H be an admissible weight function and W_m^H be defined as in (2.2). Let f be defined on B^d and $F(\mathbf{x}) = f(\mathbf{x}_1)$, where $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in S^{d+m-1}$, $\mathbf{x}_1 \in B^d$. Then*

$$\mathbf{P}_n(f, W_m^H; \mathbf{x}_1) = P_n(F, H; \mathbf{x}).$$

To state our results on orthogonal series on the ball, we first state a result on the summability of the Fourier orthogonal series with respect to a reflection invariant weight function on the sphere. Let us recall that if G is a reflection group and κ is an associated multiplicity function, then h-harmonics are orthogonal with respect to $h_\kappa^2(\mathbf{x}) d\omega$ on S^{d-1}, where h_κ is defined in (2.7).

THEOREM 3·3. *Let h_κ be the reflection invariant weight function (2.7) on $\mathbb{R}^d$ associated to a reflection group G and a multiplicity function κ. Let $f \in L^p_{h_\kappa^2}(S^{d-1})$ [resp. $C(S^{d-1})$] for $1 \le p < \infty$ [resp. $p = \infty$]. Then the expansion of f as the Fourier series with respect to h_κ^2 is (C, δ) summable in $L^p_{h_k^2}(S^{d-1})$ [resp. $C(S^{d-1})$] with $1 \le p < \infty$ [resp. $p = \infty$], provided $\delta > \gamma_\kappa + (d-2)/2$.*

Proof. The case $p = \infty$ is proved in [**26**] under the condition that the intertwining operator V is positive, which was conjectured by Dunkl and proved since then by Rösler in [**17**]. The Cesàro (C,δ) means of the Fourier orthogonal series with respect to h_κ^2 are given by

$$s_n^\delta(f,h_\kappa^2;\mathbf{x}) = \int_{S^{d-1}} f(\mathbf{y}) K_n^\delta(h_\kappa^2;\mathbf{x},\mathbf{y}) h_\kappa^2(\mathbf{y}) d\omega_d,$$

where the kernel $K_n^\delta(h_\kappa^2)$ is the (C,δ) means of the kernel $P_n(h_\kappa^2)$ in (2.8). The proof in [**26**] works by showing that the uniform norm of $s_n^\delta(f,h_\kappa^2)$ is bounded, which is the same as showing that

$$\|s_n^\delta(\cdot,h_\kappa^2)\|_\infty := \sup_{\mathbf{x}\in S^{d-1}} \int_{S^{d-1}} |K_n^\delta(h_\kappa^2;\mathbf{x},\mathbf{y})| h_\kappa^2(\mathbf{y}) d\omega < \infty$$

Since $s_n^\delta(f,h_\kappa^2)$ is a linear operator, we can show that $\|s_n^\delta(\cdot,h_\kappa^2)\|_1 = \|s_n^\delta(\cdot,h_\kappa^2)\|_\infty$, where $\|\cdot\|_1$ stands for the $L^1_{h_\kappa^2}$ norm. First we recall that if f is continuous, then $s_N^\sigma(f,h_\kappa^2)$ converges uniformly to f on S^{d-1} for $\sigma = 2|\kappa|_1 + d - 1$ and $s_N^\sigma(\cdot,h_\kappa^2)$ defines a positive operator ([**26**, p. 2972]). Now, it is straightforward to see that

$$\int_{S^{d-1}} |s_n^\delta(f,h_\kappa^2;\mathbf{y})| h_\kappa^2(\mathbf{y}) d\omega \le \|s_n^\delta(\cdot,h_\kappa^2)\|_\infty \int_{S^{d-1}} |f(\mathbf{y})| h_\kappa^2(\mathbf{y}) d\omega \tag{3·5}$$

for every $f \in L^1_{h_\kappa^2}$. We show that the equality is obtained by setting $f(\mathbf{x}) = K_N^\sigma(h_\kappa^2;\mathbf{x}^*,\mathbf{x})$, where $\mathbf{x}^* \in S^{d-1}$ is a point for which the supremum in the definition of $\|s_n^\delta(\cdot,h_\kappa^2)\|_\infty$ is attained, and letting $N \to \infty$. Indeed, for such an f, using the fact that

$$\begin{aligned} s_n^\delta(K_N^\sigma(h_\kappa^2;\mathbf{x}^*,\cdot),h_\kappa^2;\mathbf{x}) &= \int_{S^{d-1}} K_N^\sigma(h_\kappa^2;\mathbf{x}^*,\mathbf{y}) K_N^\delta(h_\kappa^2;\mathbf{x},\mathbf{y}) h_\kappa^2(\mathbf{y}) d\omega \\ &= s_N^\sigma(K_n^\delta(h_\kappa^2;\mathbf{x},\cdot),h_\kappa^2;\mathbf{x}^*), \end{aligned}$$

we see that the left hand side of (3.5) with $f = K_n^\delta(h_\kappa^2;\mathbf{x}^*,\cdot)$ becomes $\|s_n^\delta(\cdot,h_\kappa^2)\|_\infty$ as $N \to \infty$, so is the right hand side since $K_N^\sigma(h_\kappa^2;\mathbf{x}^*,\mathbf{x})$ is nonnegative and its integral is 1. Hence, $\|s_n^\delta(\cdot,h_\kappa^2)\|_1 = \|s_n^\delta(\cdot,h_\kappa^2)\|_\infty$. Consequently, we have that $\|s_n^\delta(\cdot,h_\kappa^2)\|_1$ is bounded if $\delta > |\kappa|_1 + (d-2)/2$. The case $1 < p < \infty$ follows from the standard Riesz interpolation theorem. □

From the last two theorems, we can derive a number of corollaries for orthogonal expansion on the ball.

COROLLARY 3·4. *Let h_κ^2 be as in (2.10), invariant under the octahedral group, and let $W(\mathbf{x}) = h_\kappa^2(\mathbf{x})(1-|\mathbf{x}|^2)^{\mu-1/2}$. Let $f \in L^p_W(B^d)$ [resp. $C(B^d)$] for $1 \le p < \infty$ [resp. $p = \infty$]. Then the expansion of f as the Fourier orthogonal series with respect to W is (C,δ) summable in $L^p_W(B^d)$ [resp. $C(B^d)$] for $1 \le p < \infty$ [resp. $p = \infty$], provided $\delta > d\kappa_0 + \binom{d}{2}\kappa_1 + \tau + (d-1)/2$.*

Proof. In Theorem 3.1 we take the special case that $m = 2$, $H_1 = h_\kappa^2$ and $H_2(\mathbf{x}) = c|x_{d+1}|^{2\mu}$, $\mathbf{x} = (x_1,\ldots,x_{d+1})$. Then the weight function is invariant under a reducible reflection group $G_1 \times \mathbb{Z}_2$, where G_1 is the octahedral group. Hence, the desired result follows from Theorem 3.1 and Theorem 3.3. □

It is remarkable that we can state such a theorem on the Fourier orthogonal expansion, since little is known on the orthogonal polynomials with respect to the weight function W in the corollary; in particular, no closed formulae are known for orthogonal polynomials. We may also state a corollary for the weight function $W^H = W_1^H$ defined in (2.2), with

H being the weight function h_κ^2 defined in (2.10). In this case, $m = 1$ and we do not need to factor H into $H_1 H_2$. For $d = 2$, the corresponding weight function W^H takes the form

$$W^H(\mathbf{x}) = |x_1 x_2|^{\kappa_0} |1 - 2x_1 - x_2|^{\kappa_1} |1 - x_1 - 2x_2|^{\kappa_1} / \sqrt{1 - x_1^2 - x_2^2}.$$

Again, there is little knowledge about orthogonal polynomials associated with this weight function. From Theorem 2.5, we see that the lack of information on orthogonal polynomials on B^d can be viewed as a consequence of lack of information on the h-harmonics associated to h_κ^2. Although there have been many recent studies on h-harmonics or orthogonal polynomials associated with reflection invariant weight functions, closed formulae for orthonormal bases are not known for weight functions other than those in (2.9). For the weight function in (2.9), an orthonormal basis can be given in terms of the Jacobi polynomials (see [**8**, **25**]) and the intertwining operator is given by (2.11). The explicit formula allows us to prove:

THEOREM 3·5. *Let* $W_{\kappa,\mu}(\mathbf{x}) = C_{\kappa,\mu} \prod_{i=1}^d |x_i|^{2\kappa_i} (1 - |\mathbf{x}|^2)^{\mu - 1/2}$. *Let* $f \in L^p_{W_{\kappa,\mu}}(B^d)$ *[resp.* $C(B^d)$*] for* $1 \le p < \infty$ *[resp.* $p = \infty$*]. Then the expansion of* f *as the Fourier orthogonal series with respect to* $W_{\kappa,\mu}$ *is* (C, δ) *summable in* $L^p_{W_{\kappa,\mu}}(B^d)$ *[resp.* $C(B^d)$*] for* $1 \le p < \infty$ *[resp.* $p = \infty$*], provided* $\delta > \sum_i \kappa_i + \mu + (d-1)/2$. *Moreover, if at least one* $\kappa_i = 0$, *then the expansion is not* (C, δ) *summable in* $L^p_{W_{\kappa,\mu}}(B^d)$ *for* $p = 1$ *or* $p = \infty$ *provided* $\delta \le \sum_i \kappa_i + \mu + (d-1)/2$.

Proof. In Theorem 3.1 we take $m = 1$, $H(\mathbf{x}) = c \prod_{i=1}^d |x_i|^{2\kappa_i} |x_{d+1}|^{2\mu}$, where $\kappa_{d+1} = \mu$. Then the corresponding weight function W_m^H in (2.2) becomes $W_{\kappa,\mu}$. The weight function H is invariant under the reflection group $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$; hence, the sufficient direction follows from Theorem 3.3.

We now prove that the order of the index is also necessary for $p = \infty$ and $p = 1$, provided that at least one component of κ is zero. Let us assume that $\kappa_1 = 0$. We denote $w^\lambda(t) = c(1 - t^2)^{\lambda - 1/2}$, $\lambda > -1/2$, normalized to have integral 1 on $[-1, 1]$; the orthogonal polynomials associated with w^λ are Gegenbauer polynomials $C_n^{(\lambda)}$. In the formula of Corollary 2.7, we take $\mathbf{y} = \mathbf{e}_1 = (1, 0, \ldots, 0)$. Since $\kappa_1 = 0$, the formula holds under the limit (2.12), we conclude that

$$\begin{aligned}\mathbf{P}_n(W_{\kappa,\mu}; \mathbf{x}, \mathbf{e}_1) =& \frac{n + |\kappa|_1 + \mu + (d-1)/2}{|\kappa|_1 + \mu + (d-1)/2} C_n^{(|\kappa|_1 + \mu + (d-1)/2)}(x_1) \\ =& \tilde{C}_n^{(|\kappa|_1 + \mu + (d-1)/2)}(1) \tilde{C}_n^{(|\kappa|_1 + \mu + (d-1)/2)}(x_1),\end{aligned}$$

where $\tilde{C}_n^{(\lambda)}$ denotes the orthonormal polynomials with respect to w^λ, the second equal sign follows from [**22**, p. 80]. We denote by $K_n^\delta(w^\lambda)$ the (C, δ) means of the Fourier orthogonal expansion in Gegenbauer polynomials $C_n^{(\lambda)}$. Then, it follows that $\mathbf{K}_n^\delta(W_{\kappa,\mu}; \mathbf{x}, \mathbf{e}_1)$, which is the (C, δ) means of $\mathbf{P}_n^(W_{\kappa,\mu}; \mathbf{x}, \mathbf{e}_1)$, is given by the formula

$$\mathbf{K}_n^\delta(W_{\kappa,\mu}; \mathbf{x}, \mathbf{e}_1) = K_n^\delta(w^{|\kappa|_1 + \mu + (d-1)/2}; 1, x_1).$$

Hence, upon integrating over B^d and using the easily verified formula

$$\int_{B^d} g(\mathbf{x}) d\mathbf{x} = \int_{-1}^1 \int_{B^{d-1}} g(x_1, \sqrt{1 - x_1^2}\, \mathbf{y}) d\mathbf{y} (1 - x_1^2)^{(d-1)/2} dx_1,$$

a straightforward computation shows that

$$\int_{B^d} |\mathbf{K}_n^\delta(W_{\kappa,\mu})(\mathbf{x},\mathbf{e}_1)| W_{\kappa,\mu}(\mathbf{x}) d\mathbf{x}$$
$$= c\int_{-1}^{1} |K_n^\delta(w^{|\kappa|_1+\mu+(d-1)/2};1,x_1)| w^{|\kappa|_1+\mu+(d-1)/2}(x_1) dx_1$$

where c is a constant independent of n, its value can be determined by setting $n = 0$. Therefore, the (C,δ) summability of the orthogonal expansion with respect to $W_{\kappa,\mu}$ at the point $\mathbf{e}_1$ is equivalent to the (C,δ) summability of the Fourier expansion in Gegenbauer polynomials of index $|\kappa|_1+\mu+(d-1)/2$ at the point $x = 1$. As a consequence, the desired result follows from [**22**, p. 246, Theorem 9.1.3]. This completes the proof of $p = \infty$. The proof of the case $p = 1$ follows as in the proof of Theorem 3.3. □

For $\kappa = 0$, the above theorem was proved in [**28**] with a direct but rather involved proof. The relation between the (C,δ) means of the orthogonal expansion on B^d and the (C,δ) means of expansion in the Gegenbauer polynomials goes much deeper than what has appeared in the above proof. In fact, the formulae (2.8) implies that

$$K_n^\delta(h_\kappa^2;\mathbf{x},\mathbf{y}) = V[K_n^\delta(w^{|\kappa|_1+\mu+(d-2)/2};\langle\mathbf{x},\cdot\rangle)](\mathbf{y}),$$

which also implies analogous formulae for orthogonal expansions in terms of the weight functions given in Corollary 3.4 and Theorem 3.5. This connection is used to prove the uniform convergence in Theorem 3.3 in [**26**]. It also allows us to show that (C,δ) means of the orthogonal expansion define a positive operator for a appropriate δ. For example, we have

THEOREM 3·6. *The (C,δ) means of the Fourier orthogonal series with respect to $W_{\kappa,\mu}$ on B^d define a positive linear operator, provided $\delta \ge 2|\kappa|_1 + 2\mu + d$.*

Proof. This follows easily from (3.6) and an inequality due to Kogbetliantz (cf, [**2**, p. 71]), which states that $K_n^{2\lambda+1}(w^\lambda;1,t) \ge 0$. □

We can also state a similar theorem for the weight function W in Corollary 3.4. The inequality for the Gegenbauer series is a special case of the Askey-Gasper inequality [**2**, **10**]. The latter inequality also leads to other positive summation method for orthogonal series on the sphere or on the ball. We refer to [**28**] for some examples.

For $\kappa = 0$ and μ being a half integer, the conclusion of the above two theorems can also be derived from an observation by Kogbetliantz in [**12**, **13**], which states that the summability of orthogonal series with respect to $W_\mu(\mathbf{x}) = w_\mu(1-|\mathbf{x}|^2)^{\mu-1/2}$, $\mu = (m-1)/2$, on B^d is related to the classical spherical harmonics on S^{d+m-1}. Such a relation is given precisely in Lemma 3.2, upon taking H, H_1 and H_2 as constant weight functions (see (2.3) and the discussion there). In this case, Theorem 3.5 states that (C,δ) means of the Fourier orthogonal series with respect to $W_{(m-1)/2}$ converges uniformly if and only if $\delta > (d+m-2)/2$, which is the same as the *critical index* of the Fourier expansion in the spherical harmonics on S^{d+m-1}.

For $1 < p < \infty$, the condition on the summability index in Theorem 3.5 is not sharp even when one κ_i is zero, since the partial sums of the Fourier orthogonal series ($\delta = 0$) converges in $L^2_{W_{\kappa,\mu}}(B^d)$. There are many results in the literature for summability under the critical index for multiple Fourier series (cf. [**19**, **20**] and the references therein) as well as for series in spherical harmonics (cf. [**4**, **14**, **18**, **21**] and the references therein).

The relation between orthogonal polynomials with respect to $W_{(m-1)/2}$ on B^d and the spherical harmonics allows us to state results on the Fourier orthogonal series with respect to $W_{(m-1)/2}$. For example, using the result in [**18**], we have

THEOREM 3·7. *Let m be a positive integer such that $d+m-1 \geq 3$. Suppose that p and δ satisfy the inequality*

$$|1/2-1/p| \geq 1/(d+m), \quad \delta > \delta_p := \max\{(d+m-1)|1/p-1/2|-1/2, 0\}. \tag{3·6}$$

Then, for $f \in L^p_{W_{(m-1)/2}}(B^d)$, there is a constant $A_{p,\delta}$ such that

$$\Big(\int_{B^d} |s_n^\delta(f, W_{(m-1)/2}; \mathbf{x})|^p W_{(m-1)/2}(\mathbf{x}) d\mathbf{x}\Big)^{1/p} \leq A_{p,\delta}\Big(\int_{B^d} |f(\mathbf{x})|^p W_{(m-1)/2}(\mathbf{x}) d\mathbf{x}\Big)^{1/p}.$$

In particular, the Cesàro (C,δ) means of the Fourier orthogonal series converges in $L^p_{W_{(m-1)/2}}(B^d)$ provided $\delta > \delta_p$.

Proof. Let $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in S^{d+m-1}$ with $\mathbf{x}_1 \in B^d$, and let $F(\mathbf{x}) = f(\mathbf{x}_1)$ be as in the statement of Lemma 3.2. We denote by $S_n^\delta(F)$ the (C,δ) means of the Fourier series of F in spherical harmonics on S^{d+m-1}. From Lemma 3.2 we have that $s_n^\delta(f, W_{(m-1)/2}; \mathbf{x}_1) = S_n^\delta(F; \mathbf{x})$. Using Lemma 2.1, we have

$$\int_{B^d} |s_n^\delta(f, W_{(m-1)/2}; \mathbf{x}_1)|^p W_{(m-1)/2}(\mathbf{x}_1) d\mathbf{x}_1 = \int_{S^{d+m-1}} |S_n^\delta(F; \mathbf{x})|^p d\omega_{d+m}.$$

We then apply Theorem 3.2 in [**18**] to bound the right hand side of the above equation by a constant multiple of the integral of $|F|^p$, which can be written as the integral of $|f|^p$ over B^d with another use of Lemma 2.1. This gives the desired inequality. The condition (3.6) comes from applying Theorem 3.2 in [**18**] for S^{d+m-1}. □

In the above theorem we assume that $d+m-1 \geq 3$, since we apply the result in [**18**] for S^{d-1} with $d \geq 3$. In the case $d = 2$, Theorem 3.1 in [**18**] gives an even stronger result. To apply the result in our case, we need to take $d+m-1 = 2$, which yields $m = 1$ and $d = 2$. Hence, for the weight function $W_0(\mathbf{x}) = w_0(1-|\mathbf{x}|^2)^{-1/2}$ on B^2, we can state a stronger result using Theorem 3.1 in [**18**].

One particular interesting case is perhaps the Lebesgue measure on B^d, which corresponds to $m = 2$ in Theorem 3.7. We state this case as a corollary.

COROLLARY 3·8. *Let p satisfies $|1/2-1/p| \geq 1/(d+2)$, $d \geq 2$. Then, the Cesáro (C,δ) means of the Fourier orthogonal series with respect to the Lebesgue measure converges in $L^p(B^d)$ provided $\delta > \max\{(d+1)|1/p-1/2|-1/2, 0\}$.*

Likewise, many other results on the Fourier series in spherical harmonics can be extended to the Fourier orthogonal series with respect to $W_{(m-1)/2}$. For example, various multiplier theorems (cf. [**4**, **14**, **21**]). We will not state these results for $W_{(m-1)/2}$ on B^d, since such extensions will mostly be straightforward; for example, the multiplier theorems will follow as a consequence of Theorem 3.1.

A far more interesting question is to extend these results to W_μ, $\mu \neq (m-1)/2$. More generally, we can ask the question of how to extend these and other results to other weight functions, such as $W_{\kappa,\mu}$ in Theorem 3.5 or W in Corollary 3.4. In view of Theorem 3.1, we may work with the Fourier orthogonal series on the unit sphere. Although the compact formula for the reproducing kernel of the h-harmonics is given in (2.8), we may need an explicit formula for the intertwining operator V to proceed further. At the moment, the formula of V is known in the case of the product weight function (2.9). We note, however, that even in the case of the weight function (2.9), the extensions such as Theorem 3.7 and the multiplier theorems will not be trivial. The results on expansions in spherical harmonics are often proved using the method developed for classical Fourier analysis in the Euclidean space, in which the fact that both S^{d-1} and $\mathbb{R}^d$ are homogeneous space under the orthogonal group plays an important role. For the weight function (2.9), the underlying group is no longer the orthogonal group, but a subgroup $\mathbb{Z}_2 \times \ldots \times \mathbb{Z}_2$ which no longer acts transitively on S^{d-1}.

Acknowledgements. The author thanks a referee for his careful review. This work is partially supported by the National Science Foundation under Grant DMS-9802265.

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