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ON ℓ-1 RIESZ SUMMABILITY OF THE INVERSE FOURIER INTEGRAL

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Dedicated to G.G. Lorentz on the occasion of his ninetieth birthday

ABSTRACT. On $L(\mathbb{R}^d)$, $d \in \mathbb{N}$, the ℓ-1 Riesz (R, δ) means of the inverse Fourier transform converge almost everywhere provided $\delta > 0$. The proof uses a new representation of the ℓ-1 Riesz kernel derived from the ℓ-1 Poisson kernel.

1. INTRODUCTION

Let f belong to $L(\mathbb{R}^d)$ and let $\hat{f}$ be its Fourier transform. The ℓ-1 partial integral of the inverse Fourier transform is defined by

$$S_{R;d}(f;\mathbf{x}) = \int_{|\mathbf{v}|_1 \le R} e^{i\mathbf{v}\cdot\mathbf{x}} \hat{f}(\mathbf{v})\, d\mathbf{v}, \qquad R > 0, \tag{1.1}$$

where $\mathbf{v}\cdot\mathbf{x}$ denotes the usual Euclidean inner product on $\mathbb{R}^d$ and $|\mathbf{v}|_1 = |v_1| + \cdots + |v_d|$ is the ℓ^1-norm of $\mathbf{v}$ in $\mathbb{R}^d$. In [1] we proved, among others, that the Riesz (R, δ) means, or equivalently the Cesàro (C, δ) means, of the integral define positive linear transformations on $L(\mathbb{R}^d)$ for $\delta \ge 2d - 1$; the order of summability to assure positivity is best possible. Positivity of the kernel implies norm convergence of the summability methods, but it is not necessary. In [2] we discussed specifically convergence of the ℓ-1 Riesz (R, δ) means

$$S^{\delta}_{R;d}(f;x) = \int_{|\mathbf{v}|_1 \le R} \left(1 - \frac{|\mathbf{v}|_1}{R}\right)^{\delta} e^{i\mathbf{v}\cdot\mathbf{x}} \hat{f}(\mathbf{v})\, d\mathbf{v}, \qquad R > 0; \tag{1.2}$$

we stated the following

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Theorem 1.1. *Let $\delta > 0$. If $f \in L^p(\mathbb{R}^d)$, $1 \le p < \infty$, or $f \in C_0(\mathbb{R}^d)$, $p = \infty$, then*

(a) $\lim_{R\to\infty} \|S^\delta_{R;d}(f) - f\|_p = 0,$ *and*

(b) $\lim_{R\to\infty} S^\delta_{R;d}(f;\mathbf{x}) = f(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d,$ *almost everywhere.*

We gave a proof of norm convergence; this part of the theorem, however, had to be attributed to A.N. Podkorytov [7], who in 1981 published a general theorem on summability of multiple Fourier series w.r.t. polyhedral norms, from which norm convergence of the ℓ-1 Riesz means of the inverse Fourier integral could be easily concluded. Our proof differed from the one of Podkorytov, partly due to the more special situation we were working in, but there were similar arguments and the proof was rather long. The proof of part (b) was not published, partly due to the fact that the proof we had worked out was quite involved and difficult to write down for higher dimensions. Here we present a proof of part (b) using a new representation of the Riesz kernel. The essential tool in the previous investigation was a representation of the Riesz kernel in terms of a divided difference. Now we will work with a representation given in terms of the ℓ-1 Poisson kernel; it substantially simplifies the proof. In particular, it also yields a much simpler one of part (a).

In order to put the results of Theorem 1.1 into perspective, it is enlightening to compare these with the analogue statement of the Bochner-Riesz means

$$S^{(2)}_{R;\delta;d}(f;\mathbf{x}) = \int_{|\mathbf{v}|\le R} \left(1 - \frac{|\mathbf{v}|^2}{R^2}\right)^\delta e^{i\mathbf{v}\cdot\mathbf{x}} \hat{f}(\mathbf{v})\, d\mathbf{v}, \qquad R > 0,$$

where $|\mathbf{v}|$ stands for the usual Euclidean norm of $\mathbf{v}$. The convergence of these means requires $\delta > (d-1)/2$, the so-called critical index; its proof is an easy consequence of the explicit representation of the kernel as a Bessel function and of a general convergence theorem for radial (ℓ-2, as we like to say) approximate identities in $L(\mathbb{R}^d)$, see E.M. Stein and G. Weiss [8]. In this respect, the interesting results are proved at the critical index for $1 \le p < 2$. For the ℓ-1 Riesz means there is no critical index (unless we define zero to be it), and the proofs for norm and a.e. convergence are not as straight forward.

For bivariate Fourier series a.e. convergence of the ℓ-1 Cesàro means has been proved a long time ago; already in 1944 J.G. Herriot [5] verified that the associated maximal operator is of weak type $(1,1)$ which implies a.e. convergence. His proof was based on a compact representation of the ℓ-1 Dirichlet kernel as well as on earlier work of J. Marcinkiewicz, A. Zygmund, and G. Grünwald. Even two years earlier Herriot proved analogous convergence theorems for the ℓ-∞ Cesàro means of bivariate Fourier series – he spoke of triangular and square partial sums, respectively. His results on square partial sums were rediscovered by L. V. Zhizhiashvili [9] in 1968. The paper marks in a way a new beginning of intensive investigations on summability of multiple Fourier series and integrals on the Russian hemisphere, see e.g. the surveys of B.I. Golubov [4] and M.I. D'yachenko [3].

The paper is organized as follows. In the next section we prove the new representation of the Riesz kernel and use it to give a simple proof of norm convergence. In Section 3 we prove a.e. convergence.

Note added after completion of the paper. We learned from Yong-Cheol Kim that A. Seeger in his doctoral thesis in 1985 proved a general a.e. convergence result for the Riesz (R, δ)-means of the inverse Fourier integral w.r.t. an arbitrary polyhedral norm. Seeger's proof is based on Podkorytov's proof, cf. [7] again, as well as on a method of H.S. Shapiro. We think, however, that the particular method of summation: summing w.r.t. the ℓ_1-norm, and the particular method of proof: using a representation of the ℓ-1 Dirichlet kernel via the ℓ-1 Poisson kernel, justifies a publication of the paper.
In a recent note of the second named author on "conjugate integrals and series in the ℓ-1 sense" in the *Proceedings of the Second ISAAC Congress*, Kluwer Acad. Publ., Dordrecht/Boston/London 2000, 31–39, the approach via the ℓ-1 Poisson kernel has also been successfully applied—it seems to be impossible to introduce the notation of conjugacy for general polyhedral norms on $\mathbb{R}^d$.
Finally, we would like to refer to a paper of P. Oswald in *Math. Nachr.* **133** (1987), 173–187, in which the author discusses the ℓ-∞ Riesz means on the Hardy spaces $H^p(\mathbb{R}^d)$, $0 < p \le 1$. Oswald's results aren't directly relevant to our ones, except possibly for $p = 1$. A referee pointed the paper out to us; since it is not listed in the survey's referred to above and since the results and proofs are of general interest, we want to make the reader to become aware of it.

2. Representation of the Riesz kernel

We first recall several preliminary results from [1]. The ℓ-1 partial integral $S_{R,d}$ and its Riesz means $S^\delta_{R,d}$, defined in (1.1) and (1.2), respectively, can be written as

$$S_{R,d}(f;\mathbf{x}) = D_{R,d} * f(\mathbf{x}) \quad \text{and} \quad S^\delta_{R,d}(f;\mathbf{x}) = \sigma^\delta_{R,d} * f(\mathbf{x}),$$

where $*$ stands for the usual convolution. $D_{R,d}$ is the associated ℓ-1 Dirichlet kernel which has a representation in terms of a divided difference:

$$D_{R,d}(\mathbf{x}) := \int_{|\mathbf{v}|_1 \le R} e^{i\mathbf{v}\cdot\mathbf{x}} d\mathbf{v} = [x_1^2, \ldots, x_d^2] G_{R,d}, \qquad \mathbf{x} \in \mathbb{R}^d,$$

in which the function $G_{R,d}$ is defined by

$$G_{R,d}(u) = (-1)^{[\frac{d}{2}]} 2^d \sqrt{u}^{d-2} \begin{cases} \cos R\sqrt{u}, & \text{for } d \text{ even}, \\ \sin R\sqrt{u}, & \text{for } d \text{ odd}, \end{cases}$$

and $\sigma^\delta_{R,d}$ is the associated ℓ-1 Riesz kernel of order δ which can be written in terms of the Dirichlet kernel as

$$\sigma^\delta_{R,d}(\mathbf{x}) := \int_{|\mathbf{v}|_1 \le R} \left(1 - \frac{|\mathbf{v}|_1}{R}\right)^\delta e^{i\mathbf{v}\cdot\mathbf{x}}\, d\mathbf{v} = \frac{\delta}{R^\delta} \int_0^R (R-\rho)^{\delta-1} D_{\rho,d}(\mathbf{x}) d\rho.$$

The representation of the kernel as a divided difference played an essential role in the investigation in [1]. It allowed us to prove that the kernel $\sigma^{2d-1}_{R,d}$ is nonnegative, for example. The nature of the divided difference, however, made it difficult to derive a sharp pointwise estimate for the kernel $\sigma_{R,d}^\delta$, and there lied the difficulty of the proof in [2]. We will derive another representation for $\sigma_{R,d}^\delta$ from which a sharp pointwise estimate will be derived in the following section. For this purpose, we need the ℓ-1 Poisson kernel on $\mathbb{R}^d$ defined by $\widehat{p}_d(\lambda;\mathbf{v}) = e^{-\lambda|\mathbf{v}|_1}$, $\mathbf{v} \in \mathbb{R}^d$, or

$$p_d(\lambda;\mathbf{x}) = \int_{\mathbb{R}^d} e^{-\lambda|\mathbf{v}|_1} e^{i\mathbf{v}\cdot\mathbf{x}} d\mathbf{v} = \prod_{j=1}^{d} \frac{2\lambda}{\lambda^2 + x_j^2}, \qquad \mathbf{x} \in \mathbb{R}^d \text{ and } \Re\,\lambda > 0. \tag{2.1}$$

We can also write the kernel in terms of a single integral of the Dirichlet kernel, see [1], a fact that will not be needed here. The following theorem gives the representation of the Riesz kernel in terms of the Poisson kernel.

Theorem 2.1. *For any $\sigma > 0$,*

$$\sigma_{R,d}^\delta(\mathbf{x}) = \frac{\Gamma(\delta+1)}{\pi} \frac{e^{\sigma R}}{R^\delta} \Re\Big\{ \int_0^\infty e^{iRt} \frac{p_d(\sigma+it;\mathbf{x})}{(\sigma+it)^{\delta+1}}\, dt \Big\}, \qquad \mathbf{x} \in \mathbb{R}^d \text{ and } R > 0. \tag{2.2}$$

Proof. Let $\lambda = \sigma + it$ with $\sigma > 0$, fixed, and $t \in \mathbb{R}$. We take the Laplace transform of the kernel $\sigma_{\rho,d}^\delta$, multiplied by ρ^δ, w.r.t. to the variable ρ; i.e.,

$$\begin{aligned}\int_0^\infty e^{-\lambda\rho} \rho^\delta \sigma_{\rho,d}^\delta(\mathbf{x})\, d\rho &= \int_0^\infty e^{-\lambda\rho} \int_{|\mathbf{v}|_1 \le \rho} (\rho - |\mathbf{v}|_1)^\delta e^{i\mathbf{v}\cdot\mathbf{x}} d\mathbf{v}\, d\rho \\ &= \int_{\mathbb{R}^d} e^{i\mathbf{v}\cdot\mathbf{x}} \Big(\int_{|\mathbf{v}|_1}^\infty e^{-\lambda\rho} (\rho - |\mathbf{v}|_1)^\delta d\rho \Big) d\mathbf{v}.\end{aligned}$$

Upon changing variables – setting $u = \rho - |\mathbf{v}|_1$ – the inner integral becomes the Laplace transform of the function u^δ, $u \in \mathbb{R}_+$, multiplied by $e^{-\lambda|\mathbf{v}|_1}$. Hence, by the definition of the ℓ-1 Poisson kernel we conclude that

$$\int_0^\infty e^{-\lambda\rho} \rho^\delta \sigma_{\rho,d}^\delta(\mathbf{x})\, d\rho = \frac{\Gamma(\delta+1)}{\lambda^{\delta+1}} \int_{\mathbb{R}^d} e^{i\mathbf{v}\cdot\mathbf{x}}\, e^{-\lambda|\mathbf{v}|_1} d\mathbf{v} = \frac{\Gamma(\delta+1)}{\lambda^{\delta+1}} p_d(\lambda;\mathbf{x}).$$

By the inverse Laplace transform

$$\begin{aligned}\rho^\delta \sigma_{\rho,d}^\delta(\mathbf{x}) &= \frac{\Gamma(\delta+1)}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda\rho} \frac{p_d(\lambda;\mathbf{x})}{\lambda^{\delta+1}}\, d\lambda \\ &= \frac{\Gamma(\delta+1)}{2\pi} e^{\sigma\rho} \int_{-\infty}^\infty e^{it\rho} \frac{p_d(\sigma+it;\mathbf{x})}{(\sigma+it)^{\delta+1}}\, dt, \qquad \lambda = \sigma + it.\end{aligned}$$

Formula (2.2) then follows from the representation (2.1) of the ℓ-1 Poisson kernel. $\square$

Lemma 2.2. *Let $d \geq 2$. For $\mathbf{x} \in \mathbb{R}^d$ and $R > 0$*

$$|\sigma_{R,d}^{\delta}(\mathbf{x})| \leq \frac{\mathrm{const}_\delta}{R^\delta} \int_0^\infty \frac{(t+R^{-1})^{d-\delta-1}}{\prod_{j=1}^d (|x_j+t|+R^{-1})(|x_j-t|+R^{-1})} dt, \tag{2.3}$$

where const_δ *only depends on δ (and the dimension d).*

Proof. Let $\sigma > 0$ be fixed and let $t > 0$. From

$$(\sigma+t)/\sqrt{2} \leq |\sigma + it| \leq \sigma + t \quad \text{and} \quad |(\sigma+it)^2 + x^2| = |\sigma + i(t+x)||\sigma + i(t-x)|$$

it follows that

$$\frac{2^d(\sigma+t)^d}{\prod_{j=1}^d (\sigma+|x_j+t|)(\sigma+|x_j-t|)} \leq |p_d(\sigma+it);\mathbf{x})| \leq \frac{2^{2d}(\sigma+t)^d}{\prod_{j=1}^d (\sigma+|x_j+t|)(\sigma+|x_j-t|)}, \tag{2.4}$$

and consequently,

$$|\sigma_{R,d}^{\delta}(\mathbf{x})| \leq 2^{2d+\delta+1} \frac{\Gamma(\delta+1)}{\pi} \frac{e^{\sigma R}}{R^\delta} \int_0^\infty \frac{(t+\sigma)^{d-\delta-1}}{\prod_{i=1}^d (\sigma+|x_j+t|)(\sigma+|x_j-t|)} dt.$$

– we only needed the second half of the estimate for the Poisson kernel, but we wanted to point at the strength of the estimate. Setting $\sigma = R^{-1}$ then leads to formula (2.3). □

For proving almost everywhere convergence, we will need a more refined estimate in the following section. The simple estimate (2.3), however, is strong enough to prove norm convergence, that is, part (a) of Theorem 1.1.

Proof of Theorem 1.1 part (a). A standard argument shows that it suffices to prove that the kernel $\sigma_{R,d}^\delta$ is in $L^1(\mathbb{R}^d)$ for $R = 1$. By (2.3), we have

$$\int_{\mathbb{R}^d} |\sigma_{1,d}^\delta(\mathbf{x})| d\mathbf{x} \leq \mathrm{const}_\delta \int_0^\infty (t+1)^{d-\delta-1} \left(\prod_{j=1}^d \int_0^\infty \frac{dx_j}{(|x_j+t|+1)(|x_j-t|+1)} \right) dt.$$

Upon dividing the interval $(0,\infty)$ into $(0,t]$ and $[t,\infty)$ for $t > 0$, one can easily verify that

$$\int_0^\infty \frac{dx}{(|x+t|+1)(|x-t|+1)} = \frac{1}{2}\left(\frac{1}{t} + \frac{1}{t+1}\right) \log(1+2t) \leq \frac{1}{t} \log(1+2t).$$

Hence,

$$\int_{\mathbb{R}^d} |\sigma_{1,d}^\delta(\mathbf{x})| d\mathbf{x} \leq \mathrm{const}_\delta \int_0^\infty (t+1)^{d-\delta-1} t^{-d} \log^d(1+2t) dt;$$

the integral on the right-hand side is finite for $\delta > 0$. □

One has to compare the proof with the one given in [1] to see the power of the representation (2.2) of the Riesz kernel. Its real strength, however, is seen in the following section.

3. Proof of almost everywhere convergence

At first we will derive an estimate for the Riesz kernel which will be essential for the proof of almost everywhere convergence. Because of the symmetry of the kernel, we only need to give the estimate on the set Ω defined by

$$\Omega = \{\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{R}^d : 0 < x_1 < \dots < x_d < \infty\}.$$

Theorem 3.1. *Let $0 < \delta < 1$. Then*

$$|\sigma^\delta_{R,d}(\mathbf{x})| \le \frac{\mathrm{const}_\delta}{R^\delta} \frac{1}{\prod_{l=1}^d (x_l - x_{l-1} + R^{-1})^{1+\delta/d}}, \qquad \mathbf{x} \in \Omega \quad \text{and} \quad R > 0, \tag{3.1}$$

where x_0 is set to be equal to zero.

The theorem is an easy consequence of the following more general estimate.

Lemma 3.2. *Let $d \ge 2$ and $0 < \delta < 1$. There is a constant* const_δ*, independent of R, such that for $\mathbf{x} \in \Omega$*

$$\begin{aligned}|\sigma^\delta_{R,d}(\mathbf{x})| \le \frac{\mathrm{const}_d}{R^\delta} \Bigg\{ & \frac{(x_1 + R^{-1})^{d-\delta-1} \prod_{l=1}^d (x_l + R^{-1})^{-1}}{\prod_{l=2}^d (x_l - x_1 + R^{-1})} + \\ & + \sum_{j=1}^{d-2} \frac{(x_{j+1} + R^{-1})^{d-j-\delta-1} \prod_{l=j+1}^d (x_l + R^{-1})^{-1}}{(x_{j+1} - x_j + R^{-1}) \prod_{l=1}^{j-1} (x_j - x_l + R^{-1}) \prod_{l=j+2}^d (x_l - x_{j+1} + R^{-1})} \\ & + \frac{(x_{d-1} + R^{-1})^{-\delta} (x_d + R^{-1})^{-1}}{(x_d - x_{d-1} + R^{-1}) \prod_{j=1}^{d-2} (x_{d-1} - x_j + R^{-1})} \Bigg\}.\end{aligned} \tag{3.2}$$

Proof. Let us consider the integral in the representation of the Riesz kernel given in (2.2) and let us integrate it by parts. Using the representation (2.1), we obtain

$$\begin{aligned}\int_0^\infty e^{iRt} \frac{p_d(\sigma + it; \mathbf{x})}{(\sigma + it)^{\delta+1}}\, dt = \frac{i}{R} \frac{p_d(\sigma; \mathbf{x})}{\sigma^{\delta+1}} - \\ - \frac{1}{R} \int_0^\infty e^{iRt} \frac{p_d(\sigma + it; \mathbf{x})}{(\sigma + it)^{\delta+1}} \left(\frac{d - \delta - 1}{\sigma + it} - \sum_{j=1}^d \frac{2(\sigma + it)}{(\sigma + it)^2 + x_j^2} \right) dt.\end{aligned}$$

The first term on the right-hand side of the equation is purely imaginary, so it doesn't count taking the real part of the integral; i.e.,

$$\begin{aligned}\sigma^\delta_{R,d}(\mathbf{x}) = \frac{\Gamma(\delta + 1)}{\pi} \frac{e^{\sigma R}}{R^{\delta+1}} \Re \Bigg\{ \int_0^\infty e^{iRt} \frac{p_d(\sigma + it; \mathbf{x})}{(\sigma + it)^{\delta+1}} \Bigg(- \frac{d - \delta - 1}{\sigma + it} + \\ + \sum_{j=1}^d \frac{2(\sigma + it)}{(\sigma + it)^2 + x_j^2} \Bigg) dt \Bigg\}.\end{aligned}$$

It then follows from the representation above and from the elementary estimates for the Poisson kernel given in the proof of Lemma 2.2 that

$$
\begin{aligned}
|\sigma^{\delta}_{R,d}(\mathbf{x})| \le \mathrm{const}_{\delta}\, \frac{e^{\sigma R}}{R^{\delta+1}} \int_0^{\infty} \Bigg\{ (d-\delta-1) \frac{(t+\sigma)^{d-\delta-2}}{\prod_{l=1}^{d}(|x_l+t|+\sigma)(|x_l-t|+\sigma)} + \\
+ \sum_{j=1}^{d} \frac{(t+\sigma)^{d-\delta}}{(|x_j+t|+\sigma)(|x_j-t|+\sigma)\prod_{l=1}^{d}(|x_l+t|+\sigma)(|x_l-t|+\sigma)} \Bigg\}\, dt \\
=: \mathrm{const}_{\delta}\, \frac{e^{\sigma R}}{R^{\delta+1}} \Big\{ A + \sum_{j=1}^{d} B_j \Big\}. \qquad (3.3)
\end{aligned}
$$

We will show that the term A and each term B_j are bounded by the desired estimate. To estimate A, we write

$$
A = \sum_{k=0}^{d} \int_{x_k}^{x_{k+1}} \frac{(t+\sigma)^{d-\delta-2}}{\prod_{l=1}^{d}(|x_l+t|+\sigma)(|x_l-t|+\sigma)}\, dt =: \sum_{k=0}^{d} A_k, \qquad (3.4)
$$

where we take the convention that $x_0 = 0$ and $x_{d+1} = \infty$.

Let $k = 0$. For $0 = x_0 \le t \le x_1$,

$$
\prod_{l=1}^{d}(x_l - t + \sigma) \ge \sigma \prod_{l=2}^{d}(x_l - x_1 + \sigma) \quad \text{and} \quad \prod_{l=1}^{d}(x_l + t + \sigma) \ge \prod_{l=1}^{d}(x_l + \sigma),
$$

giving

$$
\begin{aligned}
A_0 &\le \frac{\prod_{l=1}^{d}(x_l+\sigma)^{-1}}{\sigma \prod_{l=2}^{d}(x_l - x_1 + \sigma)} \int_0^{x_1} (t+\sigma)^{d-\delta-2}\, dt \\
&\le \frac{1}{d-\delta-1} \frac{(x_1+\sigma)^{d-\delta-1}\prod_{l=1}^{d}(x_l+\sigma)^{-1}}{\sigma \prod_{l=2}^{d}(x_l - x_1 + \sigma)}.
\end{aligned}
$$

Next, let $1 \le k \le d-2$. For $x_k \le t \le x_{k+1}$,

$$
\begin{aligned}
\prod_{l=1}^{d}(|x_l - t| + \sigma) &= \prod_{l=1}^{k-1}(t - x_l + \sigma)(t - x_k + \sigma)(x_{k+1} - t + \sigma) \prod_{l=k+2}^{d} (x_l - t + \sigma) \\
&\ge \sigma(x_{k+1} - x_k + \sigma) \prod_{l=1}^{k-1}(x_k - x_l + \sigma) \prod_{l=k+2}^{d} (x_l - x_{k+1} + \sigma),
\end{aligned}
$$

and

$$
\prod_{l=1}^{d}(x_l + t + \sigma) \ge (t+\sigma)^k \prod_{l=k+1}^{d} (x_l + \sigma),
$$

giving

$$A_k \le \frac{\prod_{l=k+1}^d (x_l+\sigma)^{-1}}{\sigma(x_{k+1}-x_k+\sigma)\prod_{l=1}^{k-1}(x_k-x_l+\sigma)\prod_{l=k+2}^d(x_l-x_{k+1}+\sigma)} \times \int_{x_k}^{x_{k+1}} (t+\sigma)^{d-\delta-k-2}\,dt$$
$$\le \frac{1}{d-\delta-k-1}\frac{(x_{k+1}+\sigma)^{d-\delta-k-1}\prod_{l=k+1}^d (x_l+\sigma)^{-1}}{\sigma(x_{k+1}-x_k+\sigma)\prod_{l=1}^{k-1}(x_k-x_l+\sigma)\prod_{l=k+2}^d(x_l-x_{k+1}+\sigma)}.$$

Similarly one proves that except for the factors $1/\delta$ and $1/(1+\delta)$, respectively, the terms A_{d-1} and A_d are bounded from above by

$$\frac{(x_{d-1}+\sigma)^{-\delta}(x_d+\sigma)^{-1}}{\sigma(x_d-x_{d-1}+\sigma)\prod_{l=1}^{d-2}(x_{d-1}-x_l+\sigma)}.$$

Therefore, adding it all up, we see from (3.3) that for $\sigma = R^{-1}$ the term $A\,e^{\sigma R}/R^{\delta+1}$ can be estimated as stated in (3.2).

For each j, $1\le j\le d$, the term B_j in (3.3) is estimated similarly. We write again

$$B_j = \sum_{k=0}^d \int_{x_k}^{x_{k+1}} \frac{(t+\sigma)^{d-\delta}(x_j+t+\sigma)^{-1}\prod_{l=1}^d (x_l+t+\sigma)^{-1}}{(|x_j-t|+\sigma)\prod_{l=1}^d(|x_l-t|+\sigma)}\,dt =: \sum_{j=0}^d B_{j,k}. \tag{3.5}$$

If $k\ne j$ and $k\ne j-1$, then $B_{j,k}$ can be estimated just like A_k as done above. Let us first assume that $k<j-1$. Then

$$B_{j,0} \le \frac{\prod_{l=1}^d (x_l+\sigma)^{-1}}{\sigma\ \prod_{l=2,l\ne j}^d(x_l-x_1+\sigma)}\int_0^{x_1}\frac{(t+\sigma)^{d-\delta}}{(x_j-t+\sigma)^2(x_j+t+\sigma)}dt$$
$$\le \frac{(x_1+\sigma)^{d-\delta-1}\prod_{l=1}^d (x_l+\sigma)^{-1}}{\sigma\ \prod_{l=2}^d(x_l-x_1+\sigma)},$$

except for the factor $1/(d-\delta-1)$ the estimate equals the one derived for A_0. For $1\le k\le j-2$

$$B_{j,k} \le \frac{\prod_{l=k+1}^d (x_l+\sigma)^{-1}}{\sigma\ (x_{k+1}-x_k+\sigma)\prod_{l=1}^{k-1}(x_k-x_l+\sigma)\prod_{l=k+2,l\ne j}^d(x_l-x_{k+1}+\sigma)} \times \int_{x_k}^{x_{k+1}}\frac{(t+\sigma)^{d-\delta-k}}{(x_j-t+\sigma)^2(x_j+t+\sigma)}dt,$$

while for $j<k\le d-2$,

$$B_{j,k} \le \frac{\prod_{l=k+1}^d (x_l+\sigma)^{-1}}{\sigma\ (x_{k+1}-x_k+\sigma)\prod_{l=1,l\ne j}^{k-1}(x_k-x_l+\sigma)\prod_{l=k+2}^d(x_l-x_{k+1}+\sigma)} \times \int_{x_k}^{x_{k+1}}\frac{(t+\sigma)^{d-\delta-k}}{(x_j-t+\sigma)^2(x_j+t+\sigma)}dt;$$

in both cases $(1 \le k \le d-2, k \neq j, j-1)$

$$B_{j,k} \le \frac{(x_{k+1}+\sigma)^{d-\delta-k-1}\prod_{l=k+1}^{d}(x_l+\sigma)^{-1}}{\sigma\,(x_{k+1}-x_k+\sigma)\prod_{l=1}^{k-1}(x_k-x_l+\sigma)}.$$

while for $k=d-1, k\neq j, j-1$,

$$B_{j,d-1} \le \frac{(x_{d-1}+\sigma)^{-\delta}(x_d+\sigma)^{-1}}{\sigma\,(x_d-x_{d-1}+\sigma)\prod_{l=1}^{d-2}(x_{d-1}-x_l+\sigma)\prod_{l=k+2}^{d}(x_l-x_{k+1}+\sigma)}$$

holds true. For $k=j\,(k\le d-2)$

$$B_{j,j} \le \frac{(x_{j+1}+\sigma)^{d-\delta-j-1}\prod_{l=j+1}^{d}(x_l+\sigma)^{-1}}{\prod_{l=1}^{j-1}(x_j-x_l+\sigma)\prod_{l=j+2}^{d}(x_l+x_{j+1}+\sigma)}\times$$
$$\times\int_{x_j}^{x_{j+1}}\frac{1}{(t-x_j+\sigma)^2(x_s+t+\sigma)}dt.$$

The integral can be evaluated exactly and easily estimated from above by $3/\sigma(x_{j+1}-x_j+\sigma)$; hence,

$$B_{j,j} \le \frac{3(x_{j+1}+\sigma)^{d-\delta-j-1}\prod_{l=j+1}^{d}(x_l+\sigma)^{-1}}{\sigma\,(x_{j+1}-x_j+\sigma)\prod_{l=1}^{j-1}(x_j-x_l+\sigma)\prod_{l=j+2}^{d}(x_l-x_{j+1}+\sigma)}.$$

Similarly, for $k=j-1\,(k\le d-1)$

$$B_{j,j-1} \le \frac{3(x_j+\sigma)^{d-\delta-j}\prod_{l=j}^{d}(x_l+\sigma)^{-1}}{\sigma\,(x_j-x_{j-1}+\sigma)\prod_{l=1}^{j-2}(x_{j-1}-x_l+\sigma)\prod_{l=j+1}^{d}(x_l-x_j+\sigma)},$$

while $B_{d-1,d-1}$ and $B_{d,d-1}$ can be estimated by

$$\frac{3(x_{d-1}+\sigma)^{-\delta}(x_d+\sigma)^{-1}}{\sigma\,(x_d-x_{d-1}+\sigma)\prod_{l=1}^{d-2}(x_{d-1}-x_l+\sigma)}.$$

Finally, for $1\le j\le d$

$$B_{j,d} \le \frac{(x_d+\sigma)^{-\delta-1}}{\prod_{l=1;l\neq j}^{d}(x_d-x_l+\sigma)}\int_{x_d}^{\infty}\frac{1}{(t-x_j+\sigma)^2}dt \le \frac{(x_{d-1}+\sigma)^{-\delta}(x_d+\sigma)^{-1}}{\sigma\prod_{l=1}^{d-1}(x_d-x_l+\sigma)}.$$

□

Proof of Theorem 3.1. We deduce the final estimate (3.1) from the one given in Lemma 3.1. For the terms of the sum, indexed by $j=1,\dots,d-2$ in (3.2), it follows from the elementary inequalities

$$(x_{j+1}-x_j+R^{-1})\prod_{l=1}^{j-1}(x_j-x_l+R^{-1})\prod_{l=j+2}^{d}(x_l-x_{j+1}+R^{-1}) \ge \prod_{l=2}^{d}(x_l-x_{l-1}+R^{-1})$$

and

$$\begin{aligned}\prod_{l=j+1}^{d}(x_l+R^{-1}) &= \prod_{l=j+1}^{d}(x_l+R^{-1})^{\delta/d}\prod_{l=j+1}^{d}(x_l+R^{-1})^{1-\delta/d}\\ &\geq \prod_{l=j+1}^{d}(x_l+R^{-1})^{\delta/d}(x_{j+1}+R^{-1})^{d-j-\delta}(x_{j+1}+R^{-1})^{\delta j/d}\\ &\geq (x_{j+1}+R^{-1})^{d-j-\delta-1}(x_1+R^{-1})^{1+\delta/d}\prod_{l=2}^{d}(x_l-x_{l-1}+R^{-1})^{\delta/d}.\end{aligned}$$

The first and the last term on the right-hand side of (3.2) can be estimated similarly. □

To prove almost everywhere convergence of the ℓ-1 Riesz (R,δ) means for $\delta>0$, we shall utilize the following facts about maximal functions on $\mathbb{R}^d$: Let $\mathbf{a}$ be a multi-index in $\mathbb{R}^d_+$ and let

$$K^{(\mathbf{a})}=\{\mathbf{y}\in\mathbb{R}^d\,:\,y_{j-1}-a_j<y_j<y_{j-1}+a_j, 1\leq j\leq d\},\qquad \text{with } y_0=0.$$

Since $K^{(\mathbf{a})}$ is symmetric (w.r.t. the origin) and convex, the family $\{\mathbf{x}+K^{(r\mathbf{a})}\}_{r\in\mathbb{R}_+;\mathbf{x}\in\mathbb{R}^d}$ defines a Vitali cover of $\mathbb{R}^d$ with Vitali constant equal to 2^d; i.e., if $f\in L_{\mathrm{loc}}(\mathbb{R}^d)$, the maximal function

$$\begin{aligned}M_*^{(\mathbf{a})}(f;\mathbf{x}) &= \sup_{r>0}\frac{1}{|K^{(r\mathbf{a})}|}\int_{K^{(r\mathbf{a})}}|f(\mathbf{x}+\mathbf{y})|\,d\mathbf{y}\\ &= \sup_{r>0}\frac{1}{2^d r^d a_1\cdots a_d}\int_{-ra_1}^{ra_1}\int_{y_1-ra_2}^{y_1+ra_2}\cdots\int_{y_{d-1}-ra_d}^{y_{d-1}+ra_d}|f(\mathbf{x}+\mathbf{y})|dy_d\cdots dy_1,\end{aligned}$$

is of weak type $(1,1)$. To be precise,

$$\big|\{\mathbf{x}\in\mathbb{R}^d\,:\,M_*^{(\mathbf{a})}(f;\mathbf{x})>t\}\big|\leq\frac{2^d}{t}\|f\|_1,\qquad t\in\mathbb{R}_+, \tag{3.6}$$

see N.M. Rivière [6] and also A. Zygmund [10; Vol. 2, p. 309ff]. The importance of the estimate lies in the fact that it holds uniformly w.r.t. the index set $\mathbb{R}^d_+$.

Next let $\mathbf{k}$ be a multi-index in $\mathbb{N}_0^d$, $\mathbf{2^k}=(2^{k_1},\ldots,2^{k_d})$, and $\sigma>0$, we set

$$M_\sigma^*(f;\mathbf{x})=\sup_{\mathbf{k}\in\mathbb{N}_0^d}\{M_*^{(\mathbf{2^k})}(f;\mathbf{x})2^{-\sigma|\mathbf{k}|_1}\}.$$

where, obviously, $|\mathbf{k}|_1=k_1+\ldots+k_d$. It is an easy exercise to conclude from (3.6) that

Lemma 3.3. *For $\sigma>0$, the maximal function $M_\sigma^*(\cdot)$ is of weak type $(1,1)$ with constant only dependent on the dimension d and the parameter σ.*

We are now in position to complete the

Proof of Theorem 1.1 part (b). Let f be a function in $L(\mathbb{R}^d)$. We may assume that $f(\mathbf{x}) \geq 0$ a.e. on $\mathbb{R}^d$. Since the maximal operator M_σ^* is of weak type $(1,1)$, it suffices to show that the absolute value of $S_{R,d}^\delta(f;\mathbf{x})$, defined in (1.2), is bounded by the maximal function $M_\sigma^*(f;\mathbf{x})$ on $\mathbb{R}^d$ for $\delta > 0$, uniformly in $R > 0$.

It is well-known that, if the Riesz means $S_{R,d}^\delta(f;\mathbf{x})$ converge for some δ_0, they converge for $\delta > \delta_0$. Hence, we only have to deal with the case $0 < \delta < 1$. Moreover, a standard argument shows that we only need to consider the point $\mathbf{x} = 0$; that is, we only need to establish the estimate

$$|S_{R,d}^\delta(f;0)| = \left| \int_{\mathbb{R}^d} \sigma_{R,d}^\delta(\mathbf{x}) f(\mathbf{x}) d\mathbf{x} \right| \leq \mathrm{const}_\delta \;\; M_\sigma^*(f;0)$$

for some $\sigma > 0$, where const_δ is a constant independent of R. Setting

$$h(\mathbf{x}) = \sum_{\varepsilon \in \{-1,1\}^d} f(\varepsilon_1 x_1, \dots, \varepsilon_d x_d) \tag{3.7}$$

and using the symmetry of the kernel, we see that it is sufficient to prove that

$$\left| \int_\Omega \sigma_{R,d}^\delta(\mathbf{x})\, h(\mathbf{x})\; d\mathbf{x} \right| \leq \mathrm{const}_\delta \;\; M_\sigma^*(f;0), \tag{3.8}$$

where $\Omega = \{\mathbf{x} \in \mathbb{R}^d : 0 < x_1 \leq \ldots \leq x_d < \infty\}$ as defined above. To prove the inequality we write

$$\begin{aligned}
\int_\Omega \sigma_{R,d}^\delta(\mathbf{x})\, h(\mathbf{x})\; d\mathbf{x} &= \int_0^\infty \int_{x_1}^\infty \cdots \int_{x_{d-1}}^\infty \sigma_{R,d}^\delta(\mathbf{x})\, h(\mathbf{x})\; dx_d \cdots dx_1 \\
&= \sum_{k_1=0}^\infty \cdots \sum_{k_d=0}^\infty \int_{c(k_1)/R}^{2^{k_1}/R} \int_{x_1+c(k_2)/R}^{x_1+2^{k_2}/R} \cdots \int_{x_{d-1}+c(k_d)/R}^{x_{d-1}+2^{k_d}/R} \sigma_{R,d}^\delta(\mathbf{x})\, h(\mathbf{x})\; dx_d \cdots dx_1 \\
&= \sum_{\mathbf{k} \in \mathbb{N}_0^d} \int_{\Omega_\mathbf{k}} \sigma_{R,d}^\delta(\mathbf{x})\, h(\mathbf{x})\; d\mathbf{x},
\end{aligned}$$

where we define $c(k_j) = 0$ if $k_j = 0$ and $= 2^{k_j - 1}$ if $k_j = 1, 2, \ldots$, $j = 1, 2, \ldots, d$, so that for $k_j = 0$ the relevant integral runs from x_{j-1} to $x_{j-1} + 1/R$. ¿From the estimate of the Riesz kernel $\sigma_{n,d}^\delta(\mathbf{x})$, derived in Theorem 3.1, we conclude that

$$|\sigma_{R,d}^\delta(\mathbf{x})| \leq \frac{\mathrm{const}_\delta}{R^\delta} \frac{1}{\prod_{l=1}^d (2^{k_l}/R)^{1+\delta/d}} \leq \frac{\mathrm{const}_\delta}{\prod_{l=1}^d (2^{k_l}/R)} 2^{-|\mathbf{k}|_1 \delta/d}, \quad \mathbf{x} \in \Omega_\mathbf{k}.$$

Hence,

$$\begin{aligned}
\int_\Omega |\sigma_{R,d}^\delta(\mathbf{x})|\, h(\mathbf{x})\; d\mathbf{x} &\leq \mathrm{const}_\delta \sum_{k_1=0}^\infty \cdots \sum_{k_d=0}^\infty 2^{-|\mathbf{k}|_1 \delta/d} \frac{1}{\prod_{i=1}^d (2^{k_i}/R)} \int_{\Omega_\mathbf{k}} h(\mathbf{x})\; d\mathbf{x} \\
&\leq \mathrm{const}_\delta \sum_{\mathbf{k} \in \mathbb{N}_0^d} \frac{2^{-|\mathbf{k}|_1 \delta/d}}{2^{|\mathbf{k}|_1 + d}/R^d} \int_{-2^{k_1}/R}^{2^{k_1}/R} \int_{x_1 - 2^{k_2}/R}^{x_1+2^{k_2}/R} \cdots \int_{x_{d-1}-2^{k_d}/R}^{x_{d-1}+2^{k_d}/R} f(\mathbf{x})\; d\mathbf{x},
\end{aligned}$$

where the second inequality follows from the definition of Ω_k and the one of h given in (3.7).

We conclude that

$$\int_\Omega |\sigma^\delta_{R,d}(\mathbf{x})|\, h(\mathbf{x})\, d\mathbf{x} \le \mathrm{const}_\delta \sum_{\mathbf{k}\in\mathbb{N}_0^d} 2^{-|\mathbf{k}|_1\delta/d} M_*^{(\mathbf{2^k})}(f;0)$$
$$\le \mathrm{const}_\delta\, M^*_{\delta/2d}(f;0) \sum_{\mathbf{k}\in\mathbb{N}_0^d} 2^{-|\mathbf{k}|_1\delta/2d}.$$

since the sum is finite, this is the desired inequality. □

References

1. H. Berens and Y. Xu, *ℓ-1 summability for multiple Fourier integrals and positivity*, Math. Proc. Camb. Phil. Soc. **122** (1997), 149-172.
2. H. Berens and Y. Xu, *ℓ-1 Riesz means of the inverse Fourier integrals*, Approximation Theory VIII (C.K. Chui and L.L. Schumaker, eds.), World Sci. Publ. Co., Singapore, 1995, pp. 55-62.
3. M.I. D'yachenko, *Some problems in the theory of multiple trigonometric series*, Russian Math. Surveys **47** (1992), 103–171.
4. B.I. Golubov, *Multiple Fourier series and Fourier integrals*, J. Soviet. Math. **24** (1984), 639–73.
5. J.G. Herriot, *Nörlund summability of multiple Fourier series*, Duke Math. J. **11** (1944), 735–754.
6. N.M. Rivière, *Singular integrals and multiplier operators*, Ark. Mat. **9** (1971), 243–278.
7. A.M. Podkorytov, *Summation of multiple Fourier series over polyhedra*, Vestnik Leningrad Univ. Math. **13** (1981), 69–77.
8. E.M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, 1971.
9. L.V. Zhizhiashvili, *A generalization of a certain theorem of Marcinkiewicz*, Math. USSR – Izvestija **2** (1968), 1065-1075.
10. A. Zygmund, *Trigonometric Series, 2nd ed.*, Cambridge Univ. Press, Cambridge, 1968.

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