# APPROXIMATION BY MEANS OF $h$-HARMONIC POLYNOMIALS ON THE UNIT SPHERE

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Abstract. The $h$-harmonics are analogous of the ordinary harmonics, they are orthogonal homogeneous polynomials on the sphere with respect to a weight function that is invariant under a reflection group. Two means of the orthogonal expansions, the de la Vallée Poussin means and an analog of spherical means, are defined and their approximation behaviors are studied. A weighted modulus of smoothness is defined using the modified spherial means and is proved to be equivalent to a weighted K-modulus defined using the differential-difference $h$-spherical Laplacian. A Bernstein type inequality for the $h$-spherical Laplacian is also established.

## 1. Introduction

There are many papers devoted to the study of approximation by spherical polynomials on the unit sphere $S^{d-1} = \{x : \|x\| = 1\}$ of $\mathbb{R}^d$; see, for example, [3, 4, 6, 13, 14, 15]. Ordinary spherical harmonics often serve as an important building block of the approximation processes. As it is well-known (cf. [9, 18, 20]), harmonic polynomials are homogeneous polynomials that satisfy the Laplace equation $\Delta P = 0$, where $\Delta = \partial_1^2 + \ldots + \partial_d^2$ and $\partial_i$ denotes the ordinary partial derivative with respect to $x_i$. The spherical harmonics are the restrictions of harmonic polynomials on the unit sphere and they are orthogonal polynomials with respect to the surface (Lebesgue) measure. The underline structure of orthogonal group plays a prominent role in the theory of spherical harmonics, since the surface measure is the only invariant measure on the sphere under the orthogonal group. Most of the work on approximation by spherical polynomials works with $L^p(S^{d-1})$ defined with respect to the surface measure.

There is an analog theory of $h$-harmonics with reflection groups in place of the orthogonal group, discovered by Dunkl recently ([7, 8], see [9] and the references therein). For a nonzero vector $v \in \mathbb{R}^d$, let $\sigma_v$ denote the reflection with respect to the hyperplane perpendicular to $v$,

$$x\sigma_v := x - 2(\langle x, v\rangle/\|v\|^2)v, \qquad x \in \mathbb{R}^d,$$

where $\langle x, y\rangle$ denote the usual Euclidean inner product. Let $G$ be a finite reflection group on $\mathbb{R}^d$ with a fixed positive root system $R_+$, normalized so that $\langle v, v\rangle = 2$ for all $v \in R_+$. Then $G$ is a subgroup of the orthogonal group generated by the reflections $\{\sigma_v : v \in R_+\}$. Let $\kappa$ be a nonnegative multiplicity function $v \mapsto \kappa_v$ defined on $R_+$ with the property that $\kappa_u = \kappa_v$ whenever $\sigma_u$ is conjugate to $\sigma_v$

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in $G$ ($ug = v$ for some $g \in G$); then $v \mapsto \kappa_v$ is a $G$-invariant function. The $h$-harmonics are homogeneous polynomials orthogonal with respect to the measure $h_\kappa^2 d\omega$ on $S^{d-1}$, where $h_\kappa$ is defined by

$$h_\kappa(x) = \prod_{v \in R_+} |\langle x, v\rangle|^{\kappa_v}, \qquad x \in \mathbb{R}^d. \tag{1.1}$$

The function $h_\kappa$ is a positive homogeneous function of degree $\gamma_\kappa := \sum_{v \in R_+} \kappa_v$, and it is invariant under the reflection group $G$. The simplest example is given by the case $G = \mathbb{Z}_2^d$ for which $h_\kappa$ is just the product weight function

$$h_\kappa(x) = \prod_{i=1}^d |x_i|^{\kappa_i}, \qquad \kappa_i \geq 0. \tag{1.2}$$

The theory of $h$-harmonics is comparable to the theory of the ordinary harmonics in many ways. The essential ingredient of the theory is a family of first-order differential-difference operators, $\mathcal{D}_i$, called Dunkl's operators, which generates a commutative algebra; these operators are defined by

$$\mathcal{D}_i f(x) = \partial_i f(x) + \sum_{v \in R_+} k_v \frac{f(x) - f(x\sigma_v)}{\langle x, v\rangle} \langle v, \varepsilon_i\rangle, \qquad 1 \leq i \leq d,$$

where $\varepsilon_1, \ldots, \varepsilon_d$ are the standard unit vectors of $\mathbb{R}^d$. The $h$-Laplacian is defined by $\Delta_h = \mathcal{D}_1^2 + \ldots + \mathcal{D}_d^2$. Keeping the notation $\Delta$ for the ordinary Laplacian, then $\Delta_h$ is equal to

$$\Delta_h f(x) = \Delta f(x) + \sum_{v \in R_+} \kappa_v \left[ \frac{2\langle \nabla f(x), v\rangle}{\langle x, v\rangle} - \frac{f(x) - f(x\sigma_v)}{\langle x, v\rangle^2} \right].$$

The $h$-Laplacian plays the role similar to that of the ordinary Laplacian. Let $\mathcal{P}_n^d$ denote the space of homogeneous polynomials of degree $n$ in $d$ variables and let $\mathcal{H}_n^d(h_\kappa^2) \subset \mathcal{P}_n^d$ denote the space of $h$-harmonic polynomials of degree $n$. Then

$$\int_{S^{d-1}} pq h_\kappa^2 d\omega = 0, \qquad p \in \mathcal{H}_n^d(h_\kappa^2), \quad q \in \Pi_{n-1}^d,$$

where $\Pi_m^d$ denotes polynomials of degree at most $m$ in $d$ variables. That is, the $h$-harmonics are exactly those homogeneous polynomials satisfying the equation $\Delta_h P = 0$. The spherical $h$-harmonics are the restriction of $h$-harmonics on the unit sphere. In terms of the polar coordinates $y = ry'$, $r = \|y\|$, the operator $\Delta_h$ takes the form

$$\Delta_h = \frac{\partial^2}{\partial r^2} + \frac{d + 2\gamma_\kappa - 1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \Delta_{h,0},$$

where $\Delta_{h,0}$ is the (Laplace-Beltrami) operator on the sphere. Applying $\Delta_h$ to $h$-harmonics $Y \in \mathcal{H}_n(h_\kappa^2)$ with $Y(y) = r^n Y(y')$ shows that the spherical $h$-harmonics are eigenfunctions of $\Delta_{h,0}$; that is,

$$\Delta_{h,0} Y(x) = -n(n + d + 2\gamma_\kappa - 2) Y(x), \qquad x \in S^{d-1}, \quad Y \in \mathcal{H}_n^d(h_\kappa^2).$$

The purpose of this paper is to consider approximation by $h$-harmonic polynomials. In the $L^p$ metric, it can be viewed as weighted approximation, in which the measure $d\omega$ on the sphere is replaced by $h_\kappa^2 d\omega$. The situation can be quite different from that of ordinary harmonics. Since the orthogonal group acts transitively on the sphere $S^{d-1}$, much of the results for the ordinary harmonics can be proved by considering just one point; this can also be seen from the fact that the integral kernels of the expansions are functions of the form $\phi(\langle x, y\rangle)$, where $\phi$ is a function of one variable (say, zonal harmonics). Since reflection groups do not act transitively on the sphere, the kernel functions for $h$-harmonic expansions are no longer in such simple form. However, some results can still be proved as before without dealing with the reflection symmetry, although such results are usually weaker. This is illustrated by the norm convergence of the Cesàro means, studied in [22, 26, 12] and also by a result on almost everywhere convergence of the de la Vallée Poussin means from a somewhat different point of view.

In this paper, we will use the de la Vallée Poussin means of the $h$-harmonic expansions as a motivation. The approximation behavior of these means can be described by a K-modulus, defined just as in the case of ordinary spherical harmonics with the differential operator $\Delta$ replaced by the differential-difference operator $\Delta_h$. For ordinary harmonics, the K-modulus is equivalent to a modulus of smoothness defined through a spherical mean of the function. Such a mean does not work for $h_\kappa^2 d\omega$ since the reflection symmetry is not taken into consideration. We will give an analogous definition of modulus smoothness for $h_\kappa^2 d\omega$, in which the spherical means are defined implicitly through an integral relation, and show that it is equivalent to the K-modulus.

The paper is organized as follows. The next section states the basic fact on $h$-harmonic expansions. The de la Vallèe Poussin and spherical means are discussed in the Sections 3 and 4, respectively. A Bernstein type inequality for the action of $\Delta_{h,0}$ on the spherical polynomials, which is used in Section 3, is proved in Section 5. Some other connection is pointed out in the last section.

## 2. $h$-HARMONIC EXPANSIONS

Let $h_\kappa$ be the reflection invariant weight function defined in (1.1). Throughout this paper, we denote by $L^p(h_\kappa^2)$, $1 \le p \le \infty$, the space of functions defined on $S^{d-1}$ with the finite norm

$$\|f\|_{p,\kappa} = \Big( \int_{S^{d-1}} |f(y)|^p h_\kappa^2(y) d\omega(y) \Big)^{1/p}, \qquad 1 \le p < \infty,$$

and for $p = \infty$ we assume that $L^\infty$ is replaced by $C(S^{d-1})$ with the usual uniform norm $\|f\|_\infty$. The surface measure $d\omega$ has a normalization constant $\sigma_{d-1} = \int_{S^{d-1}} d\omega = 2\pi^{d/2}/\Gamma(d/2)$, which is the surface area.

Let $N_n = \dim \mathcal{H}_n(h_\kappa^2)$. Recall that $\mathcal{P}_n^d$ denote the space of homogeneous polynomials. It is known that

$$\dim \mathcal{P}_n^d = \binom{n+d-1}{d} \qquad \text{and} \qquad N_n = \dim \mathcal{P}_n^d - \dim \mathcal{P}_{n-2}^d.$$

For $n \ge 0$ let $S_{n,i}$ be a sequence of orthonormal h-harmonics in $\mathcal{H}_n^d(h_\kappa^2)$ with $1 \le i \le N_n$. Then $\{S_{n,i}\}$ is an orthonormal basis of $\mathcal{H}_n^d(h_\kappa^2)$. The standard Hilbert space theory shows that every $f \in L^2(h_\kappa^2)$ can be expanded into an $h$-harmonic series

$$f(x) = \sum_{n=0}^{\infty} \sum_{i=1}^{N_n} a_{n,i}(f) S_{n,i}(x) \quad \text{with} \quad a_{n,i} = \int_{S^{d-1}} f(y) S_{n,i}(y) h_\kappa^2(y) d\omega.$$

For the ordinary harmonics, such an expansion is called the Laplace series (cf. [10, Chapt. 12]). The $n$-th partial sum operator of the expansion is denoted by

$S_n(h_\kappa^2; f)$, which can be written as an integral

$$S_n(h_\kappa^2; f, x) := \sum_{k=0}^{n} \sum_{i=1}^{N_k} a_{k,i}(f) S_{k,i}(x) = \int_{S^{d-1}} f(y) K_n(h_\kappa^2; x, y) h_\kappa^2(y) d\omega(y),$$

where the kernel $K_n(h_\kappa^2; x, y)$ is defined by

$$K_n(h_\kappa^2; x, y) = \sum_{k=1}^{n} P_k(h_\kappa^2; x, y) \quad \text{with} \quad P_k(h_\kappa^2; x, y) = \sum_{i=1}^{N_k} S_{k,i}(x) S_{k,i}(y).$$

The kernel $K_n(h_\kappa^2; x, y)$ is the reproducing kernel of the polynomial space $\Pi_n^d$ and $P_n(h_\kappa^2; x, y)$ is the reproducing kernel of the space of $h$-harmonics $\mathcal{H}_n^d(h_\kappa^2)$ in $L^2(h_\kappa^2)$. For later use, we also define

$$Y_n(h_\kappa^2; f, x) := \sum_{i=1}^{N_n} a_{n,i}(f) S_{n,i}(x) = \int_{S^{d-1}} f(y) P_n(h_\kappa^2; x, y) h_\kappa^2(y) d\omega(y),$$

which is the projection operator of $f$ onto $\mathcal{H}_n^d(h_\kappa^2)$. Although orthonormal bases of $\mathcal{H}_n^d(h_\kappa^2)$ are not unique, the reproducing kernel $P_n(h_\kappa^2; x, y)$ is uniquely determined. Furthermore, it enjoys a compact formula in terms of the intertwining operator between the commutative algebra generated by the partial derivatives and the one generated by Dunkl's operators. The intertwining operator $V$ is a linear operator determined uniquely by

$$V\mathcal{P}_n \subset \mathcal{P}_n, \qquad V1 = 1, \qquad \mathcal{D}_i V = V\partial_i, \qquad 1 \le i \le d.$$

The compact formula of the reproducing kernel to $h_\kappa^2$ for any finite reflection group is given by ([22])

$$P_n(h_\kappa^2; x, y) = \frac{n + \gamma_\kappa + (d-2)/2}{\gamma_\kappa + (d-2)/2} V[C_n^{\gamma_\kappa + \frac{d-2}{2}}(\langle \cdot, y \rangle)](x). \tag{2.1}$$

If all $\kappa_v = 0$, $V$ becomes the identity operator and the above formula becomes

$$P_n(x, y) = \frac{n + (d-2)/2}{(d-2)/2} C_n^{\frac{d-2}{2}}(\langle x, y \rangle), \tag{2.2}$$

the right hand is the so-called zonal harmonic. However, an explicit formula of $V$ is known only in the case of symmetric group $S_3$ for three variables and in the case of the abelian group $\mathbb{Z}_2^d$. In the latter case, $V$ is an integral operator,

$$Vf(x) = c_\kappa \int_{[-1,1]^d} f(x_1 t_1, \ldots, x_d t_d) \prod_{i=1}^{d} (1 + t_i)(1 - t_i^2)^{\kappa_i - 1} dt, \tag{2.3}$$

where $c_\kappa$ denotes the constant

$$c_\kappa = b_{\kappa_1} \ldots b_{\kappa_d}, \qquad \text{and} \quad b_r = \Big( \int_{-1}^{1} (1 - t^2)^{r-1} dt \Big)^{-1}.$$

If some $\kappa_i = 0$, then the formula holds under the limit relation

$$\lim_{\lambda \to 0} b_\lambda \int_{-1}^{1} f(t)(1-t)^{\lambda - 1} dt = [f(1) + f(-1)]/2.$$

This leads to an explicit formula for the reproducing kernel; more precisely, for the weight function $h_\kappa^2$ in (1.2), we have the compact formula ([21])

$$P_n(h_\kappa^2; x, y) = c_\kappa \frac{n + |\kappa| + (d-2)/2}{|\kappa| + (d-2)/2} \times \int_{[-1,1]^d} C_n^{|\kappa|+\frac{d-2}{2}}(x_1 y_1 t_1 + \ldots + x_d y_d t_d) \prod_{i=1}^{d} (1+t_i)(1-t_i^2)^{\kappa_i - 1} dt, \tag{2.4}$$

where $|\kappa| = \kappa_1 + \cdots + \kappa_d$.

One important property of the intertwining operator is that it is positive ([16]), that is, $Vp \ge 0$ if $p \ge 0$. We will need another important property which shows that the integral of $Vf$ has an explicit form, although a compact formula of the intertwining operator itself is not known in general. The following theorem is proved in [22]:

**Theorem 2.1.** *Let $V$ be the intertwining operator. Then*

$$\int_{S^{d-1}} Vf(x) h_\kappa^2(x) d\omega = A_\kappa \int_{B^d} f(x)(1 - \|x\|^2)^{|\kappa|+\frac{d-3}{2}} dt,$$

*for $f \in L^2(h_\kappa^2; S^{d-1})$ such that both integrals are finite; in particular, if $g : \mathbb{R} \mapsto \mathbb{R}$ is a function such that all integrals below are defined, then*

$$\int_{S^{d-1}} Vg(\langle x, \cdot \rangle)(y) h_\kappa^2(y) d\omega(y) = B_\kappa \int_{-1}^{1} g(t\|x\|)(1-t^2)^{|\kappa|+\frac{d-3}{2}} dt,$$

*where $A_\kappa$ and $B_\kappa$ are constants whose values can be determined by setting $f(x) = 1$ and $g(t) = 1$, respectively.*

This theorem plays an important role in the study of $h$-harmonic expansions, to which we now turn our attention. Just as in the case of ordinary harmonics, the partial sums of the expansion do not converge in general for $f \in L^p(h_\kappa^2)$ and $p \ne 2$. We need to consider summability methods, such as various means of the $h$-harmonic expansions.

As seen from (2.2), the kernel functions for the ordinary harmonic expansions are of the form $\phi(\langle x, y \rangle)$, a fact that plays an essential role in most of the literature. Some effect of this fact or the lacking of it in the case of $h$-harmonics will become clear in the following sections.

## 3. de la Vallée Poussin means

For $h_\kappa$ defined in (1.1) and $f \in L^1(h_\kappa^2)$, these means are defined by

$$M_n(h_\kappa^2; f, x) := \sum_{k=0}^{n} \mu_{k,n}^\lambda Y_n(h_\kappa^2; f, x) = a_\kappa \int_{S^{d-1}} m_n(h_\kappa^2; x, y) f(y) h_\kappa^2(y) d\omega(y),$$

where $a_\kappa$ is the normalization constant of $h_\kappa^2 d\omega$, $a_\kappa^{-1} = \int_{S^{d-1}} h_\kappa^2(y) d\omega(y)$ and

$$\mu_{k,n}^\lambda = \frac{n!}{(n-k)!} \frac{\Gamma(n + 2\lambda + 1)}{\Gamma(n + k + 2\lambda + 1)}, \qquad \lambda = \gamma_\kappa + \frac{d-2}{2}.$$

Using the compact formula for $Y_n(h_\kappa^2; x, y)$, it follows that the integral kernel can be written as

$$m_n(h_\kappa^2; x, y) = V\big[m_n(w_\lambda; 1, \langle x, y \rangle)\big], \tag{3.1}$$

where $w_\lambda$ is the weight function of Gegenbauer polynomials,

$$w_\lambda(t) = c_\lambda (1-t^2)^{\lambda-1/2}, \quad -1 \le t \le 1, \qquad c_\lambda^{-1} = \int_{-1}^{1} (1-t^2)^{\lambda-1/2} dt, \tag{3.2}$$

and $m_n(w_\lambda)$ is the de la Vallée Poussin means of the Gegenbauer polynomials,

$$m_n(w_\lambda; 1, t) = \sum_{k=0}^{n} \mu_{k,n}^\lambda \frac{k+\lambda}{\lambda} C_k^\lambda(t).$$

The classical de la Vallée Poussin sum for the cosine functions has an extension for the Gegenbauer polynomials (see, for example, [1, p. 11]),

$$\sum_{k=0}^{n} \frac{n!}{(n-k)!} \frac{\Gamma(n+\lambda+1/2)}{\Gamma(n+k+2\lambda+1)} \frac{k+\lambda}{\lambda} C_k^\lambda(t) = \frac{\Gamma(1/2)}{2^{2\lambda}\Gamma(\lambda+1)} \left(\frac{1+t}{2}\right)^n,$$

which shows that the kernel of the means for $h$-harmonics can be written as

$$m_n(h_\kappa^2; x, y) = \frac{\Gamma(1/2)}{2^{2\lambda}\Gamma(\lambda+1)} \frac{\Gamma(n+2\lambda+1)}{\Gamma(n+\lambda+1/2)} V^{(y)} \left[ \left( \frac{1+\langle x, y\rangle}{2} \right)^n \right], \tag{3.3}$$

where the superscript $y$ means that $V$ is acting on $y$ variable.

These means were introduced by de la Vallèe Poussin for Fourier series. They have been extended to various other series. For the Gegenbauer expansions (more generally, Jacobi expansions), they were studied in [2, 5] and they attracted considerable attention as the Bernstein-Durrmeyer polynomials in recent years. For the ordinary harmonic expansions, they were studied in [11, 4]. Several their properties are common in all cases. We list some properties below.

(1) The means $M_n(h_\kappa^2; f)$ are positive linear operators that map $\Pi_n^d$ to $\Pi_n^d$;

(2) The means $M_n(h_\kappa^2; \cdot)$ are self-adjoint with respect to the inner product

$$\langle f, g\rangle_\kappa = a_\kappa \int_{S^{d-1}} f(y) g(y) h_\kappa^2(y) d\omega;$$

(3) If $\Delta_{h,0} f \in L^1(h_\kappa^2)$, then $\Delta_{h,0} M_n(h_\kappa^2; f) = M_n(h_\kappa^2; \Delta_{h,0} f)$;

(4) For $f \in L^p(h_\kappa^2)$, $\|M_k(h_\kappa^2; f)\|_{\kappa,p} \le \|f\|_{\kappa,p}$ and $M_n(h_\kappa^2; f)$ converges to $f$ in norm.

That the means are positive follows from the positivity of the intertwining operator $V$ and (3.3). The first three properties are simple consequence of the definition. The inequality $\|M_k(h_\kappa^2; f)\|_{\kappa,p} \le \|f\|_{\kappa,p}$ follows from the fact that the means are positive and $M_n(h_\kappa^2; 1) = 1$. If $f$ is a polynomial of degree $m$, then $P_k(h_\kappa^2; f) = 0$ for $k > m$ and

$$M_n(h_\kappa^2; f) = \sum_{k=0}^{m} \mu_{k,n}^\lambda P_k(h_\kappa^2; f) \to \sum_{k=0}^{m} P_k(h_\kappa^2; f) = S_m(h_\kappa^2; f) = f, \quad n \to \infty,$$

since $\mu_{k,n}^\lambda - 1 = \mathcal{O}(n^{-1})$. Therefore, the triangle inequality gives the fourth property.

(5) For $0 \le k \le n$, $\quad \mu_{k,n}^\lambda - \mu_{k,n-1}^\lambda = \dfrac{k(k+2\lambda)}{n(n+2\lambda)} \mu_{k,n}^\lambda$;

(6) For all $f \in L^1(h_\kappa^2)$, $\quad M_n(h_\kappa^2; f) - f = \sum_{k=n+1}^{\infty} \dfrac{\Delta_{h,0} M_k(h_\kappa^2; f)}{k(k+2\lambda)}$.

The property (5) follows from the definition of $\mu_{k,n}^{\lambda}$, (6) is a consequence of (5) and (4) since the facts that $\Delta_{h,0}Y = -j(j+2\lambda)Y$ for all $Y \in \mathcal{H}_j^d(h_\kappa^2)$ and $M_n(h_\kappa^2;Y) = \mu_{j,n}^{\lambda}Y$, if $j \le n$, lead to

$$M_{k-1}(h_\kappa^2;f) - M_k(h_\kappa^2;f) = \frac{\Delta_{h,0}M_k(h_\kappa^2;f)}{k(k+2\lambda)}.$$

From these properties, the approximation behavior of the de la Vallée Poussin means can be easily described using a K-modulus. Define

$$D^p(h_\kappa^2) = \{f \in L^p(h_\kappa^2) : -k(k+2\lambda)P_k(h_\kappa^2;f) = P_k(h_\kappa^2;g) \quad \text{for some } g \in L^p(h_\kappa^2)\}.$$

If $\Delta_{h,0}f \in L^p(h_\kappa^2)$, then $f \in D^p(h_\kappa^2)$ since we can take $g = \Delta_{h,0}f$. The Peetre K-modulus between $L^p(h_\kappa^2)$ and $D^p(h_\kappa^2)$ is defined by

$$K(f,t;L^p,D^p)_\kappa := \inf\left\{\|f-g\|_{p,\kappa} + t\|\Delta_{h,0}\,g\|_{p,\kappa},\ g \in D^p(h_\kappa^2)\right\}.$$

**Theorem 3.1.** *Let $f \in L^p(h_\kappa^2)$. Then there are two constants $c_1$ and $c_2$ such that*

$$c_1 \max_{k\ge n}\|M_k(h_\kappa^2,f) - f\|_{\kappa,p} \le K(f,n^{-1};L^p,D^p)_\kappa \le c_2 \max_{k\ge n}\|M_k(h_\kappa^2,f) - f\|_{\kappa,p}.$$

*Proof.* For $g \in D^p(h_\kappa^2)$, the properties (3) – (5) show that

$$\|M_k(h_\kappa^2;g) - g\|_{\kappa,p} \le \sum_{k=n+1}^{\infty}\frac{\|M_k(h_\kappa^2;\Delta_{h,0}g)\|_{\kappa,p}}{k(k+2\lambda)} \le \delta_n\|\Delta_{h,0}g\|_{\kappa,p}, \tag{3.4}$$

where $\delta_n = \sum_{k=n+1}^{\infty}\frac{1}{k(k+2\lambda)} \sim 1/n$. Consequently, the triangle inequality

$$\|M_k(h_\kappa^2;f) - f\|_{\kappa,p} \le \|M_k(h_\kappa^2;f-g)\|_{\kappa,p} + \|M_k(h_\kappa^2;g) - g\|_{\kappa,p} + \|f-g\|_{\kappa,p}$$

gives the left hand side of the stated inequality. On the other hand, take

$$g_n = \frac{1}{\delta_n}\sum_{k=n+1}^{\infty}\frac{M_k(h_\kappa^2;f)}{k(k+2\lambda)}.$$

Then $\Delta_{h,0}\,g_n = (M_n(h_\kappa^2;f) - f)/\delta_n$ by the property (6) and $g_n \in D^p(h_\kappa^2)$. We have

$$\begin{aligned} K(f,\delta_n;L^p,D^p)_\kappa &\le \|f-g_n\|_{\kappa,p} + \delta_n\|\Delta_{h,0}\,g_n\|_{\kappa,p} \\ &\le \|M_k(h_\kappa^2;f) - f\|_{\kappa,p} + \sum_{k=n+1}^{\infty}\frac{\|M_k(h_\kappa^2;f)-f\|_{\kappa,p}}{k(k+2\lambda)}, \end{aligned}$$

from which the right hand side of the stated inequality follows. □

The properties we list above are essentially the same as the ordinary spherical harmonic (see [4]). The similarity ends as soon as we start to depart from these formal properties. In the case of ordinary harmonics, the kernel of the means $m_n(x,y)$ is given by the formula (3.3) with $V = id$, so that the de la Vallée Poussin means of the ordinary harmonic expansions take the form

$$\begin{aligned} M_n(f;x) &= \sigma_{d-1}^{-1}\int_{S_{d-1}} m_n(x,y)f(y)d\omega \\ &= \frac{\Gamma(1/2)}{2^{2\lambda}\Gamma(\lambda+1)}\frac{\Gamma(n+2\lambda+1)}{\Gamma(n+\lambda+1/2)}\int_{S^{d-1}}\Big(\frac{1+\langle x,y\rangle}{2}\Big)^n f(y)d\omega, \end{aligned}$$

where $\lambda = (d-2)/2$. The geodesic distance on $S^{d-1}$ is defined by $d(x,y) = \arccos\langle x,y\rangle$. It follows that $(1+\langle x,y\rangle)/2 = \cos^2(d(x,y)/2)$, so that the nonnegative kernel satisfies

$$\sigma_{d-1}^{-1}\int_{S^{d-1}} m_n(x,y)d\omega(y) = 1, \quad \text{and} \quad \lim_{n\to\infty} \sup_{0<\delta\le d(x,y)\le\pi} m_n(x,y) = 0.$$

These properties immediately imply that for $f \in L^1$, $M_n(f,x)$ converges to $f(x)$ for almost all $x \in S^{d-1}$ ([4]).

For the $h$-harmonics, however, the situation is quite different. Let us consider the case of $\mathbb{Z}_2^d$, that is, $h_\kappa$ given in (1.2). In this case, the kernel in (3.3) is given by

$$m_n(h_\kappa^2; x,y) = \frac{\Gamma(1/2)}{2^{2\lambda}\Gamma(\lambda+1)}\frac{\Gamma(n+2\lambda+1)}{\Gamma(n+\lambda+1/2)}$$
$$\times c_\kappa \int_{[-1,1]^d}\left(\frac{1+x_1y_1t_1+\ldots+x_dy_dt_d}{2}\right)^n \prod_{i=1}^d (1+t_i)(1-t_i^2)^{\kappa_i-1}dt.$$

The kernel is no longer a function of the form $\phi(\langle x,y\rangle)$.

**Theorem 3.2.** *If $f$ is invariant under $\mathbb{Z}_2^d$ and $f \in L^1(h_\kappa^2)$, then $M_n(h_\kappa^2; f,x)$ converges to $f(x)$ for almost all $x \in S^{d-1}$.*

*Proof.* We can assume that $x \in S_+^{d-1} := \{x \in S^{d-1} : x_1 \ge 0, \ldots, x_d \ge 0\}$. Since $f$ is invariant under $\mathbb{Z}_2^d$, we have

$$M_n(h_\kappa^2; f,x) = a_\kappa \int_{S_+^{d-1}} \sum_{\varepsilon\in\{-1,1\}^d} m_n(h_\kappa^2; x, (\varepsilon_1 y_1, \ldots, \varepsilon_d y_d)) f(y) h_\kappa^2(y) d\omega.$$

For $x \in \mathbb{R}^d$ define $\bar{x} = (|x_1|, \ldots, |x_d|)$. The definition of $m_n(h_\kappa^2)$ shows that the following estimate holds:

$$0 \le m_n(h_\kappa^2; x,y) \le \frac{\Gamma(1/2)}{2^{2\lambda}\Gamma(\lambda+1)}\frac{\Gamma(n+2\lambda+1)}{\Gamma(n+\lambda+1/2)}\Big(\frac{1+\langle\bar{x},\bar{y}\rangle}{2}\Big)^n,$$

which implies that the supremum of $m_n(x,y)$ over $0 < \delta \le d(\bar{x},\bar{y}) \le \pi$ converges to zero as $n \to \infty$. Since the right hand is invariant under sign changes, the estimate also holds for $\sum_\varepsilon m_n(h_\kappa^2; x, (\varepsilon_1 y_1, \ldots, \varepsilon_d y_d))$. Since $d(\bar{x},\bar{y}) = d(x,y)$ for $x,y \in S_+^{d-1}$, the stated convergence follows from these properties of the kernel. $\square$

By turning to functions that are invariant, we avoid dealing with the action of the underlying group. However, it is likely that the almost everywhere convergence holds for all functions in $L^1(h_\kappa^2)$, without the additional assumption of invariant. We do not see a way to prove such a result at this point. Another common proof of almost everywhere convergence is through the use of maximal function. The usual maximal function on the sphere is defined by

$$M_f(x) = \sup_{0<\theta<\pi} \frac{\int_{E(x,\theta)} |f(y)|d\omega(y)}{\int_{E(x,\theta)} d\omega(y)}, \qquad f \in L^1,$$

where $E(x,\theta) = \{y \in S^{d-1} : \langle x,y\rangle \ge \cos\theta\}$ is the spherical cap. See [4, 6] for the proof of almost everywhere convergence of the de la Vallée Poussin means and the Cesàro means, respectively, using the maximal function, in which the fact that the kernel functions are of the form $\phi(\langle x,y\rangle)$ plays an essential role. We do not know a proper definition of a maximal function for studying $h$-harmonic expansions, although replacing $d\omega$ by $h_\kappa^2(y)d\omega$ in the definition of $M_f$ gives a meaningful one from the point of view of weighted approximation.

Let us end this section with a more positive note. Let $E_n(f)_{\kappa,p}$ denote the best approximation of $f$ by spherical polynomials of degree $n$ in the $L^p(h_\kappa^2)$ metric,

$$E_n(f)_{\kappa,p} = \inf\{\|f-P\|_{\kappa,p} : P \in \Pi_n^d\}.$$

As a consequence of a Bernstein inequality proved in the next section, we follow the standard proof (cf. [4]) to conclude the following weak type estimate of the de la Vallèe Poussin means:

**Theorem 3.3.** *For $f \in L^p(h_\kappa^2)$ and $n \in \mathbb{N}_0$,*

$$\|M_n(h_\kappa^2; f) - f\|_{\kappa,p} \le \frac{c}{n+1}\sum_{k=0}^{\sqrt{n}}(k+1)E_k(f)_{\kappa,p}.$$

*Proof.* Let $P_n$ denote the spherical polynomial of degree $n$ that is a best approximation to $f$. The triangle inequality gives

$$\|M_n(h_\kappa^2; f) - f\|_{\kappa,p} \le 2E_{2^k}(f)_{\kappa,p} + \|M_n(h_\kappa^2; P_{2^k}) - P_{2^k}\|_{\kappa,p}.$$

Let $P_{-1} = 0$ and write $P_{2^k} = \sum_{j=0}^k (P_{2^j} - P_{2^{j-1}})$. Then by (3.4) and the Bernstein type inequality to be proved in Theorem 5.1,

$$\|M_n(h_\kappa^2; P_{2^k}) - P_{2^k}\|_{\kappa,p} \le \frac{c}{n+1}\|\Delta_{h,0}P_{2^k}\|_{\kappa,p} \le \frac{c}{n+1}\sum_{j=0}^k \|\Delta_{h,0}(P_{2^j} - P_{2^{j-1}})\|_{\kappa,p}$$

$$\le \frac{c}{n+1}\sum_{j=0}^k 2^{2j}\|P_{2^j} - P_{2^{j-1}}\|_{\kappa,p} \le \frac{c}{n+1}\Big[\sum_{j=0}^k 2^{2j}\|E_{2^j}(f)_{\kappa,p}\|_{\kappa,p} + \|f\|_{\kappa,p}\Big].$$

Setting $k$ such that $2^{2(k-1)} < n \le 2^{2k}$ gives the stated result. □

## 4. Spherical means

Let us return to Theorem 3.1. When $\kappa = 0$, we have $\Delta_h = \Delta$ and the K-modulus becomes

$$K(f, t; L^p, D^p) := \inf\left\{\|f-g\|_p + t\|\Delta_0\, g\|_p,\ g \in D^p\right\},$$

where $\Delta_0$ is the ordinary spherical Laplacian. This K-modulus is known to be equivalent to a modulus of smoothness defined as follows. Let $T_\theta$ denote the spherical means of $f$

$$T_\theta f(x) = \frac{1}{\sigma_{d-2}(\sin\theta)^{2\lambda}}\int_{\langle x,y\rangle = \cos\theta} f(y)d\omega(y). \tag{4.1}$$

The properties of the spherical means are well-known; see [3, 14], for example. Another expression for this means is

$$T_\theta f(x) = \frac{1}{\sigma_{d-2}}\int_{S_x^\perp} f(x\cos\theta + \sin\theta u)d\omega(u),$$

in which $S_x^\perp$ denote the equator in $S^{d-1}$ with respect to $x$, $S_x^\perp = \{y \in S^{d-1} : \langle x, y\rangle = 0\}$, it is isomorphic to the sphere $S^{d-2}$. The spherical modulus of smoothness is defined by

$$\omega(f,\delta)_p := \sup_{0<\theta<\delta}\|T_\theta f - f\|_p, \qquad f \in L^p.$$

Since $\lim_{\theta\to 0}\|T_\theta f - f\|_p = 0$, this modulus is meaningful. Moreover, it is known ([14]) that for each $f \in L^p$ and $0 < \delta < \pi$,

$$c_1\omega(f,\delta)_p \le K(f,t;L^p,D^p) \le c_2\omega(f,\delta)_p$$

for some constants $c_1$ and $c_2$, so that the two moduli are equivalent. In particular, in the analog of Theorem 3.1, the approximation behavior of the de la Vallée Poussin means for the ordinary harmonics can be described by this modulus of smoothness.

We will define a modulus of smoothness equivalent to $K(f,t;L^p,D^p)_\kappa$. One might consider a weighted spherical mean such as

$$T_\theta(h_\kappa^2;f,x) = \frac{\int_{\langle x,y\rangle=\cos\theta} f(y)h_\kappa^2(y)d\omega(y)}{\int_{\langle x,y\rangle=\cos\theta} h_\kappa^2(y)d\omega(y)}.$$

Upon writing the mean as an integral over $S^{d-1}$ by introducing a kernel function, it is easy to see that this mean satisfies the property

$$\|T_\theta(h_\kappa^2;f)\|_{\kappa,p} \le \|f\|_{\kappa,p} \quad \text{and} \quad \lim_{\theta\to 0}\|T_\theta(h_\kappa^2;f) - f\|_{\kappa,p} = 0, \qquad \forall f \in L^p(h_\kappa^2).$$

However, the definition clearly depends on the relative position of $x$ but does not take into the consideration of the reflection symmetry. We will define our spherical mean, denoted by $T_\theta^\kappa f$, in an indirected way:

**Definition 4.1.** *Let $\lambda = \gamma_\kappa + \frac{d-2}{2}$. For $0 \le \theta \le \pi$, the means $T_\theta^\kappa$ is defined by*

$$c_\lambda \int_0^\pi T_\theta^\kappa(f,x)g(\cos\theta)(\sin\theta)^{2\lambda}d\theta = a_\kappa \int_{S^{d-1}} f(y)Vg(\langle x,y\rangle)h_\kappa^2(y)d\omega(y),$$

*where $g$ is any function $[-1,1] \mapsto \mathbb{R}$ such that the integral in the right hand side is finite, $V$ is the intertwining operator, $c_\lambda$ is defined as in (3.2) and $a_\kappa$ is the normalization constant of $h_\kappa^2 d\omega$.*

To see that $T_\theta^\kappa$ is indeed an extension of the ordinary spherical means $T_\theta$, recall the integral relation

$$\int_{S^{d-1}} f(y)d\omega(y) = \int_0^\pi \int_{S^{d-2}} f(\cos\theta, \sin\theta x')d\omega(x')(\sin\theta)^{d-2}d\theta;$$

If we choose a coordinate system so that $x$ becomes the north pole and let $y$ have the spherical coordinates in this system, then the above relation shows that

$$\begin{aligned}\int_{S^{d-1}} f(y)g(\langle x,y\rangle)d\omega(y) &= \int_0^\pi g(\cos\theta)\int_{S_x^\perp} f(x\cos\theta + \sin\theta u)d\omega(u)(\sin\theta)^{d-2}d\theta \\ &= \sigma_{d-2}\int_0^\pi g(\cos\theta)T_\theta f(x)(\sin\theta)^{d-2}d\theta.\end{aligned}$$

Since $V = id$ when $\kappa = 0$, comparing with the equation in the Definition 4.1 shows that $T_\theta^0 f = T_\theta f$. Hence, we can consider $T_\theta^\kappa f$ as a proper extension of $T_\theta f$.

In order to study the properties of the $T_\theta^\kappa f$, we will need the following analog of Funk-Hecke formula for $h$-harmonics proved in [25].

**Lemma 4.2.** *Let $f$ be a continuous function on $[-1,1]$. Let $Y^h \in \mathcal{H}_n^d(h_\kappa^2)$. Then*

$$\int_{S^{d-1}} Vf(\langle \mathbf{x},\cdot\rangle)(\mathbf{y})Y^h(\mathbf{y})h_\kappa^2(\mathbf{y})d\omega = \mu_n(f)Y^h(\mathbf{x}), \qquad x \in S^{d-1},$$

*where* $\lambda = \gamma_\kappa + \frac{d-2}{2}$ *and* $\mu_n(f)$ *is defined by*

$$\mu_n(f) = c_\lambda \frac{1}{C_n^\lambda(1)} \int_{-1}^{1} f(t) C_n^\lambda(t)(1-t^2)^{\lambda-\frac{1}{2}} dt.$$

Recall that $Y_n(h_\kappa^2; f)$ denote the projection of $f$ to $\mathcal{H}_n^d(h_\kappa^2)$. The properties of $T_\theta^\kappa f$ are given in the following two propositions.

**Proposition 4.3.** *The means* $T_\theta^\kappa f$ *satisfy the following properties:*

(1) *Let* $f_0(x) = 1$, *then* $T_\theta^\kappa f_0(x) = 1$.

(2) *For* $f \in L^1(h_\kappa^2)$,

$$Y_n(h_\kappa^2; T_\theta^\kappa f) = \frac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)} Y_n(h_\kappa^2; f);$$

*in particular,* $\Delta_{h,0} T_\theta^\kappa f = T_\theta^\kappa(\Delta_{h,0} f)$ *if* $\Delta_{h,0} f \in L^1(h_\kappa^2)$.

(3) $T_\theta^\kappa : \Pi_n^d|_{S^{d-1}} \mapsto \Pi_n^d|_{S^{d-1}}$ *and*

$$T_\theta^\kappa f \sim \sum_{n=0}^{\infty} \frac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)} Y_n(h_\kappa^2; f).$$

(4) *For* $0 \le \theta \le \pi$,

$$T_\theta^\kappa f - f = \int_0^\theta (\sin s)^{-2\lambda} ds \int_0^s T_t^\kappa(\Delta_{h,0} f)(\sin t)^{2\lambda} dt.$$

*Proof.* Theorem 2.1 shows that

$$a_\kappa \int_{S^{d-1}} V g(\langle x, y\rangle) h_\kappa^2(y) d\omega(y) = c_\lambda \int_0^\pi g(\cos\theta)(\sin\theta)^{2\lambda} d\theta,$$

from which (1) follows. Let $Y^h \in \mathcal{H}_n^d(h_\kappa^2)$. Denote by $\langle f, Y^h\rangle_\kappa$ the Fourier coefficient of $f$ with respect to $Y^h$, By the Funk-Hecke formula,

$$\begin{aligned}
&\int_{S^{d-1}} \Big[a_\kappa \int_{S^{d-1}} f(y) V g(\langle x, y\rangle) h_\kappa^2(y) d\omega(y)\Big] Y^h(x) h_\kappa^2(x) d\omega(x) \\
&= \int_{S^{d-1}} f(y) \Big[a_\kappa \int_{S^{d-1}} V g(\langle x, y\rangle) Y^h(x) h_\kappa^2(x) d\omega(x)\Big] h_\kappa^2(y) d\omega(y) \\
&= \int_{S^{d-1}} f(y) Y^h(y) h_\kappa^2(y) d\omega(y) c_\lambda \int_{-1}^{1} g(t) \frac{C_n^\lambda(t)}{C_n^\lambda(1)} (1-t^2)^{\lambda-\frac{1}{2}} dt \\
&= \langle f, Y^h\rangle_\kappa c_\lambda \int_0^\pi g(\cos\theta) \frac{C_n^\lambda(\theta)}{C_n^\lambda(1)} (\sin\theta)^{2\lambda} d\theta.
\end{aligned}$$

By the definition of $T_\theta^\kappa f$, this shows that the Fourier coefficient of $T_\theta^\kappa f$ with respect to $Y^h$ satisfies $\langle T_\theta^\kappa f, Y^h\rangle_\kappa = \langle f, Y^h\rangle_\kappa C_n^\lambda(\theta)/C_n^\lambda(1)$, which gives (2). Moreover, this equation shows that if $f$ is a spherical polynomial of degree $n$, then so is $T_\theta^\kappa f$, from which (3) follows.

Let $I_\theta(\Delta_{h,0} f)$ denote the right hand side of the formula in (4). A moment reflection shows that we only need to establish the formula for polynomials, which further reduces to show that

$$P(h_\kappa^2; T_\theta^\kappa f - f) = P(h_\kappa^2; I_\theta(\Delta_{h,0} f))$$

for $f \in \mathcal{H}_n^d(h_\kappa^2)$. Since $\Delta_{h,0} f = -n(n+2\lambda) f$ for $f \in \mathcal{H}_n^d(h_\kappa^2)$, the property (2) shows that the above equation is equivalent to

$$\frac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)} - 1 = -n(n+2\lambda) \int_0^\theta (\sin s)^{-2\lambda} ds \int_0^s \frac{C_n^\lambda(\cos t)}{C_n^\lambda(1)} (\sin t)^{2\lambda} dt.$$

Taking derivatives in both side shows that this is equivalent to the differential equation satisfied by the Gegenbauer polynomial $C_n^\lambda$ ([19, p. 80]). □

For ordinary spherical means ($\kappa = 0$), these properties are proved in [3, 14]. The property (3) clearly shows that $\|T_\theta^\kappa f\|_{\kappa,2} = \|f\|_{\kappa,2}$. For $p \ne 2$, we consider $h_\kappa^2$ in (1.2) associated with $\mathbb{Z}_2^d$.

**Proposition 4.4.** *Let $h_\kappa^2$ be defined in* (1.2). *For $f \in L^p(h_\kappa^2)$, $1 \le p < \infty$, or $f \in C(S^{d-1})$,*

$$\|T_\theta^\kappa f\|_{\kappa,p} \le \|f\|_{\kappa,p} \qquad \text{and} \qquad \lim_{\theta\to 0} \|T_\theta^\kappa f - f\|_{\kappa,p} = 0.$$

*Proof.* We start with a simple relation that comes from a change of variable,

$$\begin{aligned}\int_{-1}^1 g(a+bt)(1+t)(1-t^2)^{\mu-1} dt &= \int_{-1}^1 g(a+|b|t)(1+\operatorname{sign}\{b\}t)(1-t^2)^{\mu-1} dt \\ &= \int_{-1}^1 g(z) k(a,b,z)(1-z^2)^{\lambda-\frac12} dz,\end{aligned}$$

where $k(a,b,z)$ is defined by

$$k(a,b,z) = \Big(1 - \frac{(z-a)^2}{|b|^2}\Big)^{\mu-1} \Big(1 + \frac{z-a}{|b|}\Big) \frac{1}{(1-z^2)^{\lambda-\frac12}}$$

for $|z-a| \le |b|$ and $k(a,b,z) = 0$ otherwise. We note that $k(a,b,z)$ is nonnegative. Recall the formula of $V$ in (2.3); we get with $\mu = \kappa_d$ that

$$Vg(\langle x,y\rangle) = \int_{-1}^1 g(z) K(x,y,z)(1-z^2)^{\lambda-\frac12} dz$$

where the kernel $K$ is defined by

$$K(x,y,z) = c_\kappa \int_{[-1,1]^{d-1}} k(x_1y_1 + \ldots + x_{d-1}y_{d-1}, x_dy_d, z) \prod_{i=1}^{d-1} (1+t_i)(1-t_i^2)^{\kappa_d - 1} dt.$$

We note that $K(x,y,z)$ is nonnegative and it is symmetric with respect to $x$ and $y$. This gives, in particular,

$$\begin{aligned}&a_\kappa \int_{S^{d-1}} f(y) Vg(\langle x,y\rangle) h_\kappa^2(y) d\omega(y) \\ &\qquad = \int_{-1}^1 g(z) \Big[\int_{S^{d-1}} f(y) K(x,y,z) h_\kappa^2(y) d\omega(y)\Big] (1-z^2)^{\lambda-1/2} dz,\end{aligned}$$

which shows that, by Definition 4.1, we can write

$$T_\theta^\kappa f(x) = \int_{S^{d-1}} f(y) K(x,y,\cos\theta) h_\kappa^2(y) d\omega(y).$$

Setting $f(x) \equiv 1$ shows that the integral of $K(x,y,\cos\theta)$ is 1. Hence, using the fact that $K(x,y,z)$ is nonnegative, it follows that $\|T_\theta^\kappa f\|_\infty \le \|f\|_\infty$. Furthermore, we have

$$\begin{aligned}\|T_\theta^\kappa f\|_{\kappa,1} &= \int_{S^{d-1}} \Big| \int_{S^{d-1}} f(y) K(x,y,\cos\theta) h_\kappa^2(y) d\omega(y) \Big| h_\kappa^2(x) d\omega(x) \\ &\le \int_{S^{d-1}} |f(y)| \Big[ \int_{S^{d-1}} K(x,y,\cos\theta) h_\kappa^2(x) d\omega(x) \Big] h_\kappa^2(y) d\omega(y),\end{aligned}$$

which shows, since $K(x,y,z)$ is symmetric in $x$ and $y$, that $\|T_\theta^\kappa f\|_{\kappa,1} \le \|f\|_{\kappa,1}$. The Riesz interpolation theorem gives the inequality for $1 < p < \infty$.

The norm inequality just proved and the triangle inequality gives that

$$\|T_\theta^\kappa f - f\|_{\kappa,p} \le 2\|f - P\|_{\kappa,p} + \|T_\theta^\kappa P - P\|_{\kappa,p}$$

for any polynomial $P$. The property (3) in Proposition 4.3 leads to

$$T_\theta^\kappa P - P = \Big( \frac{C_n^\lambda(\cos\theta)}{C_n^\lambda(1)} - 1 \Big) P, \qquad P \in \Pi_n^d.$$

This shows that $T_\theta^\kappa P - P$ converges to 0 as $\theta \to 0$, from which the convergence of $\|T_\theta^\kappa f - f\|_{\kappa,p}$ follows. $\square$

This proposition shows that the following definition of "modulus of smoothness" is meaningful:

**Definition 4.5.** *For $f \in L^p(h_\kappa^2)$, $1 \le p < \infty$, or $f \in C(S^{d-1})$, define*

$$\omega(f,\delta)_{\kappa,p} = \sup_{0<\theta\le\delta} \|T_\theta^\kappa f - f\|_{\kappa,p}.$$

Recall the definition of K-modulus $K(f,\delta^2; L^p, D^p)_\kappa$. We show that these two moduli are equivalent.

**Theorem 4.6.** *Let $h_\kappa^2$ be defined in* (1.2). *For $0 < \delta < \pi/2$, there exist constants $c_1$ and $c_2$ such that*

$$c_1 \omega(f,\delta)_{\kappa,p} \le K(f,\delta^2; L^p, D^p)_\kappa \le c_2 \omega(f,\delta)_{\kappa,p}.$$

*Proof.* For $g \in D^p(h_\kappa^2)$, the property (4) in Proposition 4.3 and Proposition 4.4 show that

$$\begin{aligned}\|T_\theta^\kappa g - g\|_{\kappa,p} &\le \int_0^\theta (\sin s)^{-2\lambda} ds \int_0^s \|T_t^\kappa(\Delta_{h,0} g)\|_{\kappa,p} (\sin t)^{2\lambda} dt \\ &\le \|\Delta_{h,0} g\|_{\kappa,p} \int_0^\theta (\sin s)^{-2\lambda} ds \int_0^s (\sin t)^{2\lambda} dt \\ &\le c\,\theta^2 \|\Delta_{h,0} g\|_{\kappa,p}.\end{aligned}$$

The triangle inequality and Proposition 4.4 then gives

$$\|T_\theta^\kappa f - f\|_{\kappa,p} \le 2\|f - g\|_{\kappa,p} + c\,\theta^2 \|\Delta_{h,0} g\|_{\kappa,p},$$

which shows that $\omega(f,\delta)_{\kappa,p} \le c K(f,\delta^2; L^p, D^p)_\kappa$. On the other hand, define

$$g(x) = v_\theta \int_0^\theta (\sin s)^{-2\lambda} ds \int_0^s T_t^\kappa f(x) (\sin t)^{2\lambda} dt$$

with $v_\theta^{-1} = \int_0^\theta (\sin s)^{-2\lambda} ds \int_0^s (\sin t)^{2\lambda} dt$. Then $\Delta_{h,0}\, g = v_\theta (T_\theta^\kappa f - f)$ by Proposition 4.3 so that $g \in D^p(h_\kappa^2)$. This also gives

$$\|\Delta_{h,0}\, g\|_{\kappa,p} = v_\theta \|T_\theta^\kappa f - f\|_{\kappa,p} \le c\,\theta^{-2} \|T_\theta^\kappa f - f\|_{\kappa,p},$$

since for $\theta \le \pi/2$, the inequality $(2/\pi)\theta \le \sin\theta \le \theta$ shows that $v_\theta^{-1} \sim \theta^2$. Moreover,

$$f - g = v_\theta^{-1} \int_0^\theta (\sin s)^{-2\lambda} ds \int_0^s (T_t^\kappa f - f)(\sin t)^{2\lambda} dt,$$

from which $\|f-g\|_{\kappa,p} \le \omega(f,\delta)_{\kappa,p}$ follows. Together, these give the other direction of the stated inequality. □

As a consequence of this theorem, we can replace the K-modulus in Theorem 3.1 by $\omega(f;\delta)_{\kappa,p}$. Moreover, we can use the equivalence to derive properties of the weighted modulus of smoothness. As an example, we give the following corollary:

**Corollary 4.7.** *There is a constant $c$ such that for $t > 0$,*

$$\omega(f, t\delta)_{\kappa,p} \le c \max\{t^2, 1\} \omega(f,\delta)_{\kappa,p}.$$

This property is not obvious from the definition of $\omega(f,\delta)_{\kappa,p}$, but it is evident for the K-modulus and the equivalence shows that it holds for the modulus of smoothness. The theorem and the corollary justify the definition of the weighted modulus of smoothness. However, unlike the ordinary modulus of smoothness $\omega(f,t)_p$, the lack of explicit formula such as (4.1) means that $\omega(f,t)_{\kappa,p}$ is not entirely satisfactory for describing the smoothness of the function.

## 5. A Bernstein type inequality

The main result of this section is the following Bernstein type inequality, already used in the proof of Theorem 3.3:

**Theorem 5.1.** *For $1 \le p \le \infty$ there is a constant $c$ such that*

$$\|\Delta_{h,0} P\|_{\kappa,p} \le c n^2 \|P\|_{\kappa,p}, \qquad P \in \Pi_n^d.$$

In the case of ordinary Laplacian, such an inequality is proved in [14]. In the following we use $c$ to denote a generic constant, which depends only on $d$, $p$ and $\kappa$ and whose value may be different from line to line.

Just as in the case of the ordinary harmonics, the spherical $h$-Laplacian is applied to functions defined on $S^{d-1}$. This operator can also be defined by the property

$$\Delta_{h,0} f(x) = \Delta_h f\Big(\frac{y}{\|y\|}\Big)\Big|_{y=x}, \qquad x \in S^{d-1}.$$

The proof of Theorem 5.1 will depend on several lemmas. First we need the following analog of another de la Vallée Poussin operator defined using the Cesàro means. For $\delta > 0$, the Cesàro $(C,\delta)$ means, $s_n^\delta$, of a sequence $\{s_n\}$ are defined by

$$s_n^\delta = \frac{1}{\binom{n+\delta}{n}} \sum_{k=0}^n \binom{n-k+\delta-1}{n-k} s_k = \frac{1}{\binom{n+\delta}{n}} \sum_{k=0}^n \binom{n-k+\delta}{n-k} c_k,$$

where the second equality holds if $s_n$ is the $n$-th partial sum of the series $\sum_{k=0}^\infty c_k$. We denote the $n$-th $(C,\delta)$ means of the $h$-harmonic expansion by $S_n^\delta(h_\kappa^2; f)$. The definition shows that $S_n^0(h_\kappa^2; f)$ is the $n$-th partial sum $S_n(h_\kappa^2; f)$. Then the other de la Vallée Poussin type operator is defined by

$$L_{n,m}(h_\kappa^2; f) = \frac{1}{n^m} \sum_{j=0}^m (2^j n)_m \prod_{i=0, i\ne j}^m \frac{1}{2^j - 2^i} \, S_{2^j n-1}^m (h_\kappa^2; f),$$

where $(a)_m = a(a-1)\dots(a-m+1)$. When $m = 1$, $L_{n,1} = 2S_{2n-1} - S_{n-1}$ is the de la Vallée Poussin mean in the classical Fourier analysis. Recall that $E_n(f)_{\kappa,p}$ denote the best approximation of $f$ from $\Pi_n^d$ in the $\|\cdot\|_{\kappa,p}$ norm. The means $L_{n,m}(h_\kappa^2;f)$ satisfy the following:

**Lemma 5.2.** *For all $P \in \Pi_{n-1}^d$, $L_{n,m}(h_\kappa^2;P) = P$. Moreover, let $m = [2\gamma_\kappa + d - 1] + 1$; then for $1 \le p \le \infty$,*

$$\|L_{n,m}(h_\kappa^2;f,x) - f(x)\|_{\kappa,p} \le (1 + B_m)E_{n-1}(f)_{\kappa,p},$$

*where $B_m = 1 + \sum_{j=0}^m \left[(2^j+1)^m / \prod_{i=0,i\ne j}^m |2^j - 2^i|\right]$.*

The means $L_{n,m}$ are defined so that $L_n^m(h_\kappa^2;P) = P$ for $P \in \Pi_{n-1}$. It is proved in [22] that $S_n^m(h_\kappa^2;f)$ is nonnegative, the integral of $|S_n^m(h_\kappa^2;f)|$ with respect to $h_\kappa^2 d\omega$ on $S^{d-1}$ is 1; thus it has norm 1 in the $L^p(h_\kappa^2)$ metric. The stated inequality follows from the triangular inequality just as in [9, p. 262]. From the definition of $S_n^\delta$, we can write $L_{n,m}$ as an integral operator

$$L_{n,m}(h_\kappa^2;f,x) = a_\kappa \int_{S^{d-1}} f(y) L_{n,m}(h_\kappa^2;x,y) h_\kappa^2(y)\, d\omega(y),$$

where $L_{n,m}(h_\kappa;x,y)$ is the kernel function. Using the compact formula of $P_n(h_\kappa^2;x,y)$, we can write

$$L_{n,m}(h_\kappa^2;x,y) = V[L_{n,m}(w_\lambda;\langle x,y\rangle)], \qquad \lambda = \gamma_\kappa + \frac{d-2}{2}, \tag{5.1}$$

where $w_\lambda$ is as in (3.2) and $L_{n,m}(w_\lambda;t)$ is the de la Vallée Poussin type means of the sequence $\frac{k+\lambda}{\lambda} C_n^\lambda(t)$; in other words, if $L_{n,m}(w_\lambda;g)$ is the $n$-th de la Vallée Poussin type mean built upon the Cesàro means of the Gegenbauer expansion of $g : [-1,1] \mapsto \mathbb{R}$, then

$$L_{n,m}(w_\lambda;g,t) = \int_{-1}^1 L_{n,m}(w_\lambda;s,t) g(s) w_\lambda(s)\, ds,$$

and $L_{n,m}(w_\lambda;t) = L_{n,m}(w_\lambda;1,t)$. Since the Cesàro means of the Gegenbauer expansion with respect to $w_\lambda$ is nonnegative if $\delta > 2\lambda + 1$, it follows just as the statement of Lemma 5.2 that the means $L_{n,m}(w_\lambda;g)$ are uniformly bounded. In particular, it follows that

$$\int_{-1}^1 |L_{n,m}(w_\lambda;t)| w_\lambda(t)\, dt \le c.$$

We will need a Bernstein-Markov type inequality for the derivatives of polynomials of one variable. The inequality involves the differential operator associated with the Gegenbauer polynomials. Let $\Lambda_\lambda$ be the differential operator defined by

$$\Lambda_\lambda f(t) = w_\lambda^{-1}(t)\left[(1-t^2) w_\lambda(t) f'(t)\right]'.$$

Then the Gegenbauer polynomials are eigenfunctions of $\Lambda_\lambda$; more precisely, the polynomials $C_n^\lambda$ satisfy the equation $\Lambda_\lambda f = -n(n+2\lambda) f$ ([19, p. 80]).

**Lemma 5.3.** *There is a constant $c$ such that for given $n \in \mathbb{N}$, the inequality*

$$\int_{-1}^1 |\Lambda_\lambda P(t)| w_\lambda(t)\, dt \le c\, n^2 \int_{-1}^1 |P(t)| w_\lambda(t)\, dt$$

*holds for every polynomial $P$ of degree at most $n$.*

This lemma appears in [17], it also follows from the usual Bernstein-Markov inequality

$$\int_{-1}^1 |P'(t)| \sqrt{1-t^2}\, w_\lambda(t)\, dt \le c\, n \int_{-1}^1 |P(t)| w_\lambda(t)\, dt$$

and [5, Lemma 3]. The second lemma gives the effect of $\Delta_{h,0}$ on the reproducing kernel.

**Lemma 5.4.** *Let* $\lambda = \gamma_k + (d-2)/2$. *Then for each fixed* $x$,

$$\Delta_{h,0} P_n(h_\kappa^2; x, y) = V^{(y)} \Lambda_\lambda P_n(w_\lambda; \langle x, y\rangle),$$

*where* $P_n(w_\lambda; t) = ((n+\lambda)/\lambda) C_n^\lambda(t)$.

*Proof.* As a function of $y$, $P_k(h_\kappa^2; x, y)$ is in $\mathcal{H}_k^d(h_\kappa^2)$ for each fixed $x$; hence, by (2.1), it follows that

$$\begin{aligned}
\Delta_{h,0} P_n(h_\kappa^2; x, y) &= -n(n + 2\gamma_\kappa + d - 2) P_n(h_\kappa^2; x, y) \\
&= -n(n+2\lambda) \frac{n+\lambda}{\lambda} V\left[C_n^\lambda(\langle \cdot, y\rangle)\right](y) \\
&= \frac{n+\lambda}{\lambda} V\left[\Lambda_\lambda C_n^\lambda(\langle \cdot, y\rangle)\right](y) = V^{(y)} \Lambda_\lambda P_n(w_\lambda; \langle x, y\rangle),
\end{aligned}$$

where we have used the fact that $\Lambda_\lambda C_n^\lambda = (n+2\lambda) C_n^\lambda$. ☐

*Proof of Theorem 3.1.* Let $m$ be as in the Lemma 5.2. Let $P$ be a polynomial of degree at most $n$. Then it follows from Lemma 5.2 that

$$\begin{aligned}
\Delta_{h,0} P(x) = L_{n,m}(h_\kappa^2; \Delta_{h,0} P, x) &= a_\kappa \int_{S^{d-1}} L_{n,m}(h_\kappa^2; x, y) \Delta_{h,0} P(y) h_\kappa^2(y) d\omega(y) \\
&= a_\kappa \int_{S^{d-1}} \Delta_{h,0}^{(y)} L_{n,m}(h_\kappa^2; x, y) P(y) h_\kappa^2(y) d\omega(y),
\end{aligned}$$

where the last equal sign follows from the fact that $\Delta_{h,0}$ is symmetric with respect to the integral against $h_\kappa^2 d\omega$; that is, $\langle \Delta_{h,0} f, g\rangle_\kappa = \langle f, \Delta_{h,0} g\rangle_\kappa$ (considering $P \in \mathcal{H}_k^d(h_\kappa^2)$ and using the fact that $\Delta_{h,0} P = -k(k+d+2\gamma_\kappa - 2)P$). It follows that

$$\begin{aligned}
&\int_{S^{d-1}} |\Delta_{h,0} P(x)| h_\kappa^2(x) d\omega \\
&\qquad \le a_\kappa \int_{S^{d-1}} |P(x)| \left[\int_{S^{d-1}} |\Delta_{h,0}^{(y)} L_{n,m}(h_\kappa^2; x, y)| h_\kappa^2(y) d\omega(y)\right] h_\kappa^2(x) d\omega(x).
\end{aligned}$$

Denote the integral inside the square bracket by $J_n(x)$; we show that it is uniformly bounded by a constant multiple of $n^2$ for $x \in S^{d-1}$. This would prove the Bernstein type inequality for $p = 1$. Moreover, it would also prove the inequality for $p = \infty$, and the case $1 < p < \infty$ would then follow from the Riesz interpolation theorem.

We now prove the inequality $|J_n(x)| \le c n^2$. From Lemma 5.4 it follows that

$$\Delta_{h,0}^{(y)} L_{n,m}(h_\kappa^2; x, y) = V^{(y)} \Lambda_\lambda L_{n,m}(w_\lambda; \langle x, y\rangle).$$

Using the positivity of the intertwining operator and Theorem 2.1 we get

$$\begin{aligned}
|J_n(x)| &= \int_{S^{d-1}} |V^{(y)} \Lambda_\lambda L_{n,m}(w_\lambda; \langle x, y\rangle)| h_\kappa^2(y) d\omega(y) \\
&\le \int_{S^{d-1}} V^{(y)}\left[\, |\Lambda_\lambda L_{n,m}(w_\lambda; \langle x, y\rangle)| \,\right] h_\kappa^2(y) d\omega(y) \\
&= B_\kappa \int_{-1}^{1} |\Lambda_\lambda L_{n,m}(w_\lambda; s)| (1-s^2)^{\gamma_\kappa + \frac{d-3}{2}} ds.
\end{aligned}$$

Consequently, recall that $\lambda = \gamma_\kappa + (d-2)/2$, we get

$$|J_n(x)| \le c n^2 \int_{-1}^{1} |L_{n,m}(w_\lambda; t)| w_\lambda(t) dt \le c n^2,$$

by Lemma 5.3. □

## 6. Further remarks

Although we only considered approximation by $h$-harmonic polynomials on the sphere, it should be pointed out that there is a close relation between $h$-spherical harmonics on $S^{d+1}$ and orthogonal polynomials on the unit ball $B^d$ of $\mathbb{R}^d$ ([23, 27]). In fact, many results for orthogonal expansions on $B^d$ follow from those of the $h$-spherical harmonic expansions; for example, the results with respect to the weight function

$$W^B_{\kappa,\mu}(x) = \prod_{i=1}^{d} |x_i|^{2\kappa_i}(1-\|x\|^2)^{\mu-1/2}, \qquad \kappa_i \ge 0, \quad \mu \ge 0,$$

follow from those for $h_\kappa(x) = \prod_{i=1}^{d+1} |x_i|^{\kappa_i}$ with $\mu = \kappa_{d+1}$ ([26, 12]). Furthermore, there is also a relation between $h$-harmonics and orthogonal polynomials on the simplex $T^d = \{x \in \mathbb{R}^d : x_1 \ge 0, \dots, x_d \ge 0, 1 - x_1 - \dots - x_d \ge 0\}$ ([24, 27]). In particular, the orthogonal polynomials with respect to the weight function

$$W_\kappa(x) = x_1^{\kappa_1 - 1/2} \cdots x_d^{\kappa_d - 1/2}(1 - x_1 - \dots - x_d)^{\kappa_d - 1/2}, \qquad \kappa_i \ge 0,$$

are related to the $h$-harmonics associated to $h_\kappa(x) = \prod_{i=1}^{d+1} |x_i|^{\kappa_i}$. For orthogonal expansions with respect to $W_\kappa$, see [12].

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