POLYNOMIAL INTERPOLATION ON THE UNIT SPHERE AND ON THE UNIT BALL

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ABSTRACT. The problem of interpolation on the unit sphere S^d by spherical polynomials of degree at most n is shown to be related to the interpolation on the unit ball B^d by polynomials of degree n. As a consequence several explicit sets of points on S^d are given for which the interpolation by spherical polynomials has a unique solution. We also discuss interpolation on the unit disc of $\mathbb{R}^2$ for which points are located on the circles and each circle has an even number of points. The problem is shown to be related to interpolation on the triangle in a natural way.

1. INTRODUCTION

In this paper we consider polynomial interpolation on the unit sphere $S^d = \{x : \|x\| = 1\}$ of $\mathbb{R}^{d+1}$ and on the unit ball $B^d = \{x : \|x\| \le 1\}$ of $\mathbb{R}^d$, where $\|x\|$ is the Euclidean norm in the corresponding space. One of the results shows that these two are in fact closely related.

Let Π_n^d denote the space of polynomials of degree at most n in d variables and let $\mathcal{P}_n^d$ denote the space of homogeneous polynomials of degree n in d variables. The notation $\Pi_n(S^d)$ denote the spherical polynomials of $d+1$ variables, which are the restriction of polynomials in Π_n^{d+1} on S^d. It is known that

$$\dim \Pi_0(S^d) = 1 \quad \text{and} \quad \dim \Pi_n(S^d) = \binom{n+d}{d} + \binom{n+d-1}{d}, \qquad n \ge 1.$$

Let $N_n^d = \dim \Pi_n(S^d)$. We study the following polynomial interpolation problem on S^d:

Problem 1. Let $X = \{\mathbf{a}_i : 1 \le i \le N_n^d\}$ be a set of distinct points on S^d. Find conditions on X such that there is a unique polynomial $S \in \Pi_n(S^d)$ satisfying

$$S(\mathbf{a}_i) = f_i, \qquad \mathbf{a}_i \in X, \quad 1 \le i \le N_n^d,$$

for any given data $\{f_i\}$.

If there is a *unique* solution to the interpolation problem, we say that the problem is *poised* and that X solves Problem 1. It is known that the interpolation problem is poised for almost all choices of X. On the other hand, in order to show that a set of given points is poised amounts to check that the determinant of a linear system of equations is nonzero, which is often difficult. The question is often formulated as

Date: December 28, 2001, revised June 29, 2002.

1991 *Mathematics Subject Classification.* 41A05, 41A63, 65D05.

Key words and phrases. Interpolation by polynomials, unit sphere, unit ball.

Work partially supported by the National Science Foundation under Grant DMS-0201669.

looking for formations of points on the sphere for which the interpolation problem is poised.

It is known that $\dim \Pi_n^d = \binom{n+d}{n}$. Let $M_n^d = \binom{n+d}{n}$. For $d \ge 2$, the interpolation problem on B^d is given as follows:

Problem 2. Let E be a set of M_n^d points on B^d. Find conditions on E such that there is a unique polynomial $Q \in \Pi_n^d$ satisfying

$$Q(x_i) = f_i, \qquad x_i \in E, \quad 1 \le i \le M_n^d,$$

for any given data $\{f_i\}$.

Again, if E is a set for which unique interpolation exists, then we say the interpolation problem is poised and that E solves Problem 2.

In the following section, we prove a relation between Problem 1 and Problem 2, which shows that many sets of points that allow unique polynomial interpolation on B^d can be used to generate symmetric points on S^d that solve Problem 1, where symmetry means that the points are symmetric with respect to a coordinate plane. Our main examples are given in the case of $d = 2$. The polynomial interpolation on B^2 has been studied in [1, 2] recently; the study gives many sets of points on B^2 that solve Problem 2, among them one family leads to solutions of Problem 1. The points in [1, 2] are located on circles inside the unit disc, and each circle contains an odd number of equidistant points. The method of proving these results can also be used to prove the poisedeness of other sets of points; an example is given in Section 2 to illustrate this point. For recent studies and applications of polynomial interpolation on the sphere, we refer to [4, 6, 8]. ‘ In Section 3, we consider the problem of interpolation with even number of equidistant nodes on the circles; such a set of points has more symmetry than that of odd number points. In this case, we show that the interpolation can be related to interpolation on the standard triangle. However, the interpolation is unique not in Π_n^2 but in a subspace of Π_n^2. This is related to the so-called minimal interpolation. In this generality, the notion of Lagrange interpolation is as follows: for a given set of X and data $\{f(x_i) : x_i \in X\}$, find a subspace $\mathcal{P} \in \Pi^d$, often depending on the point set X, such that there is a unique polynomial $P \in \mathcal{P}$ that agrees with f on X (see [3, 5] and the references therein). In our case, we will define a polynomial subspace explicitly for the points that we specify.

2. Interpolation on the sphere and on the ball

Polynomials on S^d that are symmetric with respect to x_{d+1} are related to polynomials on B^d. This fact has been used for studying orthogonal structures of polynomials and for studying cubature formula (see, for example, [10]). It is also useful for studying interpolation. We show that there is a close relation between Problem 1 and Problem 2.

A set of points $X \subset S^d$ is *symmetric* with respect to a plane L that passes through the origin if, choosing the coordinates so that L has the equation $x_{d+1} = 0$, the point $x = (x', x_{d+1}) \in X$ implies that $(x', -x_{d+1}) \in X$, where $x' = (x_1, \ldots, x_d)$. In the following, we always assume that L is given by $x_{d+1} = 0$. Define $S_+^d = \{(x', x_{d+1}) \in S^d : x_{d+1} \ge 0\}$, the upper half sphere of S^d.

Theorem 2.1. *Let E be a set of M_n^d points on B^d that solves Problem 2. Assume that E contains exactly $M_n^d - M_{n-1}^d$ points on the boundary S^{d-1} of B^d and that*

$E^\circ := E \setminus (E \cap S^{d-1})$ solves Problem 2 for Π_{n-1}^d. Define

$$X_E = \{(x',0) : x' \in E \cap S^{d-1}\} \bigcup \Big\{(x', \pm x_{d+1}) : x_{d+1} = \sqrt{1-\|x'\|^2}, \quad x' \in E^\circ\Big\}.$$

Then X_E solves Problem 1. On the other hand, if X solves Problem 1, X is symmetric and there are exactly $M_n^d - M_{n-1}^d$ points on the hyperplane $\{x \in \mathbb{R}^{d+1} : x_{d+1} = 0\}$, then $E_X = \{x' : (x', x_{d+1}) \in X \cap S_+^{d+1}\}$ solves Problem 2.

Proof. Let E be a set of points in B^d satisfying the assumption. Then the set X_E contains $2|E| - (M_n^d - M_{n-1}^d) = M_n^d + M_{n-1}^d = N_n^d$ distinct points. Assume that S is a spherical polynomial of degree n that vanishes on X_E. For $x = (x', x_{d+1}) \in S^{d+1}$, define

$$S_e(x') = S(x', \sqrt{1-\|x'\|^2}) + S(x', -\sqrt{1-\|x'\|^2}).$$

since $S(x', x_{d+1}) + S(x', -x_{d+1})$ is an even polynomial of x_{d+1}, the relation $x_{d+1}^2 = 1 - \|x'\|^2$ shows that $S_e(x')$ is a polynomial of degree n in $x' = (x_1, \ldots, x_d)$; that is, $S_e \in \Pi_n^d$. Moreover, the symmetry of X_E shows that S_e vanishes on E; hence, since E solves Problem 2, we conclude that $S_e(x') \equiv 0$; that is, $S(x', x_{d+1}) + S(x', -x_{d+1}) \equiv 0$. Furthermore, define

$$S_o(x') = \Big[S(x', \sqrt{1-\|x'\|^2}) - S(x', -\sqrt{1-\|x'\|^2})\Big] \Big/ \sqrt{1-\|x'\|^2}.$$

Since $S(x', x_{d+1}) - S(x', -x_{d+1})$ is an odd polynomial of x_{d+1}, the relation $x_{d+1}^2 = 1 - \|x'\|^2$ shows that $S_o(x')$ is a polynomial of degree $n-1$ in x'; that is, $S_o \in \Pi_{n-1}^d$. The symmetry of X_E shows that S_o vanishes on E°; hence, since E° solves Problem 2 for Π_{n-1}^d, we conclude that $S_o(x') \equiv 0$. In particular, this shows that $S(x', x_{d+1}) - S(x', -x_{d+1}) \equiv 0$. Consequently, $S(x) \equiv 0$.

On the other hand, assume that X is symmetric and X solves Problem 1. Let P be a polynomial of degree n that vanishes on E_X. Considering P as a polynomials of $d+1$ variables on the sphere, it follows from the symmetry of X that P vanishes on X. Consequently, $P(x) \equiv 0$. ☐

By construction, the interpolation points on S^d are symmetric with respect to the equator. All of our examples are given in the case of $d = 2$. Let us note that

$$M_n^2 = \dim \Pi_n^2 = \frac{(n+1)(n+2)}{2} \quad \text{and} \quad N_n^2 = \dim \Pi_n(S^2) = (n+1)^2.$$

The set E in the theorem contains $(n+2)(n+1)/2$ distinct points of B^2 of which exactly $n+1$ points are on the boundary S^1 of B^2. The notation $S^2(a) := \{(x,y,z) : (x,y,z) \in S^2,\ z = a\}$ denotes the circle on S^2 resulted from the intersection of S^2 with the plane $z = a$ (called latitude at z).

We start with a simple result that is a consequence of the classical Bezout theorem:

Theorem 2.2. *Let E be a set of $(n+1)(n+2)/2$ points that lie on $n+1$ lines, say $\ell_0, \ell_1, \ldots, \ell_n$, and there are exactly $k+1$ points on the line ℓ_k and no point is in the intersection of two lines. If each line has exactly one point lying on the unit circle and all other points lying inside the unit disc, then the set*

$$X_E = \{(x,y,z) : (x,y) \in E, z = \pm\sqrt{1-x^2-y^2}\}$$

solves Problem 1.

Proof. Let E be a set as in the statement. It follows from the classical Bezout theorem that E solves the interpolation problem $P(x_i) = f(x_i)$, $x_i \in E$, uniquely for $P \in \Pi_n^d$. If the additional assumption holds, then $E^\circ = E \setminus (E \cap S^1)$ contains points lying on n lines, $\ell_1, \ldots, \ell_n$ and the line ℓ_i contains exactly i points. Hence E° solves Problem 2 for Π_{n-1}^2 so that we can apply Theorem 2.1. □

In this example, the points on S^2 are symmetric with respect to the equator. Evidently, it is possible to extend such a result to d dimensional. The use of the Bezout theorem in proving the poisedeness of interpolation is folklore. For interpolation by polynomials of d variables, see [5] for example.

Let us mention that one set of interpolation points on S^d is given in [6]. For S^2 the points are located on $n+1$ latitudes, say $S^2(z_i)$, $i = 0, 1, \ldots, n$, and the k-th latitude $S^2(z_k)$ contains $2k+1$ points. However, since the number of points on the circles are all different, there is little symmetry in such a distribution of points. In our next example, the points are not only symmetric with respect to the equator, they can also have a rotation symmetry with respect to the axis that joins the north and south poles.

For a positive number $r > 0$, denote the circle of radius r centered at the origin by $C(r) = \{(x, y) : x^2 + y^2 = r^2\}$. Let $\Theta_{\alpha,m}$ denote a set of $2m+1$ points that define equidistant points on the unit circle,

$$\Theta_{\alpha,m} = \{\theta_j^\alpha : \theta_j^\alpha = (2j + \alpha)\pi/(2m+1), \quad j = 0, 1, \ldots, 2m, \quad \alpha \in [0, 2)\}. \tag{2.1}$$

Let $[t]$ denote the integer part of t. The following theorem is proved in [1].

Theorem 2.3. *Let n be a positive integer, $m = [(n+1)/2]$. Let $r_1, \ldots, r_{[n/2+1}$ be distinct numbers in $[0, 1]$. Problem 2 based on the equidistant points*

$$(x_{l,j}, y_{l,j}) = (r_l \cos\theta_j, r_l \sin\theta_j), \quad \theta_j \in \Theta_{0,m},$$

on the circles $C(r_l)$, $1 \le l \le [n/2] + 1$, is poised.

We note that the number of interpolation conditions equals the $\dim \Pi_n^2$, as

$$\left(2\left[\frac{n+1}{2}\right] + 1\right)\left(\left[\frac{n}{2}\right] + 1\right) = \frac{(n+1)(n+2)}{2}.$$

Points on each circle are equidistant, but points on different circles can differ by a rotation. Each circle in Theorem 2.3 contains precisely $2m+1 = 2[(n+1)/2]+1$ points. In particular, if $n = 2m$ then each circle contains $n+1$ points.

Theorem 2.4. *Let $1 = r_1 > r_2 > \cdots > r_{m+1} > 0$ and let*

$$X = \{(x_{l,l}, y_{l,j}, z_l) : x_{l,j} = r_l \cos\theta_j,\ y_{l,j} = r_l \sin\theta_j,\ z_l = \pm\sqrt{1 - r_l^2}, \\ \theta_j \in \Theta_{0,m}, 1 \le l \le m+1\}.$$

Then X solves Problem 1 of interpolation by $\Pi_{2m}(S^2)$.

Proof. Let $n = 2m$ and let E be the set of interpolation points in Theorem 2.3 with $r_0 = 1 > r_2 > \cdots > r_m > 0$. Then E solve Problem 2 and E has $n+1$ points on the boundary. Furthermore, the set $E - (E \cap S^1)$ contains $m = [(n-1)/2] + 1$ circles and each circle has $2m+1 = 2([(n-1)/2]+1)+1$ points, which solves Problem 2 for Π_{n-1}^2 by Theorem 2.3. Hence, the assumption of Theorem 2.1 is satisfied and the stated result follows from that theorem. □

The points in the set X are distributed on $2m+1$ latitudes on the sphere S^2, each circle contains precisely $2m+1$ points. These latitudes are symmetric with respect to the equator ($r_0 = 1$) but are otherwise free; that is, there is no restriction on the positive numbers $r_1, \ldots, r_m$. Also, the points on different circles can differ up to a rotation.

The simplest case is $m = 1$. In this case there are 6 points on the disc B^2, three on the boundary S^1 and three on another circle inside B^2. The next case is $m = 2$, for which E consists of three circles with 5 equidistant points on each circle.

We note that the theorem holds only for interpolation in Π_n^2 for $n = 2m$. For $n = 2m-1$, Theorem 2.3 also holds but the requirement that there should be $n+1$ points on the boundary of B^2 in Theorem 2.1 fails, since the set of interpolation points in Theorem 2.3 always consists of odd number of points on each circle.

The results in [2] are far more general than those in Theorem 2.3; the points are located on several groups of concentric circles, circles of the same group contain same number of equidistant points (though points on different circles may differ by a rotation), but circles in different groups contain different number of points. Theorem 2.3 is the special case of one group of circles. The proof of the general result is based on a factorization method. However, the requirements that there should be $n+1$ points on the boundary of B^2 and the points on E° should solve Problem 2 for Π_{n-1}^2 are satisfied only in the special case of Theorem 2.3; so the general result cannot be combined with Theorem 2.4 to yield points on S^2 that solve Problem 1. On the other hand, however, it is possible to use the ideal of factorization to prove directly a more general result on points on S^2. This will appear in [12] and the main result there is stated below.

Theorem 2.5. *Let n and σ be positive integers, such that $n+1-\sigma$ is an even integer and $\sigma \le n+1$. Let $\lambda_1, \ldots, \lambda_\sigma$ be nonnegative integers such that*

$$\lambda_1 + \ldots + \lambda_\sigma = \frac{n+1-\sigma}{2}. \tag{2.2}$$

Define $n_k = n_{k-1} - (2\lambda_k + 1)$ for $1 \le k \le \sigma - 1$ with $n_0 = n$. The set X consists of points located on σ groups of latitudes, $\{S^2(z_0, k), S^2(z_{1,k}), \ldots, S^2(z_{2\lambda_k,k})\}$, $1 \le k \le \sigma$, and on each latitude $S^2(z_{j,k})$ there are $2(n_{k-1} - \lambda_k) + 1$ equidistant points in $\Theta_{\alpha_k, n_{k-1}-\lambda_k}$ with $\alpha_k \in [0, 2)$, where $\{z_{j,k} : 0 \le j \le 2\lambda_k, 1 \le k \le \sigma\}$ is a set of distinct points in $(-1, 1)$. Then the set X solves Problem 1 in $\Pi_n(S^2)$.

For a fixed n this theorem contains a number of different interpolation processes; Theorem 2.4 is just a special case. In fact, for each σ satisfying the condition that $n+1-\sigma$ is a nonnegative even integer, every nonnegative integer solution of the equation (2.2) leads to a set of interpolation points that solves Problem 1; moreover, the order of $\lambda_1, \ldots, \lambda_\sigma$ matters. The number of solutions of such an equation grows exponentially as n goes to infinity. We note that the points are usually not symmetric with respect to the equator. In fact, the special case in Theorem 2.4 is the only case that the points are symmetric with respect to the equator.

To study interpolation on the unit disc B^2, it is often more convenient to use the polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta, \qquad 0 \le \theta \le 2\pi, \quad 0 \le r \le 1.$$

For $P \in \Pi^2$ we write $\widetilde{P}(r,\theta) = P(r\cos\theta, r\sin\theta)$. In the polar coordinates, every polynomial $P \in \Pi_n^2$ can be written in the form ([1])

$$\widetilde{P}(r,\theta) = A_0(r^2) + \sum_{j=1}^{n} \left[r^j A_j(r^2) \cos j\theta + r^j B_j(r^2) \sin j\theta \right], \tag{2.3}$$

where $A_j(t)$ and $B_j(t)$ are polynomials of degree $[(n-j)/2]$. The proof of Theorem 2.5 uses the fact that at equidistant points $\theta \in \Theta_{0,m}$, where $n = 2m$ or $n = 2m-1$, the sum in $\widetilde{P}$ can be written as a sum over $j = 1, \ldots, m$. This idea can be used for obtaining other sets of nodes that solve Problem 2. We illustrate this ideal with the following example of $n = 4$, in which we keep 5 points on the boundary so that the set may be used to obtain a set of points on S^2 that solve Problem 1 by Theorem 2.1. Other points are chosen to locate on 2 circles with 3 and 7 points each. For a set of points $E \subset \mathbb{R}^2$, we denote by $\widetilde{E}$ the set $\widetilde{E} = \{(r,\theta) : (r\cos\theta, r\sin\theta) \in E\}$.

Proposition 2.6. *. Let r_1 and r_2 be two numbers in the interval $(0,1)$. Let*

$$\widetilde{E} = \{(1,\theta) : \theta \in \Theta_{0,2}\} \cup \{(r_1,\theta) : \theta \in \Theta_{1,3}\} \cup \{(r_2,\theta) : \theta \in \Theta_{0,1}\}.$$

If $(r_1^2-1)r_2^3 + (r_1r_2+1)(r_1^2-r_2^2) \neq 0$ and $(r_1^2-1)r_2^3 + (r_1r_2-1)(r_1^2-r_2^2) \neq 0$, then E solves Problem 2 for Π_4^2; furthermore, X_E defined in Theorem 2.1 solves Problem 1 for $\Pi_4(S^2)$.

Proof. We need to prove that if P vanishes on E then $P \equiv 0$. Assume that P is written as in (2.3) with $n = 4$. Using the fact that for $\theta \in \Theta_{0,2}$, $\cos(5-j)\theta = \cos j\theta$ and $\sin(5-j)\theta = \sin j\theta$, it follows that for $\theta \in \Theta_{0,2}$,

$$\widetilde{P}(1,\theta) = A_0(1) + \sum_{j=1}^{2} \left[(A_j(1) + A_{5-j}(1)) \cos j\theta + (B_j(1) - B_{5-j}(1)) \sin j\theta \right] = 0.$$

Using the fact that interpolation on $2m+1$ points in $[0,2\pi)$ by trigonometric polynomial of degree has a unique solution, we conclude that

$$A_0(1) = 0, \quad A_j(1) + A_{5-j}(1) = 0, \quad B_j(1) - B_{5-j}(1) = 0, \quad j = 1,2.$$

Similarly, for $\theta \in \Theta_{1,3}$, $\cos 4\theta = \cos 3\theta$ and $\sin 4\theta = -\sin 3\theta$, it follows that for $r = r_1$ and $\theta \in \Theta_{1,3}$,

$$\begin{aligned} \widetilde{P}(r_1,\theta) = & A_0(r_1^2) + \sum_{j=1}^{2} (A_j(r_1) r_1^j \cos j\theta + B_j(r_1) r_1^j \sin j\theta) \\ & + r_1^3 (A_3(r_1^2) + r_1 A_4(r_1^2)) \cos 3\theta + r_1^3 (B_3(r_1^2) - r_1 B_4(r_1^2)) \sin 3\theta = 0, \end{aligned}$$

which shows, using the uniqueness of the trigonometric interpolation, that

$$\begin{aligned} & A_0(r_1^2) = 0, \quad A_j(r_1^2) = 0, \quad B_j(r_1^2) = 0, \quad j = 1,2, \\ & A_3(r_1^2) + r_1 A_4(r_1^2) = 0, \quad B_3(r_1^2) - r_1 B_4(r_1^2) = 0. \end{aligned}$$

Furthermore, for $\theta \in \Theta_{0,1}$, $\cos 2\theta = -\cos\theta$, $\sin 2\theta = \sin\theta$, $\cos 3\theta = -1$, $\sin\theta = 0$, and $\cos 4\theta = -\cos\theta$, $\sin 4\theta = -\sin\theta$, it follows that for $r = r_2$ and $\theta \in \Theta_{0,1}$,

$$\begin{aligned} \widetilde{P}(r_2,\theta) = A_0(r_2^2) - r_2^3 A_3(r_2^2) + (r_2 A_1(r_2^2) - r_2^2 A_2(r_2^2) - r_2^4 A_4(r_2^2)) \cos\theta \\ + (r_2 B_1(r_2^2) + r_2^2 B_2(r_2^2) - r_2^4 B_4(r_2^2)) \sin\theta, \end{aligned}$$

which implies, once again by the uniqueness of the trigonometric interpolation, that

$$A_0(r_2^2) - r_2^3 A_3(r_2^2) = 0, \quad A_1(r_2^2) - r_2 A_2(r_2^2) - r_2^3 A_4(r_2^2) = 0,$$
$$B_1(r_2^2) + r_2 B_2(r_2^2) - r_2^3 B_4(r_2^2) = 0.$$

Since A_j and B_j are polynomials of degree $[(4-j)/2]$, the conditions on A_j implies that $A_0(t) = a_0(t-1)(t-r_1^2)$, $A_1(t) = a_1(t-r_1^2)$, $A_2(t) = a_2(t-r_1^2)$, $A_3(t) = a_3$ and $A_4(t) = a_4$, where the constants a_i, $0 \le i \le 4$, satisfy

$$a_3 + r_1 a_4 = 0, \quad a_1(1-r_1^2) + a_4 = 0, \quad a_2(1-r_1^2) + a_3 = 0$$
$$a_0(r_2^2-1)(r_2^2-r_1^2) - r_2^3 a_3 = 0, \quad a_1(r_2^2-r_1^2) - a_2 r_2(r_2^2-r_1^2) - a_4 r_2^3 = 0.$$

This system has zero solution if the determinant is nonzero, which gives the condition $(r_1^2-1)r_2^3 + (r_1 r_2 + 1)(r_1^2 - r_2^2) \neq 0$. Similarly, the other condition ensures that there is a unique solution for $B_j(t)$.

To show that the set X_E satisfies the condition of Theorem 2.1, we need to show that E° solves Problem 2 for Π_3^2. Here E° consists of 10 points, 7 equidistant points on one circle and 3 equidistant points on another circle. Such a set solves Problem 2 for Π_3^2 is a special case of Theorem 3.4 of [2]. □

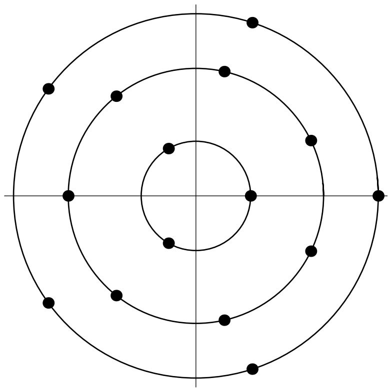

Figure 2

One set of such points is shown in Figure 2, where we choose $r_1 = 1/3$ and $r_2 = 2/3$. There are values of r_1 and r_2 for which the conditions of this proposition are not satisfied. For example, $r_1 = 1/2$ and r_2 is the positive root of $2 + t - 8t^2 - 10t^3$ (about 0.443303).

Clearly, it is possible to extend the proof of this proposition for more general cases. The solution, however, likely depends on complicated nonlinear equations on r_i.

3. Interpolation on the unit disc

The points in [1, 2] that solve Problem 2 are located on various circles and each circle has an odd number of points. The proof uses the fact that interpolation on $2n+1$ points by trigonometric polynomials of degree n has a unique solution, which is why we need an odd number of points on each circle, as illustrated in the proof of Proposition 2.6. From the symmetry point of view, an even number of points on each circle may appear more appealing since one can choose such points to be symmetric with respect to sign changes (symmetric with respect to reflection in coordinate axes). Such symmetry is often an important feature when we consider cubature formulae.

In the following, we consider sets of points for which points are located on circles and each circle has an even number of points. The number of total interpolation points, however, is not equal to the dimension of the polynomial space Π_n^2, and we consider interpolation by polynomials belonging to a subspace of Π_n^2 whose dimension agrees with the number of points. As we mentioned in the introduction, the subspace depends on the interpolation points. Let E be a set of distinct points and f be an arbitrary function defined on E. If $\mathcal{P}$ is a subspace of polynomials such that there is a unique polynomial $P \in \mathcal{P}$ that agrees with f on E, then we say that E admits a unique interpolation in the space $\mathcal{P}$.

Just like in the proof of Theorem 2.1, we will explore symmtry of the points and polynomial spaces, starting from a set of points on the triangle

$$T^2 = \{(x,y) : x \ge 0, y \ge 0, 1-x-y \ge 0\}.$$

The relation between polynomials on B^d that are symmetric with respect to sign changes and polynomials on the simplex T^d has been used for studying orthogonal structures of polynomials and for studying cubature formula (see, for example, [11]). Here we use this relation for interpolation.

Let F^* be a set of $\dim \Pi_{n-2}^2$ points on the triangle $\{(x,y) : x > 0, y > 0, 1-x-y \ge 0\}$; that is, neither coordinates of the the points in F^* are zero. Let $\{x_i : 1 \le i \le n\}$ and $\{y_i : 1 \le i \le n\}$ be two sets of distinct numbers in the interval $(0,1]$. Define the set

$$F = \{(0,0)\} \cup \{(x_i,0) : 1 \le i \le n, x_i \in (0,1]\} \cup \{(0,y_i) : 1 \le i \le n, y_i \in (0,1]\} \cup F^*$$

Theorem 3.1. *Let F be defined as above and assume that F^* admits unique solution for the interpolation problem by Π_{n-2}^2. We define*

$$E = \{(\pm\sqrt{x}, \pm\sqrt{y}) : (x,y) \in F\} \subset B^2.$$

Then E admits a unique interpolation in the polynomial subspace

$$\Pi_{2n}^* = \Pi_{2n-1}^2 \bigcup \operatorname{span}\{x^{2n-2k}y^{2k} : 0 \le k \le n\}.$$

Proof. First of all, the number of points in E is easily seen to be equal to

$$4 \dim \Pi_{n-2}^2 + 4n + 1 = n(2n+1) + (n+1) = \dim \Pi_{2n}^*,$$

so that the number of interpolation points is equal to the dimension of the polynomial subspace.

The assumption on F^* implies that the set F admits unique interpolation in Π_n^2. Indeed, assume that P is a polynomial of degree n and P vanishes on F; then $P(x,0)$ vanishes on $n+1$ distinct points $0, x_1, \ldots, x_n$. The classical Bezout theorem shows that $P(x,y) = xP_{n-1}(x)$, where P_{n-1} is a polynomial of degree $n-1$. Furthermore, $P(0,y)$, hence $P_{n-1}(0,y)$, vanishes on n distinct nonzero points $y_1, \ldots, y_n$, so that we can use Bezout theorem again to conclude that $P(x,y) = xyP_{n-2}(x,y)$, where $P_{n-2} \in \Pi_{n-2}^2$. Hence, by the definition of F^*, the polynomial P, hence P_{n-2} vanishes on F^*, and the assumption that F^* admits unique interpolation in Π_{n-2}^2 shows that $P(x,y) \equiv 0$. Furthermore, the same argument clearly shows that both

$$F_x = \{(x_i,0) : 1 \le i \le n\} \cup F^*, \quad \text{and} \quad F_y = \{(0,y_i) : 1 \le i \le n\} \cup F^*$$

admit unique solution in Π_{n-1}^2.

Let $P \in \Pi_{2n}^*$. Then P is a polynomial of degree $2n$ and we write P as

$$P(x,y) = \sum_{0 \le k+j \le 2n-1} c_{k,j} x^k y^j + \sum_{k=0}^{n} c_{2k,2n-2k} x^{2k} y^{2n-2k}.$$

Since E is symmetric with respect to the map $(x,y) \mapsto (\pm x, \pm y)$, it follows that the polynomial

$$Q_1(x,y) := P(x,y) + P(x,-y) + P(-x,y) + P(-x,-y)$$

vanishes on E. Since $Q_1(x,y)$ is clearly an even function in both x and y, it follows that we can write $Q_1(x,y) = q_1(x^2,y^2)$, where q_1 is a polynomial of degree n. The definition of E shows that q_1 vanishes on E, which implies that q_1 is identically zero, so that $Q_1(x,y) \equiv 0$. Next we consider the polynomial

$$Q_2(x,y) := \frac{P(x,y) - P(x,-y)}{2y} + \frac{P(-x,y) - P(-x,-y)}{2y}.$$

This is again a polynomial that is even in both x and y. Moreover, the monomial terms $c_{k,j} x^k y^j$ in P do not appear in Q_2 if both k and j are even or both are odd. Consequently, we can write $Q_2(x,y) = q_2(x^2,y^2)$, where q_2 is a polynomial of degree 2. The definition of E shows that q_2 vanishes on F_x, which shows that $q_2(x,y) \equiv 0$, hence $Q_2(x,y) \equiv 0$. Similar consideration shows that the polynomial

$$Q_3(x,y) := \frac{P(x,y) - P(-x,y)}{2x} + \frac{P(x,-y) - P(-x,-y)}{2x}$$

is in fact a zero function. Finally, we consider the polynomial

$$Q_4(x,y) := \frac{P(x,y) - P(-x,y) - P(x,-y) + P(-x,-y)}{4xy}.$$

Evidently, the polynomial $xyQ_4(x,y)$ is odd in both x and y, so that Q_4 is a polynomial that is even in both x and y and we can write $Q_4(x,y) = q_4(x^2,y^2)$. Furthermore, by the structure of the polynomial space Π_{2n}^*, q_4 is a polynomial of degree $n-2$ instead of a polynomial of degree $n-1$. The definition of E shows that q_4 vanishes on F^*, so that $Q_4(x,y) \equiv 0$.

Putting together, it follows from the relation

$$P(x,y) = (Q_1(x,y) + 2yQ_2(x,y) + 2xQ_3(x,y) + 4xyQ_4(x,y))/4$$

that $P(x,y) \equiv 0$. This completes the proof. □

Example 1. We can take F as the regular lattice points in the triangle T^2. The Figure 3 shows both the points in the triangle and the points in the disc.

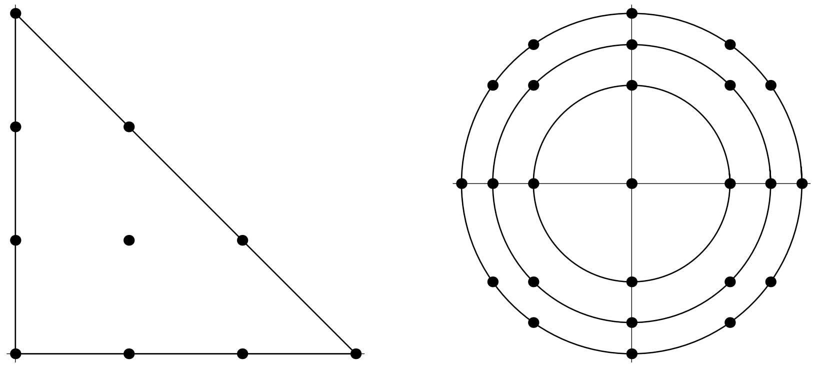

Figure 3

Another more symmetric set of points on the disc are given in Figure 4.

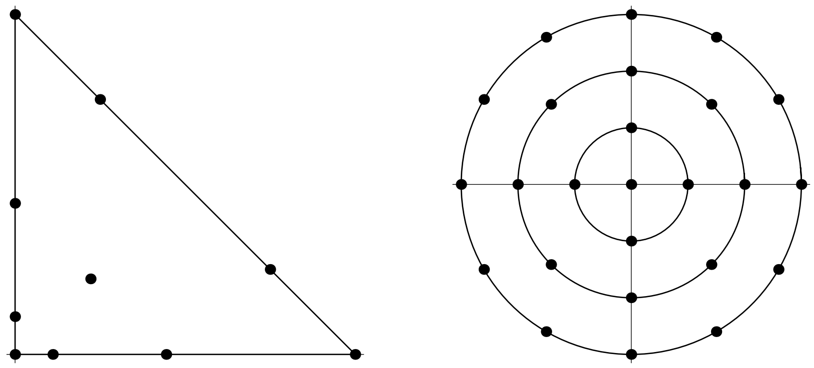

Figure 4

In the second example, the points on the triangle are chosen so that the corresponding points on the disc are equidistant on the circles. □

The origin $(0,0)$ is always in the set of interpolation points in Theorem 3.1. It is possible to remove this point from the set and uses a subspace of polynomials that is one dimension less as the space for interpolation. Such a subspace is easy to identify as shown in the following:

Theorem 3.2. *Let E be the point set defined in Theorem 3.1. Then the set $E \setminus \{(0,0)\}$ admits a unique interpolation in the polynomial subspace*

$$\Pi_{2n}^{\dagger} = \Pi_{2n-1}^2 \bigcup \operatorname{span}\{x^{2n-2k}y^{2k} : 0 \le k \le n, \ \ k \ne 0 \ \ or \ \ k \ne n\}.$$

Proof. Assume that x^{2n} is not in $\Pi_{2n}^{\dagger}$. First we prove that the point set $F \setminus \{(0,0)\}$ admits unique interpolation in the polynomial subspace $\Pi_n^* := \Pi_n^2 \setminus \operatorname{span}\{x^n\}$. Let $P_n \in \Pi_n^*$ vanish on $F \setminus \{(0,0)\}$. Then we can write P_n as

$$P_n(x,y) = yP_{n-1}(x,y) + q_{n-1}(x), \qquad P_{n-1} \in \Pi_{n-1}^2, \quad q_{n-1} \in \Pi_{n-1}^1.$$

By the definition of F, the set E contains n distinct points, $(x_i, 0)$, $1 \le i \le n$ and $x_i \in (0,1)$. It follows that on these points $q_{n-1}(x_i) = P_n(x_i, 0) = 0$. This implies that $q_{n-1}(x) \equiv 0$ so that $P(x,y) = yP_{n-1}(x,y)$. Evidently, we can now follow the proof of Theorem 3.1 and use Bezout's theorem to finish the proof.

Let $P \in \Pi_{2n}^{\dagger}$ vanish on $E \setminus \{(0,0)\}$. Following the proof of Theorem 3.1 we consider again the function $Q_1(x,y)$ which can be written as $Q_1(x,y) = q_1(x^2,y^2)$ with $q_1 \in \Pi_n^*$. The definition of E shows that q vanishes on $F \setminus \{(0,0)\}$, so that the previous paragraph shows that q_1 is identically zero, hence $Q_1(x,y) \equiv 0$. The rest of the proof is identical to that of Theorem 3.1 since the point $(0,0)$ does not play a role in dealing with Q_2, Q_3 and Q_4. □

The reason we consider such interpolation nodes is explained by the following example, in which points are the nodes of a minimal cubature formula of degree 7. A minimal cubature formula is a formula that has the smallest number of nodes among all cubature formulae of the same degree.

Example 2. For the integral $\int_{B^2} f(x,y)dxdy$, a minimal cubature formula of degree 7 has 12 nodes. One formula given in [9, p. 281, S_2:7-1] has the nodes

$$(\pm r, 0), \qquad (0, \pm r) \qquad (\pm s, \pm s) \qquad (\pm t, \pm t),$$

where r, s and t are numbers in $(0,1)$ given by

$$r^2 = \frac{3}{4} \qquad s^2 = \frac{27 - 3\sqrt{29}}{104}, \qquad t^2 = \frac{27 + 3\sqrt{29}}{104}.$$

The points are clearly symmetric with respect to both x and y. It corresponds to a cubature formula of degree 3 on the triangle T^2 with 5 points. Since the integral is rotationally symmetric, a rotation of 45° degree gives another cubature formula with 12 points

$$(\pm r/\sqrt{2}, \pm r/\sqrt{2}), \quad (0, \pm\sqrt{2}s), \quad (\pm\sqrt{2}s, 0), \quad (0, \pm\sqrt{2}t), \quad (\pm\sqrt{2}t, 0).$$

These points admit unique interpolation in the space $\Pi_3^2 \cup \mathrm{span}\{x^4, x^2y^2\}$ according to Theorem 3.2. The points are depicted in Figure 5. □

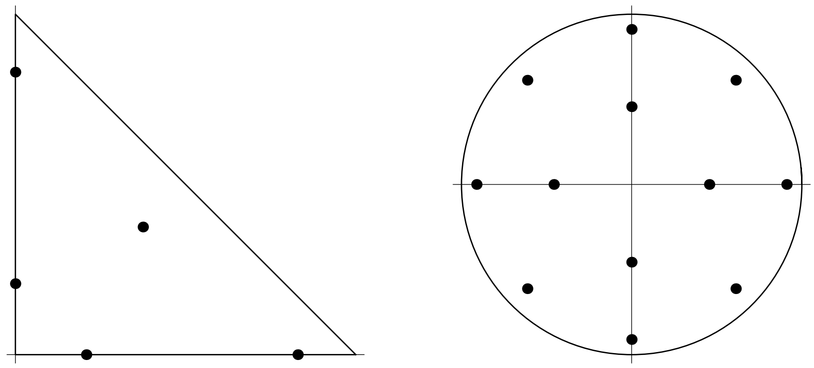

Figure 5

It is interesting to note that the interpolation in this example leads to a minimal cubature formula. In general, one may ask the question of how to choose the nodes so that the interpolation polynomial in Thereom 3.1 or Thereom 3.2 leads to a cubature formula that has highest degree of precision.

Recall that if E is a set of points in B^2, then E° denotes the set $E \setminus (E \cap S^1)$. The set X_E defined by

$$X_E = \{(x,y,0) : (x,y) \in E \cap S^1\} \bigcup \{(x,y,\pm\sqrt{1-x^2-y^2}) : (x,y) \in E^\circ\}$$

is a set of points on the sphere S^2. Let E be the set in Theorem 3.1. As an extension of Theorem 2.1, one may ask what is the subspace of spherical polynomials in which the set X_E admits a unique interpolation. The answer turns out to depend on the number of points in E that lies on the boundary of B^2.

Since the space of spherical polynomials can be considered as $\mathbb{R}[x,y,z]/(x^2+y^2+z^2-1)$, we can write a polynomial $T \in \Pi_n(S^2)$ as a sum of $T(x,y,z) = P(x,y) + zQ(x,y)$. That is, we use $z^2 = 1 - x^2 - y^2$ whenever there is an even power of z. We introduce the following notation: If $\mathcal{P}$ is a polynomial subspace of Π^2, we denote by $\mathcal{P}^*[x,y]$ the subspace $\{P(x,y) \in \Pi_n(S^2) : P \in \mathcal{P}\}$ of $\Pi_n(S^2)$. Using this notation, we clearly have $\Pi_n(S^2) = \Pi_n^2 \cup z\Pi_{n-1}^2$.

Theorem 3.3. *Let E be the set given in Theorem 3.1. If E° admits unique interpolation in a polynomial subspace $\mathcal{P}$, then X_E admits unique interpolation in the subspace $\Pi_{2n}^*[x,y] \cup z\mathcal{P}[x,y]$ of spherical polynomials.*

Proof. Let $P \in \Pi_{2n}^*[x,y] \cup z\mathcal{P}[x,y]$ vanish on X_E. Then $P(x,y,z) = p_1(x,y) + zp_2(x,y)$, where $p_1 \in \Pi_{2n}^*$ and $p_2 \in \mathcal{P}$. Since $p_1(x,y) = (P(x,y,z) + P(x,y,-z))/2$, it follows that $p_1(x,y)$ vanishes on E, so that $p_1(x,y) \equiv 0$. Furthermore, since $p_2(x,y) = (P(x,y,z) - P(x,y,-z)/(2z)$, it follows that $p_2(x,y)$ vanishes on E°, so that $p_2(x,y) \equiv 0$. □

We give two examples of this theorem, corresponding to the two extremal cases: in the first E has no points on the boundary of B^2, and in the second one E has maximal number of points on the boundary of B^2.

Corollary 3.4. *Let E be the set given in Theorem 3.1. If E contains no point on the boundary of B^2, then X_E admits unique interpolation in the subspace $\Pi_{2n}^*[x,y] \cup z\Pi_{2n}^*[x,y]$ of spherical polynomials.*

Indeed, in this case $E^\circ = E$, so that we can take $\mathcal{P} = \Pi_{2n}^*$. The set X_E contains $(2n+1)^2 + 1$ points, and the subspace can be written explicitly as

$$\Pi_{2n}^*[x,y] \cup z\Pi_{2n}^*[x,y] = \Pi_{2n-1}(S^2) \cup z\mathcal{P}_{2n-1}^2 \cup \{1,z\}\,\mathrm{span}\{x^{2k}y^{2n-2k} : 0 \le k \le n\},$$

where $\{1,z\}\mathcal{P}$ means $\mathcal{P} \cup z\mathcal{P}$. Evidently, $\Pi_{2n}^*[x,y] \cup z\Pi_{2n}^*[x,y]$ is a subspace of $\Pi_{2n+1}(S^2)$.

In Theorem 3.1 the set E comes from the set F in the triangle T^2. The requirement that F^* admits unique interpolation by polynomial of degree $n-2$ means that it can have at most $n-1$ points on the boundary $x + y - 1 = 0$. Consequently, it is easy to see that X_E can contain at most $4n$ points on the boundary of B^2.

Corollary 3.5. *Let E be the set given in Theorem 3.1. If E contains $4n$ points on the boundary of B^2 and E° admits unique interpolation in Π_{2n-2}^*, then X_E admits unique interpolation in the subspace $\Pi_{2n}^*[x,y] \cup z\Pi_{2n-2}^*[x,y]$ of spherical polynomials.*

In this case, the set X_E has $4n^2+2$ points and the space $\Pi_{2n}^*[x,y] \cup z\Pi_{2n-2}^*[x,y]$ is a subspace of $\Pi_{2n}(S^2)$. The examples of E include those in Example 3.2.

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