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On Internal Formation Theory*

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Summary

This paper describes analogues to locally induced formations which are defined entirely within a given finite solvable group. The methods are used to construct complements to normal subgroups and to determine when all such complements are conjugate.

1. Introduction

Locally induced formations are more useful than general formations, partly because they are saturated and partly because they are easy to construct. The main ingredient in defining a locally induced formation is a notion of good chief factor. One then defines the formation $\mathscr{F}$ to be the class of groups all of whose chief factors are good. As a byproduct, it is possible to separate good chief factors from bad in groups which are not in the formation $\mathscr{F}$. The $\mathscr{F}$-residual subgroup $G^{\mathscr{F}}$ of G is characterized by the property that every chief factor of G above $G^{\mathscr{F}}$ is good, but no chief factor $G^{\mathscr{F}}/K$ is good.

The main theme of [8] is that many properties of residual subgroups $G^{\mathscr{F}}$ for locally induced formations also hold for normal subgroups satisfying weaker conditions. The results depend on classification of the chief factors of a given group G as suitably central or suitably eccentric. In [9] the idea is extended to define a class of good sections of G which play the role of the $\mathscr{F}$-sections in the formation theory. In this note we continue the development and show that several apparently isolated results have a natural setting in the theory which evolves.

This work is based on the earlier ideas of Gaschütz [2], Carter and Hawkes [1], and Prentice [4]. An alternate approach due to Wielandt, [6], also constructs an internal formation theory, but since it requires homomorphic images of good factors to be good and since one of our main applications fails to have this property, we have followed the locally induced development instead.

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2. Residual Subgroups

Throughout the paper G is a finite solvable group with distinguished normal subgroup H. Notation is standard and continues that of [8] and [9].

Our first goal is to define a notion of good chief factor of G in terms of H and determine conditions on H under which every chief factor of G above H is good and every chief factor of form H/K is not good. In order to get all we need eventually, we begin with an elaborate definition, made in [9] and repeated here for reference.

A *sector* on G is a function $\mathbf{Y}$ defined on the set of subgroups of G such that $\mathbf{Y}(U) \trianglelefteq U$ whenever $U \leqq G$. A *sector screen* is a function $\mathscr{Y}$ from a set $\pi(\mathscr{Y})$ of primes to the set of sectors of G. We denote the value of $\mathscr{Y}$ at p by $\mathbf{Y}_p$. If $U \leqq G$, a p-chief factor V/W of U is *$\mathscr{Y}$-central* in case $p \in \pi(\mathscr{Y})$ and $\mathbf{Y}_p(U) \leqq C_U(V/W)$; it is *$\mathscr{Y}$-eccentric* otherwise. Let $d(\mathscr{Y})$ be the set of sections U/V of G such that every U-chief factor of U/V is $\mathscr{Y}$-central in U. For a subgroup U of G denote by $U^{d(\mathscr{Y})}$ the smallest normal subgroup of U with factor group in $d(\mathscr{Y})$.

Let $\mathbf{Y}$ be a sector on G. If $U \leqq G$, then the restriction of $\mathbf{Y}$ to U is a sector on U, denoted $\mathbf{Y} \cap U$. If $N \trianglelefteq G$, then $\mathbf{Y}$ yields a sector $\mathbf{Y}/N$ on G/N given by $(\mathbf{Y}/N)(E/N) = \mathbf{Y}(E)\,N/N$ whenever $N \trianglelefteq E \leqq G$. For $\mathscr{Y}$ a sector screen, one defines $\mathscr{Y} \cap U$ and $\mathscr{Y}/N$ by $(\mathscr{Y} \cap U)(p) = (\mathbf{Y}_p \cap U)$ and $(\mathscr{Y}/N)(p) = \mathbf{Y}_p/N$. Then $d(\mathscr{Y} \cap U)$ is the set $\big(d(\mathscr{Y})\big) \cap U$ of sections of U in $d(\mathscr{Y})$, and $d(\mathscr{Y}/N)$ is the set $d(\mathscr{Y})/N$ of sections $(U/N)/(V/N)$ with $U/V \in d(\mathscr{Y})$ and $V \geqq N$.

The normal subgroup H yields a canonical sector screen on G which we denote by $\mathscr{H}$ and define as follows:

(i) $\pi(\mathscr{H}) = \pi(G:H)$, the set of primes dividing $|G:H|$;

(ii) $\mathbf{H}_p(G)/H = \mathbf{O}_{p'}(G/H)$ for p in $\pi(\mathscr{H})$;

(iii) $\mathbf{H}_p(U) = U \cap \mathbf{H}_p(G)$ for p in $\pi(\mathscr{H})$, $U \leqq G$.

[In [8] the notation ${}^pH$ was used for the group $\mathbf{H}_p(G)$.]

Since every p-chief factor of G above H is centralized by $\mathbf{H}_p(G)$, $G/H \in d(\mathscr{H})$; i.e., $H \geqq G^{d(\mathscr{H})}$. Our first result characterizes the subgroups H of form $G^{d(\mathscr{Y})}$.

Proposition. *The following conditions on the normal subgroup H of G are equivalent.*

(1) $H = G^{d(\mathscr{Y})}$ *for some sector screen $\mathscr{Y}$ on G.*

(2) *For each p in $\pi(G:H)$ every p-chief factor H/K of G is $\mathbf{H}_p(G)$-eccentric.*

(3) $H = G^{d(\mathscr{H})}$.

Proof. Suppose that (1) holds. Then every chief factor of G above H is $\mathscr{Y}$-central, so $\pi(\mathscr{Y}) \geqq \pi(G:H)$ and $\mathbf{Y}_p(G) \cdot H/H \leqq \mathbf{O}_{p'p}(G/H)$ for each p

in $\pi(G:H)$. If H/K is p-chief in G with p in $\pi(G:H)$, then H/K is $\mathbf{Y}_p(G)$-eccentric. If H/K were $\mathbf{H}_p(G)$-central then it would be L-central, where $L/H=\mathbf{O}_{p'p}(G/H)$. Thus H/K is $\mathbf{H}_p(G)$-eccentric.

Suppose now that (2) holds. Since $H \geqq G^{d(\mathscr{H})}$ but $G/K \notin d(\mathscr{H})$ for H/K chief by (2), (3) holds.

Clearly, (3) implies (1).

We say that H is *residual* in G and write $H \mathbf{r} G$ in case H satisfies (1), (2) and (3) of the proposition. This notation is consistent with the usage in [8].

Remarks. (1) For p in $\pi(G:H)$, let $\mathbf{X}_p(G)/H=\mathbf{O}_{p'p}(G/H)$, and let $\pi(\mathscr{X})=\pi(G:H)$. Then the $\mathscr{X}$-central chief factors of G are precisely the $\mathscr{H}$-central ones, and $H \geqq G^{d(\mathscr{Y})}$ precisely if $\pi(\mathscr{Y}) \supseteq \pi(G:H)$ and $\mathbf{Y}_p(G) \leqq \mathbf{X}_p(G)$ for each p in $\pi(G:H)$. Hence, among all sector screens $\mathscr{Y}$ with $H \geqq G^{d(\mathscr{Y})}$, $\mathscr{H}$ yields as few central factors as possible. Sector screens $\mathscr{Y}$ for which each $\mathbf{Y}_p(G)$ contains $G^{d(\mathscr{Y})}$ can be viewed as "integrated". (If, in addition, $\mathbf{Y}_p(U)=U \cap \mathbf{Y}_p(G)$ for each subgroup U of G and p in $\pi(\mathscr{Y})$, then $\mathscr{Y}$ is integrated in the sense of Prentice [4].) The choice of $\mathscr{H}$ among all $\mathscr{Y}$'s with $G^{d(\mathscr{Y})} \leqq H$ can be likened to a choice of a full integrated screen to induce a formation. The choice is always possible, and it simplifies proofs.

(2) Let $K \trianglelefteq G$ with $K \leqq H$, and let K yield the sector screen $\mathscr{K}$. Then $\pi(\mathscr{K}) \supseteq \pi(\mathscr{H})$ and $\mathbf{K}_p(G) \leqq \mathbf{H}_p(G)$ for p in $\pi(\mathscr{H})$, so every $\mathscr{H}$-central chief factor of G is $\mathscr{K}$-central. In particular, every chief factor between K and $K \cap G^{d(\mathscr{H})}$ is $\mathscr{K}$-central, so if $K \mathbf{r} G$, then $K \leqq G^{d(\mathscr{H})}$. It follows that $G^{d(\mathscr{H})}$ is the unique largest residual subgroup of G contained in H.

(3) Let $H^*=G^{d(\mathscr{H})}$, and let H^* yield $\mathscr{H}^*$. Since all chief factors between H and H^* are $\mathscr{H}$-central, H/H^* is nilpotent. Moreover, for each p in $\pi(G:H)$ and p-Sylow group Σ_p of G,

$$\mathbf{H}_p(G)/H^*=(\mathbf{H}_p^*(G)/H^*) \times ((H \cap \Sigma_p)\, H^*/H^*)$$

with both factors normal in G/H^*. Hence, the $\mathscr{H}^*$-central chief factors are the $\mathscr{H}$-central ones, and $\Sigma^p \cap \mathbf{H}_p^*(G)=\Sigma^p \cap \mathbf{H}_p(G)$ for each p in $\pi(G:H)$ and each p-complement Σ^p of G.

3. Getting a Complement to H in G

One of the basic results of [8] is that if H is normal in G and if every chief factor of G below H is $\mathscr{H}$-eccentric then H is complemented in G by an $\mathscr{H}$-normalizer of G, a subgroup of form

$$D_{H,\Sigma}=\Sigma_\pi \cap \bigcap_{p\in\pi} N_G(\mathbf{H}_p(G) \cap \Sigma^p),$$

where Σ is a Sylow system of G. This result follows from the fact that $D_{H,\Sigma}$ covers $\mathscr{H}$-central chief factors and avoids $\mathscr{H}$-eccentric ones.

In unpublished work, [3], Hawkes has shown the following extended result.

Theorem 1. *If $H \trianglelefteq G$ and if every uncomplemented chief factor of G below H is $\mathscr{H}$-eccentric, then $D_{H,\Sigma}$ contains a complement to H.*

One interpretation of this theorem is that if enough complementation is imposed by fiat so that the remaining factors are suitably eccentric then H has a complement in G. Example 1 below shows that one cannot go too far in this direction.

The account which follows provides a proof of Theorem 1 which exposes the role of the uncomplemented chief factors as well as the reason why $D_{H,\Sigma}$ should contain a complement to H.

As in Remark (3) above, let $H^* = G^{d(\mathscr{H})}$. By Remark (3), H^* satisfies the hypotheses of Theorem 1 and $D_{H,\Sigma} = D_{H^*,\Sigma}$. Suppose that we can show that $D_{H^*,\Sigma}$ complements H^* in G. By hypothesis, every chief factor between H and H^* is complemented. Let Y be the intersection of a set of complements for factors between H and H^* in some chief series. Then $Y \cdot H = G$ and $Y \cap H = H^*$, so $Y \cap D_{H^*,\Sigma}$ complements H in G, as desired. Thus it will be sufficient to prove the following.

Theorem 1′. *If $H \,\mathbf{r}\, G$ and if every uncomplemented chief factor of G below H is $\mathscr{H}$-eccentric, then $D_{H,\Sigma}$ complements H.*

The proof is based on the following elementary observation.

Lemma. *Let $H \trianglelefteq G$, and suppose that G has an $\mathscr{H}$-central p-chief factor below H. Let L/M be a p-chief factor of G below H such that*

(1) *L/M is $\mathscr{H}$-central,*

(2) *every p-chief factor between H and L is $\mathscr{H}$-eccentric, and*

(3) *$|L|$ is maximal subject to* (1) *and* (2).

Then L/M is the unique minimal G-normal subgroup of H/M. Moreover, if K/L is a chief factor of G with $H \geqq K$, then K/M is a special p-group and L/M is uncomplemented in G.

Proof. If $X/M \cdot\!\lhd G/M$ with $L \neq X \leqq H$, then XL/X satisfies (1) and (2), which contradicts (3). Thus L/M is unique. Now suppose there is a K/L. Since L/M is H-central and unique, K/M is a p-group with at most one characteristic subgroup. If K/M is non-abelian, it must be special. If K/M is abelian but not special (i.e., elementary), then $\Omega_1(K/M) = L/M$. But since $\mathbf{H}_p(G) \cap \Sigma^p$ acts trivially on L/M it would then act trivially on K/M, making K/L $\mathscr{H}$-central, contrary to assumption. Thus K/M is special. If $X \cap L = M$ and $XL = G$ for some X, then $M < X \cap K \lhd XL = G$, in conflict with uniqueness of L.

Example 2 below shows that K/M, when it exists, can be either elementary abelian or non-abelian.

Now suppose Theorem 1′ is false and take L and M as in the lemma. Since $H\,\mathbf{r}\,\mathbf{G}$, $H>L$ and so a K exists. Then L/M is uncomplemented but $\mathscr{H}$-central, which is ruled out by hypothesis.

The lemma also yields a new proof of an earlier result.

Theorem 2. (Corollary 2.2 of [8].) *If $H\,\mathbf{r}\,G$ and H has abelian Sylow groups for all primes in $\pi(G:H)$, then $D_{H,\Sigma}$ complements H in G.*

Proof. Suppose not, and take K, L and M as in the lemma. Then K/M is an elementary abelian p-group. Let $Q=\Sigma^p \cap \mathbf{H}_p(G)$. Then

$$K/M = C_{K/M}(Q) \times [K, Q]\, M/M,$$

and both factors are normal in G/M, since $\Sigma_p \cap \mathbf{H}_p(G)$ is abelian. But both factors are non-trivial, which contradicts uniqueness of L/M.

4. Covering Groups as Complements

If $\mathscr{Y}$ is a sector screen on G, a *$d(\mathscr{Y})$-covering group* of G is a subgroup E of G for which $E^{d(\mathscr{Y})}=1$ and $E \cdot U^{d(\mathscr{Y})}=U$ whenever $E \leqq U \leqq G$. This concept is an obvious extension of the notion of $\mathscr{F}$-covering subgroup in formation theory. For an arbitrary sector screen $\mathscr{Y}$ on G there may not exist $d(\mathscr{Y})$-covering groups, but if H is normal in G and yields $\mathscr{H}$, then G has $d(\mathscr{H})$-covering groups. The theory of $d(\mathscr{Y})$-covering groups is developed in [9], and we refer the reader to that paper for background results, as well as proofs of the following facts.

If E is $d(\mathscr{Y})$-covering in G and if $E \leqq U \leqq G$, then E is $d(\mathscr{Y} \cap U)$-covering in U. By Theorem 1 of [9], if each $\mathbf{Y}_p$ satisfies two conditions, called (A) and (B), if $N \trianglelefteq G$ and if E is $d(\mathscr{Y})$-covering in G, then EN/N is $d(\mathscr{Y}/N)$-covering in G/N. Proposition 4 of [9] says that if we choose a normal subgroup $\mathbf{Y}_p(G)$ arbitrarily for each p in π and let $\mathbf{Y}_p(U)= U \cap \mathbf{Y}_p(G)$ whenever $U \leqq G$, then $\mathscr{Y}$ satisfies (A) and (B). In particular, if the normal subgroup H yields the sector screen $\mathscr{H}$, then $\mathscr{H}$ satisfies (*A*) and (*B*). This is the first point at which condition (iii) of the definition of $\mathscr{H}$ is used.

The natural extension of Theorem 3.1 of [5] to the present setting is the following.

Theorem 3. *Let $\mathscr{Y}$ be a sector screen on G such that each $\mathbf{Y}_p$ satisfies conditions* (A) *and* (B) *of* [9]. *If E is a $d(\mathscr{Y})$-covering group of G which complements the normal subgroup N of G, then $N=G^{d(\mathscr{Y})}$ and all complements to N are conjugate.*

Proof. As just noted, EN/N is $d(\mathscr{Y}/N)$-covering in G/N, so $G/N \in d(\mathscr{Y}/N)$ and thus $N \geqq G^{d(\mathscr{Y})}$. Since $G=E \cdot G^{d(\mathscr{Y})}$ and $E \cap N=1$, $N=G^{d(\mathscr{Y})}$.

Now let G be a minimal counterexample and let X be a complement to N not conjugate to E. Let $A \cdot\!\lhd G$ with $A \leqq N$. We may assume that $XA = EA$. Since E is $d(\mathscr{Y} \cap EA)$-covering in EA, $EA = G$ and $A = N \cdot\!\lhd G$. (The conditions (A) and (B) are inherited by subgroups and factor groups.) Hence, E is maximal but not coreless.

Let $B \cdot\!\lhd G$ with $B \leqq E$. If $B \leqq X$, then E/B and X/B are conjugates. Thus $B \cap X = 1$. Then $X \cap (AB) \lhd XA = G$,

$$G/X \cap (AB) = E(X \cap (AB))/X \cap (AB) \in d(\mathscr{Y})$$

and so $X \cap (AB) \geqq G^{d(\mathscr{Y})} = A$, a final contradiction.

The proof just given makes use of only the most elementary facts about $d(\mathscr{Y})$-covering groups, but at the time [8] was written the theory of $d(\mathscr{Y})$-covering groups had not been worked out. While promising something like Theorem 3 for the future, [8] contained the following result instead.

Theorem 4. (Theorem 3.1 of [8].) *Let $H \trianglelefteq G$ and suppose that $X \leqq G$ with $HX = G$ and $H \cap X = 1$. Suppose that for each prime p dividing $|G:H|$ the subgroup $\mathbf{O}_{p'}(X)$ acts without a fixed point on each p-chief factor of G below H. Then all complements to H in G are conjugate.*

Theorem 5 below shows that Theorem 4 really is the special case of Theorem 3 appropriate to the context of [8], although the relationship was certainly not recognized in [8].

Theorem 5. *Let $H \trianglelefteq G$ and let X complement H in G. Then X is a $d(\mathscr{H})$-covering group of G if and only if for each prime p dividing $|G:H|$ the subgroup $\mathbf{O}_{p'}(X)$ acts without a fixed point on each p-chief factor of G below H.*

Proof. Suppose that X is $d(\mathscr{H})$-covering in G. We claim that if p is in $\pi(G:H)$ and K/L is p-chief in G below H then $\mathbf{O}_{p'}(X)$ acts hypereccentrically on K/L. Since XL/L is $d(\mathscr{H}/L)$-covering in G/L, we may assume that $L = 1$. Since X is $d(\mathscr{H} \cap XK)$-covering in XK, and $\mathbf{H}_p(G) \cap XK = \mathbf{O}_{p'}(X) \cdot K$, K is $\mathbf{O}_{p'}(X)$-hypereccentric, as desired. (Here we use the fact, noted after Theorem 5 of [9], that a $d(\mathscr{H} \cap XK)$-covering group contains an $\mathscr{H} \cap XK$-normalizer.)

Now suppose that for each p in $\pi(G:H)$, $\mathbf{O}_{p'}(X)$ acts hypereccentrically on p-chief factors below H. Then all such factors are $\mathbf{H}_p(G)$-eccentric, so H is complemented by an $\mathscr{H}$-normalizer D of G. Since Theorem 4 is true, X is conjugate to D. We may assume that $X = D$. Let

$$G = H_0 \rhd H_1 \rhd \cdots \rhd H_n = 1$$

be a chief series for G through H, and let D be determined by the Sylow system Σ of G. For $i = 0, \ldots, n$ choose subgroups G_i and D_i as in Theorem 5 of [9] such that

(1) $G_0 = G$,

(2) D_i is the $\mathscr{H}$-normalizer of G_i determined by Σ, and

(3) $G_i = D_{i-1} H_i$.

The remarks following the proof of Theorem 5 of [9] show that Σ reduces into each successive D_i and G_i, and $D_0 \leqq D_1 \leqq \cdots \leqq D_n$. Moreover, D_n is $d(\mathscr{H})$-covering in G.

If $H_{m+1} \geqq H$, then $G_{m+1} = D_m H_{m+1} \geqq DH = G$, so $D_{m+1} = D_m = D$. Suppose $H > H_{m+1}$ and assume inductively that $D_m = D$. Then $G_{m+1} = D \cdot H_{m+1}$ and for p in $\pi(G:H)$

$$\mathbf{H}_p(G) \cap G_{m+1} = (\mathbf{H}_p(G) \cap D) \cdot H_{m+1} = \mathbf{O}_{p'}(D)\, H_{m+1}.$$

Thus each p-chief factor of G_{m+1} below H_{m+1} is $\mathscr{H} \cap G_{m+1}$-eccentric. Hence, $D_{m+1} \cap H_{m+1} = 1$, and so $D_{m+1} = D$. It follows by induction that $D = D_n$, a $d(\mathscr{H})$-covering group of G.

5. Another Condition for Conjugate Complements

Results such as the ones above on complementation by covering groups can be viewed as extensions of Sylow's second theorem by the use of suitable definitions. A second sort of conjugacy result is based on cohomological considerations or a detailed analysis of permutation representations. Section 3 of [8] extended the work of [5] in this direction, ending with an open question which we now answer.

Consider the following hypothesis on G and its subgroups X and H:

(*)
(1) $G = XH$, $X \cap H = 1$, $H \,\mathbf{r}\, G$,
(2) *for all p in $\pi(G:H)$ the p-Sylow groups of X act Frobeniusly on p'-chief factors of G below H, and*
(3) X is quaternion-free.

Theorem 3.4 of [8] states that if D is an $\mathscr{H}$-normalizer of G and if G, D and H satisfy (*), then all complements to H are conjugate. The following result extends that theorem as well as Theorem 4.1 of [5].

Theorem 6. *If (*) holds for G, X and H, then X is contained in an $\mathscr{H}$-normalizer of G. Hence, if some $\mathscr{H}$-normalizer of G complements H then all complements to H are conjugate.*

Proof. The second statement follows from the first and Theorem 3.4 of [8].

Let G be a minimal counterexample. Suppose that H is complemented by an $\mathscr{H}$-normalizer D. Let $A \cdot\!\!\lhd G$ with $A \leqq H$. We may assume that $XA = DA$. If A is a p-group, then for $q \neq p$ the q-Sylow groups of D are conjugate to those of X, while the p-Sylow groups of D act on chief factors between H and A just as the ones of X do. Hence G, D and H satisfy (*), and Theorem 3.4 of [8] makes D and X conjugate, contrary to assumption.

Thus H is not complemented by an $\mathscr{H}$-normalizer, and there is a p in $\pi(G:H)$ and a p-chief factor of G below H centralized by $\mathbf{O}_{p'}(X)$. Thus $\mathbf{O}_{p'}(X)=1$, p is unique, and $\mathbf{O}^p(H)=H=\mathbf{H}_p(G)$.

Let $A \cdot\!\lhd G$ with $A \leqq H$. Since G/A, XA/A and H/A satisfy (*), $X \leqq DA$ for some $\mathscr{H}$-normalizer D of G. Then $D<DA$, so A is $\mathscr{H}$-eccentric, $D \cap A=1$ and $DA<G$. Say A is a q-group.

If $q \neq p$, then A is $\mathscr{H} \cap XA$-hypereccentric, by (*). Thus X and $D \cap (XA)$ are conjugate in XA, since XA, X and A satisfy (*). This contradiction means that $q=p$. Let D be determined by Σ and let $Q=H \cap \Sigma^p$. Since the p-Sylow group $\Sigma_p \cap N_G(Q)$ of D avoids A, $N_A(Q)=1$, and so A is QA-hypereccentric. Since $D \leqq N_G(Q)$, $X \leqq XA \leqq N_G(QA)$. Let $T=QAX$. Then QA is $\mathscr{H} \cap T$-hypereccentric and is complemented in T by X and $D \cap T$. Theorem 3.4 of [8] applies again to give a final contradiction.

Example 3 below shows that even if $H=G^{\mathscr{N}}$ and (*) holds for G, X and H the complements to H need not all be conjugate.

6. Examples

Example 1. Let $U=\langle d, e \mid d^4=e^4=[d,e]^2=[d,e,e]=[d,e,d]=1\rangle$, the free group of exponent 4 and class 2 on 2 generators. Let α be the automorphism of U such that $d\alpha=d^2e^3$ and $e\alpha=d^3e[d,e]$. Then $\alpha^3=1$. Let $H=U]\langle f\rangle$, with $u^f=u\alpha$ for all u in U and $f^3=1$. Let G be the central product $\langle g\rangle H$ with $|g|=4$ and $g^2=[d,e]$. Let $L=\langle d^2[d,e], e^2[d,e]\rangle$ and $X=\langle gd, L\rangle$. Then $L \cdot\!\lhd G$, $|L|=4$, $X \cap H=L$, $X \cdot H=G$, L is H-hypereccentric, but H has no complement in G. A more complicated example with $|L|=p$, $|H|=2 \cdot p^{2p}$ and $|X|=p^2$ can be constructed for p odd.

Example 2. Let H be extra-special of exponent p ($p>2$) and order p^{2n+1} ($p^n>3$). As shown in the proof of Theorem 2 of [7], H has a faithful group X of automorphisms with the following properties:

(1) X fixes the elements of H';

(2) $X=\langle w, r\rangle$ with $|w|=p^n-1$, $|r|=2^k$ for some k and $w^r=w^{-1}$;

(3) if $V=H/H'$ then $V=V_1 \oplus V_2$ with V_1 and V_2 inequivalent faithful irreducible $\mathbf{Z}_p[w]$-modules, X is irreducible on V, and for $i=1,2$, $V_i=U_i/H'$, where U_i is an abelian normal subgroup of H of order p^{n+1}.

Let $\langle z\rangle=Z(H)$, and let $G_1=(H]X)\langle t\rangle$ and $G_2=(H]\langle w\rangle)\langle t\rangle$, where $[t, HX]=1$ and $t^p=z^m$ for some integer m. Then $H=G_1^{\mathscr{N}\mathscr{N}}=G_2^{\mathscr{A}}$, $\mathbf{H}_p(G_1)=HX$ and $\mathbf{H}_p(G_2)=H\langle w\rangle$. Moreover, H/H' is G_1-chief and U_1/H' and

U_2/H' are G_2-chief. The only choice for (K, L, M) in the lemma applied to G_1 is $(H, H', 1)$, and the only choices for G_2 are $(U_1, H', 1)$ and $(U_2, H', 1)$. Thus K/M can be H or the elementary abelian group U_1 of order p^{n+1}. Moreover G_1 and G_2 split over H precisely if $m \equiv 0 \mod p$. (The case $p^n = 3$ can be handled similarly.)

Example 3. Let P be the central product of groups $\langle x_i, y_i \rangle$ for $i = 1, 2, 3$ with $\langle x_i, y_i \rangle$ of order 27 and exponent 3 and with $[x_1, y_1] = [x_2, y_2] = [x_3, y_3] = z$. Then P has automorphisms μ and α with

$$\begin{array}{ll} x_1 \alpha = x_1 x_2 & y_1 \alpha = y_2 y_3^{-1} \\ x_2 \alpha = x_2 x_3 & y_2 \alpha = y_1^{-1} y_2 \\ x_3 \alpha = x_1 x_2 x_3 & y_3 \alpha = y_1 y_2^{-1} y_3 \\ x_1 \mu = x_1 & y_1 \mu = y_1 y_2^{-1} y_3 \\ x_2 \mu = x_1 x_2 & y_2 \mu = y_2 y_3 \\ x_3 \mu = x_1 x_2^{-1} x_3 & y_3 \mu = y_3 . \end{array}$$

Then $\mu^3 = 1$, $\alpha^{13} = 1$ and $\mu^{-1} \alpha \mu = \alpha^3$. Let $G = P] \langle a \rangle] \langle m \rangle$ with $x^a = x \alpha$, $x^m = x \mu$ for x in P, $a^m = a^3$, $a^{13} = 1$ and $m^3 = 1$. Let $H = G^{\mathcal{N}} = P] \langle a \rangle$ and let $X = \langle m \rangle$. Then G, X and H satisfy (*), but $\langle m z \rangle$ is not conjugate to X, as one can check.

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