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An Internal Approach to Covering Groups*

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This note gives an account of normalizers and covering groups more general than both the $\mathscr{F}$-normalizers and $\mathscr{F}$-covering groups of Carter, Hawkes and Gaschütz and the $\mathscr{X}$-normalizers and $\mathscr{X}$-covering groups given by M. J. Prentice. The extended notions are used in an iterative scheme which produces covering groups from normalizers, and the rate of convergence of this method in the case of $\mathscr{F}$-normalizers is compared with the rates of related procedures developed by Yen and by Fischer, Mann, and Graddon. The context is the theory of finite solvable groups.

1. INTRODUCTION

Carter and Hawkes, in [1], define the concept of an $\mathscr{F}$-normalizer, where $\mathscr{F}$ is a locally induced formation, and give a number of connections between $\mathscr{F}$-normalizers and the $\mathscr{F}$-covering groups first considered by Gaschütz. Theorem 5, below, gives an iterative process which can be used to construct an $\mathscr{F}$-covering group of a group G from $\mathscr{F}$-normalizers of certain easily obtained subgroups of G. This paper had its genesis in the observation that the proof of Theorem 5 for $\mathscr{F}$-normalizers and $\mathscr{F}$-covering groups also produces $\mathscr{X}$-covering subgroups from $\mathscr{X}$-normalizers in the sense of Prentice, [3], although the two sorts of normalizers and covering groups are defined quite differently. In this paper the term "normalizer" is used to denote a subgroup whose properties resemble those of a system normalizer or $\mathscr{F}$-normalizer, in a sense to be made precise later. Since Prentice's approach is entirely from within the fixed group G, and since the determination of subgroups of G, even when related to $\mathscr{F}$, ultimately depends on G and not on having the complete list of groups in $\mathscr{F}$ in view, it seems desirable to have some sort of internal formation theory developed entirely within a given group which unifies both previous theories.

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Sections 2 and 3 give the details of such a theory. Desirable sections of G are determined by considering their actions on their chief factors somewhat as in the theory of locally induced formations. Section 3 shows that, with suitable natural restrictions on the actions allowed, the normalizers and covering groups obtained have the familiar properties that seem to make the theory of $\mathscr{F}$-normalizers and $\mathscr{F}$-covering groups work.

Section 4 gives an iterative method for obtaining covering groups from normalizers in an axiomatic setting. The main point of Section 3 is that a number of examples satisfy the axioms. Section 5 treats still another example.

Finally, Sections 6 and 7 compare the method of Theorem 5 with two other iterative processes. In Section 6 an axiomatic treatment similar to the one given to Theorem 5 also yields a generalization of Yen's method of [6] to our setting.

Notation is intended to be standard, and is that of [4] and [5]. The notation $A \cdot \lhd G$ means that A is a minimal normal subgroup of G, while $M < \cdot G$ means that M is a maximal subgroup of G. All groups considered are finite and solvable. Although some of the results given hold under less restrictive hypotheses than solvability, we sacrifice utmost generality for the sake of clarity of exposition.

2. Good Sections

In this part of the paper we develop general notions of normalizer and covering group defined within a given group G. To have a notion of covering group one needs some classification of good sections to be covered. A notion of normalizer, on the other hand, leads to chief factors which are covered by normalizers (and hence are good) and chief factors avoided by normalizers. A section should be good if all of its chief factors are good, and in this way normalizers define good sections and so also determine what covering groups should be. Our development has obvious roots in the work of Carter and Hawkes [1] and Prentice [3].

Throughout, let G be a fixed group. A *dissection* of G is a set of sections of G. If $\mathscr{D}$ is a dissection of G and if $H \leqslant G$, then $\mathscr{D}$ induces a dissection $\mathscr{D}_H$ of H given by

$$\mathscr{D}_H = \{S \in \mathscr{D} \mid S \text{ is a section of } H\}.$$

If $N \trianglelefteq G$, $\mathscr{D}$ also induces a dissection $\mathscr{D}/N$ of G/N defined by

$$\mathscr{D}/N = \{(U/N)/(V/N) \mid U/V \in \mathscr{D} \text{ and } V \geqslant N\}.$$

We shall often denote the section $U/1$ by U and identify $\mathscr{D}/1$ with $\mathscr{D}$. It follows from the definition that if $N \trianglelefteq G$ and $N \leqslant C \leqslant G$, then

$$\mathscr{D}_C/N = (\mathscr{D}/N)_{C/N}.$$

If $\mathscr{D}$ is a dissection of G, the subgroup E of G is a *$\mathscr{D}$-covering* subgroup of G in case

(i) $E \in \mathscr{D}$ and

(ii) $EV = U$ whenever $E \leqslant U \leqslant G$ and $U/V \in \mathscr{D}$.

For each section U/V of G let $\mathrm{Cov}_{\mathscr{D}}(U/V)$ be the set of $\mathscr{D}_U/V$-covering subgroups of U/V. For future reference, we list some immediate consequences of these definitions.

PROPOSITION 1. (a) *If $N \trianglelefteq H \leqslant G$, then $H/N \in \mathscr{D}$ if and only if $H/N \in \mathrm{Cov}_{\mathscr{D}}(H/N)$.*

(b) *If $N \trianglelefteq G$, if $C/N \in \mathrm{Cov}_{\mathscr{D}}(G/N)$, and if $C \leqslant U \leqslant G$, then $C/N \in \mathrm{Cov}_{\mathscr{D}}(U/N)$.*

We need to produce dissections in some systematic way which leads to a rich enough theory. A sort of local induction is one answer.

A *sector* on G is a function $\mathbf{Y}$ defined on the set of all subgroups of G such that $\mathbf{Y}(H) \trianglelefteq H$ for every subgroup H of G.

If $\mathbf{Y}$ is a sector on G and H is a subgroup of G, the restriction of $\mathbf{Y}$ to the set of subgroups of H is a sector on H, denoted $\mathbf{Y}_H$.

If $\mathbf{Y}$ is a sector on G and $N \trianglelefteq G$, $\mathbf{Y}$ yields a sector $\mathbf{Y}/N$ on G/N defined by

$$(\mathbf{Y}/N)(E/N) = \mathbf{Y}(E)N/N \qquad \text{whenever} \quad N \leqslant E \leqslant G.$$

Let π be a set of primes. A *sector screen* of G with support π is a function $\mathscr{Y}$ from π into the set of sectors of G. Denote $\mathscr{Y}(p)$ by $\mathbf{Y}_p$. For H a subgroup of G call the p-chief factor U/V of H *$\mathscr{Y}$-central* in H in case $p \in \pi$ and $\mathbf{Y}_p(H) \leqslant C_H(U/V)$, and call U/V *$\mathscr{Y}$-eccentric* in H otherwise.

Let $d(\mathscr{Y})$ be the set of sections H/K of G such that each H-chief factor of H/K is $\mathscr{Y}$-central in H. Call $d(\mathscr{Y})$ the *dissection* of G *induced* by $\mathscr{Y}$.

Note 1. Each subgroup E of G has a smallest normal subgroup K with the property that $E/K \in d(\mathscr{Y})$. We denote this subgroup by $E^{d(\mathscr{Y})}$.

In making these definitions we have in mind two examples. If $\mathscr{L}$ is a screen with support π, the set $\{\mathscr{L}(p) \mid p \in \pi\}$ of formations locally induces a formation $\mathscr{F}$. If we define sectors $\mathbf{Y}_p$ for p in π by $\mathbf{Y}_p(H) = H^{\mathscr{L}(p)}$, then $d(\mathscr{Y})$ is the set of sections of G which belong to $\mathscr{F}$.

The d($\mathscr{Y}$)-covering groups of G are the $\mathscr{F}$-covering groups in this case, and the $\mathscr{Y}$-central chief factors are the $\mathscr{L}$-central ones.

Another quite different example, due to Prentice ([3]) is obtained by arbitrarily choosing a normal subgroup $X(p)$ of G for each p in π and defining $\mathbf{Y}_p(H) = H \cap X(p)$ whenever $H \leqslant G$. In Prentice's notation, $d(\mathscr{Y}) = \bar{\mathscr{X}}$ and the $d(\mathscr{Y})$-covering groups are the $\mathscr{X}$-covering subgroups of G.

If $H \leqslant G$ and $N \trianglelefteq G$, the sector screen $\mathscr{Y}$ on G induces sector screens $\mathscr{Y}_H$ and $\mathscr{Y}/N$ on H and G/N, respectively, with

$$(\mathscr{Y}_H)(p) = (\mathbf{Y}_p)_H \qquad \text{and} \qquad (\mathscr{Y}/N)(p) = \mathbf{Y}_p/N.$$

From these definitions it is immediate that $\mathscr{Y}_H$ induces $d(\mathscr{Y})_H$ on H and $\mathscr{Y}/N$ induces $d(\mathscr{Y})/N$ on G/N.

Let $\mathscr{Y}$ be a sector screen of G with support π, and let Σ be a Sylow system for G. The *$\mathscr{Y}$-normalizer* of G determined by Σ is the subgroup

$$\textstyle\sum_\pi \cap \bigcap_{p \in \pi} N_G(Y_p(G) \cap \sum^p).$$

The proof of the following result is essentiallly the proof of Theorem 3 of [4] (and is really due to P. Hall).

PROPOSITION 2. *Let $\mathscr{Y}$ be a sector screen of G, and let D be the $\mathscr{Y}$-normalizer of G determined by the Sylow system Σ of G. Then*

(1) *D covers every $\mathscr{Y}$-central chief factor of G and avoids every $\mathscr{Y}$-eccentric chief factor of G, and*

(2) *DK/K is the $\mathscr{Y}/K$-normalizer of G/K determined by $\Sigma K/K$, whenever $K \trianglelefteq G$.*

If $\mathscr{Y}$ is a sector screen of G and U/V is a section of G, let $\mathrm{Nor}_{\mathscr{Y}}(U/V)$ be the set of $\mathscr{Y}_U/V$-normalizers of U/V. Then Proposition 2(2) says that if $D \in \mathrm{Nor}_{\mathscr{Y}}(U)$, then $DV/V \in \mathrm{Nor}_{\mathscr{Y}}(U/V)$.

We single out an easy consequence of Proposition 2(1) for later reference.

PROPOSITION 3. *Let $\mathscr{Y}$ be a sector screen of G. If $U/V \in d(\mathscr{Y})$, then* $\mathrm{Nor}_{\mathscr{Y}}(U/V) = \{U/V\}$.

Each of the properties of $\mathrm{Nor}_{\mathscr{Y}}$ and $\mathrm{Cov}_{d(\mathscr{Y})}$ that we have verified should hold for any definitions of Nor and Cov which can be viewed as generalizations of the familiar notions, and in fact we have used little more than natural definitions and P. Hall's covering-avoiding results so far. The other conditions on Cov and Nor that we need are not true without some additional restrictions on the sectors, however. Examples of various sorts of pathology are not difficult to construct out of S_3, A_4, K_4 and $S_3 \times \mathbf{Z}_2$.

3. Sectors that Work

If the sectors Y_p for p in π are required to be well-behaved, then one can expect the families $\mathrm{Cov}_{d(\mathscr{Y})}(U/V)$ and $\mathrm{Nor}_{\mathscr{Y}}(U/V)$ to have many of the familiar properties of their antecedents. This section shows that if each $\mathbf{Y}_p$ satisfies three conditions, labeled (A), (B), and (C), then the properties needed in Section 4 hold. Since the sectors used to define $\mathscr{L}$-izers and those used by Prentice satisfy (A), (B), and (C), it follows that our convergence process in Section 4 applies to both settings as well as to mixtures of the two.

The three conditions on a sector $\mathbf{Y}$ on G which we consider are the following:

(A) If E and K are subgroups of G with $E \leqslant N_G(K)$, then $\mathbf{Y}(EK) \leqslant E \Rightarrow \mathbf{Y}(EK) \cdot K = \mathbf{Y}(E) \cdot K$.

(B) If E and K are subgroups of G with $E \leqslant N_G(K)$, then $K \leqslant \mathbf{Y}(EK) \Rightarrow \mathbf{Y}(EK) \cdot K = \mathbf{Y}(E) \cdot K$.

(C) If $E \leqslant G$ and if $g \in G$, then $\mathbf{Y}(E)^g = \mathbf{Y}(E^g)$.

Proposition 4. (1) *If $\mathscr{F}$ is a formation, and if $\mathbf{Y}(U) = U^{\mathscr{F}}$ for every subgroup U of G, then $\mathbf{Y}$ satisfies* (A), (B) *and* (C).

(2) *If $X \trianglelefteq G$, and if $\mathbf{Y}(U) = X \cap U$ for every subgroup U of G, then $\mathbf{Y}$ satisfies* (A), (B) *and* (C).

Thus both of our motivating examples, the screens and Prentice's localized formation theory, satisfy all three conditions.

Proof of the Proposition. (1) Since $(EK)^{\mathscr{F}} \cdot K/K = (EK/K)^{\mathscr{F}} = E^{\mathscr{F}}K/K$, $\mathbf{Y}(EK) \cdot K = \mathbf{Y}(E)K$, regardless of where $\mathbf{Y}(EK)$ lies. (C) is also clear.

(2) We have $\mathbf{Y}(EK) \cdot K = [(EK) \cap X]K = (EK) \cap (XK) = K \cdot [E \cap (XK)]$. If $\mathbf{Y}(EK) \leqslant E$, then $(EK) \cap X = E \cap X = \mathbf{Y}(E)$. Thus (A) holds. If $\mathbf{Y}(EK) \geqslant K$, then $K \leqslant X$ and so $K \cdot [E \cap (XK)] = K \cdot \mathbf{Y}(E)$. Hence (B) is true. Again, (C) is obvious, since $X \trianglelefteq G$.

We begin with some elementary consequences of (A), (B) and (C).

Lemma 1. *Conditions* (A), (B), *and* (C) *are inherited by subgroups and factor groups of G.*

Proof. By their nature, the conditions are inherited by subgroups. Clearly (C) is also inherited by factor groups. Let E, K and L be subgroups of G with $L \trianglelefteq G, L \leqslant E \cap K$ and $E \leqslant N_G(K)$.

Suppose that the sector $\mathbf{Y}$ satisfies (A) on G. If $(\mathbf{Y}/L)(EK/L) \leqslant E/L$, then by definition $\mathbf{Y}(EK)L/L = (\mathbf{Y}/L)(EK/L) \leqslant E/L$, so that $\mathbf{Y}(EK) \leqslant E$ and thus $\mathbf{Y}(EK) \cdot K = \mathbf{Y}(E) \cdot K$, by (A). Hence $(\mathbf{Y}/L)(E/L) \cdot (K/L) = \mathbf{Y}(E)K/L = \mathbf{Y}(EK)K/L = (\mathbf{Y}/L)(EK/L) \cdot (K/L)$, as desired.

Suppose that $\mathbf{Y}$ satisfies (B). If $K/L \leqslant (\mathbf{Y}/L)(EK/L)$, then $K \leqslant \mathbf{Y}(EK)L$, so that $K = [\mathbf{Y}(EK) \cap K]L$. Let $K_1 = \mathbf{Y}(EK) \cap K$. Then $K_1 \trianglelefteq EK = EK_1L = EK_1$. Since $K_1 \leqslant \mathbf{Y}(EK_1)$, (B) implies that $\mathbf{Y}(E)\, K_1 = \mathbf{Y}(EK_1)\, K_1 = \mathbf{Y}(EK)$. Thus $(\mathbf{Y}/L)(EK/L) = \mathbf{Y}(EK)L/L = \mathbf{Y}(E)\, K_1L/L = \mathbf{Y}(E)K/L = (\mathbf{Y}/L)(E/L) \cdot (K/L)$ as claimed.

Lemma 2. *Suppose the sector* $\mathbf{Y}$ *on* G *satisfies* (A) *and* (B). *If* E *and* K *are subgroups of* G *with* $E \leqslant N_G(K)$, *then*

$$\mathbf{Y}(E) \leqslant \mathbf{Y}(EK) \cdot K.$$

Proof. Let G be a minimal counterexample. Then $G = EK$ by Lemma 1. If $A \cdot \lhd G$ with $A \leqslant E \cap [\mathbf{Y}(G)K]$, then by minimality of G, using Lemma 1 again,

$$\begin{aligned}\mathbf{Y}(E)A/A &= (\mathbf{Y}/A)(E/A)\\ &\leqslant (\mathbf{Y}/A)(EK/A) \cdot K/A\\ &= [\mathbf{Y}(EK) \cdot K]/A,\end{aligned}$$

which is not true. Thus $E \cap [\mathbf{Y}(G)K]$ is coreless.

Since $\mathbf{Y}(E) \nleqslant \mathbf{Y}(EK)K$, $K > 1$. Let $B \cdot \lhd G$ with $B \leqslant K$. Then $B \nleqslant E$, so that $E < BE$. We may choose E maximal subject to $\mathbf{Y}(E) \nleqslant \mathbf{Y}(G)K$ and then choose K minimal. Then $\mathbf{Y}(BE) \leqslant \mathbf{Y}(G)K$. If $BE < G$, then $\mathbf{Y}(E) \leqslant \mathbf{Y}(BE) \cdot B \leqslant \mathbf{Y}(G) \cdot K$, a contradiction. Thus $BE = G$ and $E < \cdot G$. The minimal choice of K forces $B = K$.

If $\mathbf{Y}(G) \leqslant E$, then (A) yields $\mathbf{Y}(E)K = \mathbf{Y}(EK)K$, which is not true. Thus $E \cdot \mathbf{Y}(G) = G$. If $K \leqslant \mathbf{Y}(G)$, then (B) yields a contradiction. Hence $K \cap \mathbf{Y}(G) = 1$. So $E \cap [\mathbf{Y}(G)K] \lhd EK = G$, and thus $E \cap [\mathbf{Y}(G)K] = 1$, giving a final contradiction.

The next few results show some of the consequences to $\mathrm{Cov}_{d(\mathscr{Y})}$ and $\mathrm{Nor}_{\mathscr{Y}}$ of imposing (A), (B) and (C) on the sectors $\mathbf{Y}_p$.

Theorem 1. *Let* $\mathscr{Y}$ *be a sector screen with support* π. *Suppose that each sector* $\mathbf{Y}_p$ *for* p *in* π *satisfies* (A) *and* (B). *If* $E \in \mathrm{Cov}_{d(\mathscr{Y})}(G)$ *and if* $N \trianglelefteq G$, *then* $EN/N \in \mathrm{Cov}_{d(\mathscr{Y})}(G/N)$.

Proof. Since $EV = U$ whenever $EN \leqslant U$ and $U/V \in d(\mathscr{Y})$, $EN/N \cdot V/N = U/N$ whenever $EN \leqslant U$ and $(U/N)/(V/N) \in d(\mathscr{Y}/N)$. Thus we need only show that $EN/N \in d(\mathscr{Y})$.

This is not immediate, since Prentice's Example 2.1 in [3] shows that if $H \leqslant G$ and $H \in d(\mathscr{Y})$ it need not be true that $HN/N \in d(\mathscr{Y})$ even when each $\mathbf{Y}_p$ satisfies (A), (B), and (C).

To complete the proof of Theorem 1 it will be enough to establish the following.

PROPOSITION 5. *Let $\mathcal{Y}$ be a sector screen with support π such that each sector* $\mathbf{Y}_p$ *satisfies* (A) *and* (B). *If $E \in d(\mathcal{Y})$, if $N \trianglelefteq G$, and if $E \cdot (EN)^{d(\mathcal{Y})} = EN$, then $EN/N \in d(\mathcal{Y})$.*

Proof. Let G be a minimal counterexample. Then $G = EN$, by Lemma 1, but $G/N \notin d(\mathcal{Y})$. By an easy reduction, we may suppose $N \cdot \lhd G$.

There is a prime p in π and a p-chief factor H/K of G above N with $\mathbf{Y}_p(G) \nleqslant C_G(H/K)$. Since $G = EN$, $(H \cap E)/(K \cap E)$ is E-isomorphic to H/K, and since $E \in d(\mathcal{Y})$, $\mathbf{Y}_p(E) \leqslant C_G(H/K)$. Hence $\mathbf{Y}_p(G) \nleqslant \mathbf{Y}_p(E) \cdot F(G)$.

If $\mathbf{Y}_p(G) \leqslant E$, then by (A), $\mathbf{Y}_p(G) \leqslant \mathbf{Y}_p(E) \cdot N$, which is false. Thus $G = \mathbf{Y}_p(G) \cdot E$.

If $N \leqslant \mathbf{Y}_p(G)$, we again get a contradiction, by (B). Thus $N \cap \mathbf{Y}_p(G) = 1$. Hence $E \cap \mathbf{Y}_p(G) \lhd EN = G$. If $E \cap \mathbf{Y}_p(G) = 1$, then $\mathbf{Y}_p(G) \leqslant F(G)$, which is false. Let $B \cdot \lhd G$ with $B \leqslant E \cap \mathbf{Y}_p(G)$. Then $E/B \in d(\mathcal{Y})$ and so by minimality of G, $G/NB \in d(\mathcal{Y})$. Equally, $(E\mathbf{Y}_p(G)/B)/(\mathbf{Y}_p(G)/B) = (G/B)/(\mathbf{Y}_p(G)/B) \in d(\mathcal{Y}/B)$, which means $G/\mathbf{Y}_p(G) \in d(\mathcal{Y})$. Thus $G^{d(\mathcal{Y})} \leqslant (NB) \cap \mathbf{Y}_p(G) = B$. Then $G = EB = E$, a final contradiction.

THEOREM 2. *Let $\mathcal{Y}$ be a sector screen with support π. Suppose that each sector* $\mathbf{Y}_p$ *for p in π satisfies* (A), (B) *and* (C). *Then all $d(\mathcal{Y})$-covering groups of G are conjugate.*

Proof. Let G be a minimal counterexample. Then $1 < G$ and $G \notin d(\mathcal{Y})$.

Let $A \cdot \lhd G$, and let E and F be nonconjugate $d(\mathcal{Y})$-covering groups of G. By Theorem 1, EA/A and FA/A are $d(\mathcal{Y}/A)$-covering in G/A. By minimality of G, $EA = F^gA$ for some g, and since F^g is $d(\mathcal{Y})$-covering in G by (C), we may assume that $EA = FA$.

Now E and F are $d(\mathcal{Y})$-covering in EA, and so by minimality of G, $EA = G$. Since $G/A = EA/A \in d(\mathcal{Y})$ by Theorem 1, $A = G^{d(\mathcal{Y})}$ and A is unique. Since E and F complement A in G, E and F are conjugate, a final contradiction.

THEOREM 3. *Let $\mathcal{Y}$ be a sector screen with support π, and suppose that each sector* $\mathbf{Y}_p$ *satisfies* (A), (B) *and* (C). *If $N \trianglelefteq G$ and if $C/N \in \mathrm{Cov}_{d(\mathcal{Y})}(G/N)$, and $H \in \mathrm{Cov}_{d(\mathcal{Y})}(C)$, then $H \in \mathrm{Cov}_{d(\mathcal{Y})}(G)$.*

Proof. Let G be a minimal counterexample. Then $N > 1$. There is a section U/V of G with $VH < U$ and $U/V \in d(\mathcal{Y})$. If it is true that $UN/VN \in d(\mathcal{Y})$ and $(C \cap U)/(C \cap V) \in d(\mathcal{Y})$, a short argument leads to a contradiction. Since we cannot, in general, make these assertions, a longer proof seems to be necessary.

Suppose that E is a $d(\mathcal{Y})$-covering group of G. By Theorem 1, EN/N is $d(\mathcal{Y}/N)$-covering in G/N and so, by Theorem 2, EN is conjugate to C. We may assume that $EN = C$. Then E and H are $d(\mathcal{Y})$-covering in C, so that E

and H are conjugate in C by Theorem 2. Then H is $d(\mathscr{Y})$-covering in G, which is false. Thus G has no $d(\mathscr{Y})$-covering subgroup.

An easy reduction using Theorem 1 lets us suppose that $N = A\cdot \lhd G$. We also have $C = HN < G$, since H is $d(\mathscr{Y})$-covering in C but not in G. Let $B\cdot \lhd G$.

Suppose that $CB = G$. Then $G/NB = CB/NB \in d(\mathscr{Y})$ by Theorem 1, so that $G^{d(\mathscr{Y})} \leqslant NB$. Let D be a $\mathscr{Y}$-normalizer of G. Then D covers G/NB, so $DNB = G$. Moreover, $DN < G$, and so $DN < \cdot G$. Since $G/N \notin d(\mathscr{Y})$, $B \leqslant G^{d(\mathscr{Y})} \leqslant NB$. Intersection of a chief series for G through B with DN yields a chief series for DN, where the DN-factors above N come from G-factors above NB and hence above $G^{d(\mathscr{Y})}$. By Lemma 2, $\mathbf{Y}_p(DN) \leqslant \mathbf{Y}_p(G) \cdot B \leqslant C_G(H/K)$ for every p in π and p-chief factor H/K of G above NB. Hence, $DN/N \in d(\mathscr{Y})$. Because DN covers $G/G^{d(\mathscr{Y})}N$, DN/N is $d(\mathscr{Y})/N$-covering in G/N. We may suppose that $C = DN$ by Theorem 2.

Now $G = CB$, so that $N\cdot \lhd C$ and $N = C^{d(\mathscr{Y})}$. Thus $D < \cdot C$ and D is $d(\mathscr{Y})$-covering in C. We may suppose that $D = H$, by Theorem 2. Since D avoids both N and B, $G^{d(\mathscr{Y})} = NB$.

Because D is not $d(\mathscr{Y})$-covering in G, $D \cdot U^{d(\mathscr{Y})} < U$ for some U with $D \leqslant U \leqslant G$. Since $D = H \in d(\mathscr{Y})$, $D < U$, and since $NB = G^{d(\mathscr{Y})}$, $U < G$. Now $U \cap (BN) \trianglelefteq UBN = G$. Let $K = U \cap (BN)$. Since $KD = U$, $K > 1$. If $N \leqslant K$, then $C = DN \leqslant U$, and hence $C = U = D \cdot C^{d(\mathscr{Y})}$, which is not true. Thus $1 < K < BN$, so that $K\cdot \lhd G$ and BN/K is G-chief.

Now DK/K is a $\mathscr{Y}$-normalizer of G/K by Proposition 2(2). Since $K\cdot \lhd UBN$, $K\cdot \lhd U$, and so $D < \cdot U$. Because U/K is U-isomorphic to UBN/BN, by Lemma 2 $U/K \in d(\mathscr{Y})$. If $K = U^{d(\mathscr{Y})}$, then $D \cdot U^{d(\mathscr{Y})} = U$, contrary to the choice of U. Thus $U^{d(\mathscr{Y})} = 1$, and since $U < \cdot G$ and U covers $G/G^{d(\mathscr{Y})}$, U is itself $d(\mathscr{Y})$-covering in G, contradicting what has already been shown. It follows that $CB < G$.

Since C/N is $d(\mathscr{Y})/N$-covering in CB/N and H is $d(\mathscr{Y})$-covering in C, by minimality of G, H is $d(\mathscr{Y})$-covering in CB. By Theorem 1, HB/B is $d(\mathscr{Y})/B$-covering in CB/B. Since CB/NB is $d(\mathscr{Y})/NB$-covering in G/NB by Theorem 1, the minimal choice of G forces HB/B to be $d(\mathscr{Y})/B$-covering in G/B.

Since $G \notin d(\mathscr{Y})$, B can be taken to be contained in $G^{d(\mathscr{Y})}$. Because HB/B covers $G/G^{d(\mathscr{Y})}$, $HG^{d(\mathscr{Y})} = G$.

Since H is not $d(\mathscr{Y})$-covering in G, $H \cdot U^{d(\mathscr{Y})} < U$ for some U with $H < U < G$. If $UN < G$, then H is $d(\mathscr{Y})$-covering in UN, by minimality of G, although H is not $d(\mathscr{Y})$-covering in U. Thus $UN = G$, and so $U < \cdot G$ and $U \cap N = 1$.

Let $V = U^{d(\mathscr{Y})}$. If $VN \geqslant G^{d(\mathscr{Y})}$, then $G = CVN = HVN$ and so $U = HV(N \cap U) = HV$, which is false. Thus $G/VN \notin d(\mathscr{Y})$. But $G/VN = UN/VN \cong_U U/V \in d(\mathscr{Y})$, so that G/VN is a π-group. For some p in π there is a G-chief p-factor above VN which is centralized by $\mathbf{Y}_p(U)$ but not by $\mathbf{Y}_p(G)$. Hence,

$$\mathbf{Y}_p(G) \nleqslant \mathbf{Y}_p(U) \cdot F(G).$$

If $\mathbf{Y}_p(G) \leqslant U$, then by (A), $\mathbf{Y}_p(G) \cdot N = \mathbf{Y}_p(U) \cdot N$, which is not true. Thus $G = \mathbf{Y}_p(G) \cdot U$. If $N \leqslant \mathbf{Y}_p(G)$, then we get a contradiction from (B). So $N \cap \mathbf{Y}_p(G) = 1$. Then also $U \cap \mathbf{Y}_p(G) \trianglelefteq U \cdot N = G$. If $U \cap \mathbf{Y}_p(G) = 1$, then since $U < \cdot G$, $\mathbf{Y}_p(G) \cdot \lhd G$ which gives $\mathbf{Y}_p(G) \leqslant F(G)$, a contradiction. Thus $1 < U \cap \mathbf{Y}_p(G)$.

Let $A \cdot \lhd G$ with $A \leqslant U \cap \mathbf{Y}_p(G)$. Since $HAN = CA < G$, as we showed above, H is $d(\mathscr{Y})$-covering in CA by minimality of G. Thus H is $d(\mathscr{Y})$-covering in HA. Then since HA/A is $d(\mathscr{Y})$-covering in U/A by Theorem 1, H is $d(\mathscr{Y})$-covering in U by minimality of G. But H is not $d(\mathscr{Y})$-covering in U by the choice of U. This contradiction completes the proof.

THEOREM 4. *Let $\mathscr{Y}$ be a sector screen with support π, and suppose that each sector Y_p satisfies* (A), (B) *and* (C). *If D is a $\mathscr{Y}$-normalizer of G and if $G^{d(\mathscr{Y})} \leqslant F(G)$, then D is a $d(\mathscr{Y})$-covering subgroup of G.*

Proof. By Proposition 2(1), $D \cdot F(G) = G$. By Lemma 2, $\mathbf{Y}_p(D) \leqslant \mathbf{Y}_p(G) \cdot F(G)$ for each p in π, and so the intersection with D of a chief series for G is a $\mathscr{Y}$-central chief series for D. Hence, $D \in d(\mathscr{Y})$.

Let G be a minimal counterexample, and let $A \cdot \lhd G$. By Proposition 2(2), DA/A is a $\mathscr{Y}$-normalizer of G/A. By minimality of G, DA/A is $d(\mathscr{Y})/A$-covering in G/A. By Theorem 3, D is not $d(\mathscr{Y})$-covering in DA. Thus $D < DA$ and so D avoids A. Thus D is coreless. Moreover, $A \cdot\lhd DA$, since $G = D \cdot F(G)$, so that $D < \cdot DA$. Since $D \in d(\mathscr{Y})$ but D is not $d(\mathscr{Y})$-covering in DA, $D \geqslant (DA)^{d(\mathscr{Y})}$. Since $DA/A \in d(\mathscr{Y})$, $(DA)^{d(\mathscr{Y})} \leqslant A \cap D = 1$. Thus A is $\mathscr{Y}$-central in DA. Moreover, DA is $d(\mathscr{Y})$-covering in G by Theorem 3.

Suppose that B is another minimal normal subgroup of G distinct from A. Then DB is $d(\mathscr{Y})$-covering in G, too. By Theorem 2, DA and DB are conjugate, from which we get $A \leqslant DB$ and thus $DA = DB$. Then $[(AB) \cap D]A = AB$, so that $(AB) \cap D > 1$. Then $1 < (AB) \cap D \lhd ABD = G$, contradicting the fact that D is coreless. It follows that A is unique. Thus $F(G) = \mathbf{O}_p(G)$ for some prime p in π.

Since A is $\mathscr{Y}$-central in DA, $\mathbf{Y}_p(DA) \leqslant C_G(A)$. If $G^{d(\mathscr{Y})} \leqslant \mathbf{Y}_p(G)$, then since $D \cdot G^{d(\mathscr{Y})} = G$, $\mathbf{Y}_p(G) \leqslant \mathbf{Y}_p(DA) \cdot G^{d(\mathscr{Y})}$ by (B) and then $\mathbf{Y}_p(G) \leqslant C_G(A)$, which is false. Thus $G^{d(\mathscr{Y})} > G^{d(\mathscr{Y})} \cap \mathbf{Y}_p(G)$, and there is a chief factor $G^{d(\mathscr{Y})}/W$ with $W \geqslant \mathbf{Y}_p(G) \cap G^{d(\mathscr{Y})}$. Then $G^{d(\mathscr{Y})}/W$ is a p-chief factor centralized by $\mathbf{Y}_p(G)$, contradicting the definition of $G^{d(\mathscr{Y})}$.

4. Convergent Chains of Subgroups

Let G be a given group. For each section U/V of G let $\mathrm{Nor}(U/V)$ and $\mathrm{Cov}(U/V)$ be sets of subgroups of U/V, called *normalizers* and *covering groups*, respectively, of U/V, and consider the following conditions:

I. If $V \trianglelefteq G$ and if $C/V \in \mathrm{Cov}(G/V)$, then $C/V \in \mathrm{Cov}(C/V)$.

II. If $V \trianglelefteq U \leqslant G$ and if $D \in \mathrm{Nor}(U)$, then $DV/V \in \mathrm{Nor}(U/V)$.

III. If $V \trianglelefteq U \leqslant G$ and if $U/V \in \mathrm{Cov}(U/V)$, then $\mathrm{Nor}(U/V) = \{U/V\}$,

IV. If $N \trianglelefteq G$ and $V \trianglelefteq G$, and if $C/N \in \mathrm{Cov}(G/NV)$ and $H/V \in \mathrm{Cov}(C/V)$, then $H/V \in \mathrm{Cov}(G/V)$.

V. If $V \trianglelefteq U \leqslant G$, if $D/V \in \mathrm{Nor}(U/V)$ and if $(D/V) \cdot F(U/V) = U/V$, then $D/V \in \mathrm{Cov}(U/V)$.

Proposition 6. *If $\mathscr{Y}$ is a sector screen such that $\mathbf{Y}_p$ satisfies* (A), (B) *and* (C) *for each p in the support of $\mathscr{Y}$, then* $\mathrm{Nor}_{\mathscr{Y}}$ and $\mathrm{Cov}_{d(\mathscr{Y})}$ *satisfy conditions* I–V *in the role of* Nor *and* Cov, *respectively.*

Proof. The conditions follow from Proposition 1, Proposition 2(2), Proposition 3, Theorem 3 and Theorem 4, in that order.

Theorem 5. *In the setting just described, let $G = H_0 \geqslant H_1 \geqslant \cdots \geqslant H_n = 1$ with $H_i \trianglelefteq G$ and H_{i-1}/H_i nilpotent for $i = 1,\ldots, n$. Suppose the subgroups $D_0, \ldots, D_n$ and $G_0, \ldots, G_n$ satisfy:*

(1) $G_0 = G$,

(2) $D_i \in \mathrm{Nor}(G_i)$ *for $i = 0,\ldots, n$, and*

(3) $G_i = D_{i-1}H_i$ *for $i = 1,\ldots, n$.*

If I–V *hold, then $G_i/H_i \in \mathrm{Cov}(G/H_i)$ for $i = 1,\ldots, n$. In particular, $G_n/1 \in \mathrm{Cov}(G/1)$.*

Proof. Since $D_0 \in \mathrm{Nor}(G)$, $G/G = D_0G/G \in \mathrm{Nor}(G/G)$ by II, and so $G_0/H_0 = G/G \in \mathrm{Cov}(G/G)$ by V with $U = V = G$.

Assume inductively that $G_{i-1}/H_{i-1} \in \mathrm{Cov}(G/H_{i-1})$. Then by I, $G_{i-1}/H_{i-1} \in \mathrm{Cov}(G_{i-1}/H_{i-1})$ and so, by III, $\{G_{i-1}/H_{i-1}\} = \mathrm{Nor}(G_{i-1}/H_{i-1})$.

Now $G_i/H_i = D_{i-1}H_i/H_i \in \mathrm{Nor}(G_{i-1}/H_i)$ by (2) and II. Also by II, $D_{i-1}H_{i-1}/H_{i-1} \in \mathrm{Nor}(G_{i-1}/H_{i-1}) = \{G_{i-1}/H_{i-1}\}$, and so $G_iH_{i-1} = D_{i-1}H_{i-1} = G_{i-1}$.

Since

$$H_{i-1}/H_i \leqslant F(G_{i-1}/H_i) = F(G_iH_{i-1}/G_{i-1}),$$

V implies that $G_i/H_i \in \mathrm{Cov}(G_{i-1}/H_i)$. But then $G_i/H_i \in \mathrm{Cov}(G/H_i)$ by IV and the inductive assumption. The result follows by finite induction.

This theorem has the obvious corollary that if I–V hold and if $\mathrm{Nor}(U/1)$ is nonempty for every subgroup U of G then $\mathrm{Cov}(G)$ is nonempty. It thus provides an alternate proof of Theorem 4.9 [3].

Clearly, $G_{i-1} \geqslant G_i$ for each i, but it may not be possible to choose D_i's such that $D_{i-1} \leqslant D_i$ for all i.

Let $\mathscr{L}$ be a screen and let Σ be a Sylow system for G which reduces into the subgroup H of G. Then by Proposition 1 of [4], Σ also reduces into the $\mathscr{L}$-izer of H with respect to Σ. Thus if we use $\mathscr{L}$-izers for Nor we can choose D_i's such that Σ reduces into each of them as well as into each G_i.

Example 1 of [1] shows that even in this setting the D_i's need not increase. But if each $\mathscr{L}(p)$ is subgroup-closed they do, since then $G_i^{\mathscr{L}(p)} \leqslant G_{i-1}^{\mathscr{L}(p)}$ for each i and so

$$\begin{aligned} D_{i-1} &= \textstyle\sum_\pi \cap G_i \cap \bigcap_{p\in\pi} N_G(G_{i-1}^{\mathscr{L}(p)} \cap \textstyle\sum^p) \\ &\leqslant \textstyle\sum_\pi \cap G_i \cap \bigcap_{p\in\pi} N_G(G_i^{\mathscr{L}(p)} \cap \textstyle\sum^p) \\ &= D_i . \end{aligned}$$

The D_i's also increase if they are $\mathscr{F}$-normalizers and $\mathscr{F} \geqslant \mathscr{N}$. In this case, since G_i/H_i is an $\mathscr{F}$-normalizer of G_{i-1}/H_i, Theorem 4.7 of [1] shows that G_i can be joined to G_{i-1} by an $\mathscr{F}$-critical maximal chain. By repeated use of Theorem 4.3 of [1], $D_{i-1} \leqslant D_i$.

In the Prentice setting, too, the D_i's can be chosen to have Σ reduce into each D_i and G_i. Then

$$\begin{aligned} D_{i-1} &= \textstyle\sum_\pi \cap G_i \cap \bigcap_{p\in\pi} N_G(G_{i-1} \cap \mathbf{Y}_p(G) \cap \textstyle\sum^p) \\ &\leqslant \textstyle\sum_\pi \cap G_i \cap \bigcap_{p\in\pi} N_G(G_i \cap \mathbf{Y}_p(G) \cap \textstyle\sum^p) \\ &= D_i . \end{aligned}$$

5. Some Examples of Sectors Satisfying (A), (B) and (C)

Sectors given by $\mathbf{Y}_p(H) = H^{\mathscr{F}}$ or $\mathbf{Y}_p(H) = H \cap X$ with $X \trianglelefteq G$ satisfy (A), (B) and (C), as noted in Proposition 4. Mixtures involving sectors of the first type for some primes and of the second type for others yield normalizers and covering groups not considered in either [1] or [3]. To get still more normalizers and covering groups we need other sectors satisfying (A), (B), and (C). The following result is easy to verify.

PROPOSITION 7. *Let $\mathscr{F}$ be a formation, and let $X \trianglelefteq G$. If* $\mathbf{Y}(H) = H^{\mathscr{F}} \cap X$ *for all subgroups H of G, then $\mathbf{Y}$ satisfies* (B) *and* (C).

As the example at the end of this section shows, condition (A) can fail for this choice of $\mathbf{Y}$. To get sufficient conditions for (A) we use results (and notation) from [5].

PROPOSITION 8. *Let E, N, X and H be subgroups of G such that $X \trianglelefteq G$, $N \trianglelefteq E$ and H **e** E. Then*

$$X \cap (HN) \leqslant (X \cap H)N,$$

and so $(XN) \cap (HN) = (X \cap H)N$.

Proof. Let G be a minimal counterexample. Then $G = E$. A routine reduction using Proposition 1.1(ii) of [5] lets us assume that $N \cdot \lhd G$. Then $X \cap N = H \cap N = 1$. It is easy to check that $H \cap X = 1$.

Let $L = X \cap (HN)$. Then $1 \neq L \cong_G HL/H \leqslant HN/H$, so $L \cong_G N$ and $HL = HN$. Also, $(NL) \cap H \cong_G L$. If $N \leqslant \mathbf{O}_p(G)$, then since LH/H is ${}^pH$-central, so are L and N, and thus $(NL) \cap H$ is, too, contradicting the fact that H **e** G.

THEOREM 6. *Let $\mathscr{F}$ be a subgroup-closed, locally induced formation with support σ. Let G be a group whose Sylow subgroups are abelian for the primes in σ, and let $X \trianglelefteq G$.*

If $\mathbf{Y}(U) = U^{\mathscr{F}} \cap X$ for all subgroups U of G, then

$$(\mathbf{Y}_E/N)(E/N) = (E/N)^{\mathscr{F}} \cap (XN/N)$$

whenever $N \trianglelefteq E \leqslant G$, and $\mathbf{Y}$ satisfies (A).

Proof. By Corollary 2.3 of [5], $E^{\mathscr{F}}$ **e** E. Hence, by Proposition 8,

$$(XN) \cap (E^{\mathscr{F}}N) = (X \cap E^{\mathscr{F}})N \tag{1}$$

whenever $N \trianglelefteq E \leqslant G$. Since $(\mathbf{Y}_E/N)(E/N) = \mathbf{Y}(E)N/N = (X \cap E^{\mathscr{F}})N/N$, the first assertion follows.

Suppose that $E \leqslant N_G(K)$ and $(EK)^{\mathscr{F}} \cap X \leqslant E$. For (A), we want $[(EK)^{\mathscr{F}} \cap X]K = [E^{\mathscr{F}} \cap X]K$. Since $\mathscr{F}$ is subgroup-closed, $E/E \cap (EK)^{\mathscr{F}} \in \mathscr{F}$, and $(EK)^{\mathscr{F}} \cap X \geqslant E^{\mathscr{F}} \cap X$.

Let $N = E \cap K$ in (1). Then

$$X \cap [E^{\mathscr{F}}(E \cap K)] \leqslant (X \cap E^{\mathscr{F}})(E \cap K),$$

and so

$$\begin{aligned} X \cap (EK)^{\mathscr{F}} &\leqslant X \cap [(EK)^{\mathscr{F}} \cdot K] \cap E \\ &= X \cap [E^{\mathscr{F}}K] \cap E \\ &= X \cap [E^{\mathscr{F}}(K \cap E)] \\ &\leqslant (X \cap E^{\mathscr{F}})(K \cap E), \end{aligned}$$

as desired.

The first conclusion of Theorem 6 simply says that $\mathbf{Y}_E/N$ is defined the way $\mathbf{Y}$ is.

COROLLARY. *Let G be an A-group, and let $X \trianglelefteq G$. Then G has a subgroup E satisfying*

i) $(E' \cap X)' = 1$

and

ii) $E \cdot (H' \cap X)' = H$ *whenever* $E \leqslant H \leqslant G$.

Moreover, all such subgroups of G are conjugate.

Proof. Let $\mathbf{Y}_p(H) = H^{\mathscr{N}} \cap X$ for all p and H. Then (i) and (ii) characterize the $d(\mathscr{Y})$-covering subgroups of G.

The following example shows that the conclusion of Theorem 6 can fail if its hypotheses are weakened too much.

EXAMPLE. Let $G = K \times Y = SL_2(3) \times \mathbf{Z}_4$, and say $Y = \langle y \rangle$ and $Z(K) = \langle z \rangle$. Let $N = \langle zy^2 \rangle$ and $R = \langle y^2 \rangle$. Then $(G^{\mathscr{N}}N) \cap (RN) = \langle z, y^2 \rangle > \langle zy^2 \rangle = (G^{\mathscr{N}} \cap R)N$, so the first conclusion of Theorem 6 fails with $\mathscr{F} = \mathscr{N}$, $E = G$ and $X = R$. The second conclusion fails for a more complicated $\mathscr{F}$. Let $\mathscr{L}$ be the formation generated by A_4, and let $\mathscr{F}$ be the formation of groups for which $G^{\mathscr{L}}$ has exponent 2. Let $E = \mathbf{O}_2(G)$. Then $(EK)^{\mathscr{F}} \cap N = 1$, but $E^{\mathscr{F}} \cap N = N \nleqslant [(EK)^{\mathscr{F}} \cap N]K$. Thus (A) fails with $X = N$.

6. AN EXTENSION OF YEN'S METHOD

Theorem 5 provides a method for obtaining $\mathscr{F}$-covering groups of G by considering $\mathscr{F}$-normalizers of suitably chosen subgroups of G, where $\mathscr{F}$ is a locally induced formation. Yen, in [6], has given another method for constructung sequences leading to $\mathscr{F}$-covering groups. He defines subgroups U_i and V_i by:

(1) $V_0 = G$,

(2) U_i is an $\mathscr{F}$-normalizer of V_i for $i = 1, 2, \ldots$, and

(3) $V_i = U_i \cdot V_{i-1}^{\mathcal{N}\mathscr{F}}$ for $i = 1, 2, \ldots$.

He then shows that

$$U_0 \leqslant U_1 \leqslant \cdots \leqslant U_{r-1} = C = V_r \leqslant \cdots \leqslant V_1 \leqslant V_0$$

for some r, and that C is an $\mathscr{F}$-covering group of G.

In this section we show that this iterative process, too, generalizes to $\mathrm{Nor}_{\mathscr{Y}}$ and $\mathrm{Cov}_{d(\mathscr{Y})}$, and we then show that in the case actually considered by Yen our process from Theorem 5 converges more slowly than his.

In the same way as in Section 4, we consider the following five conditions on sets $\mathrm{Nor}(U/V)$ and $\mathrm{Cov}(U/V)$ of subgroups of sections of G.

VI. If $C \in \mathrm{Cov}\, G$ and $C \leqslant U \leqslant G$, then $C \in \mathrm{Cov}\, U$.

VII. If $V \trianglelefteq U \leqslant G$ and if $C \in \mathrm{Cov}\, U$, then $CV/V \in \mathrm{Cov}(U/V)$.

VIII. If $V \trianglelefteq U \leqslant G$, then $\mathrm{Cov}(U/V)$ is nonempty.

IX. If $V \trianglelefteq U \leqslant G$, then every U-conjugate of a member of $\mathrm{Cov}(U/V)$ is in $\mathrm{Cov}(U/V)$, and all members of $\mathrm{Cov}(U/V)$ are conjugate in U.

X. If $U \leqslant G$ there is a unique smallest normal subgroup U^* of U supplemented by all members of $\mathrm{Cov}\, U$.

PROPOSITION 9. *If $\mathscr{Y}$ is a sector screen such that $\mathbf{Y}_p$ satisfies* (A), (B), *and* (C) *for each p in the support of $\mathscr{Y}$, then* $\mathrm{Nor}_{\mathscr{Y}}$ *and* $\mathrm{Cov}_{d(\mathscr{Y})}$ *satisfy* VI–X *in the roles of* Nor *and* Cov, *respectively.*

Proof. Proposition 1 and Theorem 1 (together with Lemma 1) give VI and VII. We have already observed that VIII follows from Theorem 5. The first part of IX follows from (C) (and Lemma 1), while Theorem 2 gives the second statement. By Note 1, $U^{d(\mathscr{Y})}$ exists, and by VIII there is some C in $\mathrm{Cov}_{d(\mathscr{Y})}(U)$. If $K \trianglelefteq U$ and $U = C \cdot K$, then CK/K is $d(\mathscr{Y})$-covering in U/K by VII, and so $U/K \in d(\mathscr{Y})$ and $K \geqslant U^{d(\mathscr{Y})}$. Clearly $C \cdot U^{d(\mathscr{Y})} = U$. Thus X holds.

THEOREM 7. *Suppose that* Nor *and* Cov *are defined on sections of G and satisfy* II, III, V, *and* VI–X. *Define subgroups U_i and V_i of G by:*

(1) $V_0 = G$,

(2) $U_i \in \mathrm{Nor}(V_i)$ *for* $i = 1, 2, \ldots$, *and*

(3) $V_{i+1} = U_i \cdot (V_i{}^*)^{\mathcal{N}}$.

If $G^* \in \mathcal{N}^t$, *then* $U_{t-1} = V_t \in \mathrm{Cov}(G)$.

Proof. We proceed in a series of steps.

CLAIM 1. For each i, $U_i \cdot V_i^* = V_i$.

Using VIII, let $C \in \mathrm{Cov}(V_i)$. Then $CV_i^* = V_i$ by X and so, by VII, $V_i/V_i^* \in \mathrm{Cov}(V_i/V_i^*)$. From III $\{V_i/V_i^*\} = \mathrm{Nor}(V_i/V_i^*)$, and thus $U_iV_i^* = V_i$, by II.

CLAIM 2. For each i, $V_{i+1}/(V_i^*)^{\mathscr{N}} \in \mathrm{Cov}(V_i/(V_i^*)^{\mathscr{N}})$.

By II, $U_i(V_i^*)^{\mathscr{N}}/(V_i^*)^{\mathscr{N}} \in \mathrm{Nor}(V_i/(V_i^*)^{\mathscr{N}})$. Since $V_i^*/(V_i^*)^{\mathscr{N}} \leqslant F(V_i/(V_i^*)^{\mathscr{N}})$,

$$V_{i+1}/(V_i^*)^{\mathscr{N}} = U_i(V_i^*)^{\mathscr{N}}/(V_i^*)^{\mathscr{N}} \in \mathrm{Cov}(V_i/(V_i^*)^{\mathscr{N}}),$$

using V.

CLAIM 3. If $C_i \leqslant V_i$ and $C_i \in \mathrm{Cov}(G)$, then $C_i(V_i^*)^{\mathscr{N}}$ is conjugate to V_{i+1} in V_i.

By VI, $C_i \in \mathrm{Cov}(V_i)$. From VII, $C_i(V_i^*)^{\mathscr{N}}/(V_i^*)^{\mathscr{N}} \in \mathrm{Cov}(V_i/(V_i^*)^{\mathscr{N}})$. Claim 3 follows from Claim 2 and IX.

CLAIM 4. For each i, V_i contains a member, C_i, of $\mathrm{Cov}(G)$.

This is clear if $i = 0$, by VIII. Assume inductively that $C_i \leqslant V_i$ with $C_i \in \mathrm{Cov}(G)$. By Claim 3, $C_i^v \leqslant V_{i+1}$ for some v in V_i, and by IX $C_i^v \in \mathrm{Cov}(G)$. The claim follows by induction.

CLAIM 5. For each i, $(V_i^*)^{\mathscr{N}} \geqslant V_{i+1}^*$.

By Claim 3, $V_{i+1} = V_i \cdot (V_i^*)^{\mathscr{N}} = C_i^v \cdot (V_i^*)^{\mathscr{N}}$ for some v, and $C_i^v \in \mathrm{Cov}(G)$. By VI, $C_i^v \in \mathrm{Cov}(V_{i+1})$, and so $(V_i^*)^{\mathscr{N}} \geqslant V_{i+1}^*$, by X.

CLAIM 6. If $G^* \in \mathscr{N}^t$ with $t \geqslant 1$, then $U_{t-1} = V_t \in \mathrm{Cov}(G)$.

By repeated use of Claim 5, since $V_0^* \in \mathscr{N}^t$, $V_{t-1}^* \in \mathscr{N}$. Thus $U_{t-1} = V_t$. Then $C_t \leqslant V_t \leqslant V_{t-1}$, so by VI, $C_t \in \mathrm{Cov}(V_{t-1})$. By V, $U_{t-1} \in \mathrm{Cov}(V_{t-1})$. Then $| C_t | = | U_{t-1} | = | V_t |$, by IX, and $C_t = V_t \in \mathrm{Cov}(G)$.

We can compare the two convergence processes in the case in which Nor and Cov are $\mathrm{Nor}_{\mathscr{Y}}$ and $\mathrm{Cov}_{d(\mathscr{Y})}$, respectively, and each Y_p is a formation.

PROPOSITION 10. *Let the formation $\mathscr{F}$ be locally induced by the screen $\mathscr{L}$, and let Σ be a Sylow system of G. Define subgroups U_i and V_i as in Theorem 7 and D_i and G_i as in Theorem 5, taking $\mathscr{L}$-izers relative to Σ as normalizers. Then $V_iH_i = G_i$ for $i = 0, 1,\ldots$, and hence $C = V_t \leqslant V_i \leqslant G_i$ for each i.*

Proof. Since G_i/H_i is $\mathscr{F}$-covering in G/H_i by Theorem 5, and since Σ reduces into exactly one $\mathscr{F}$-covering group of G by Corollary 2.1 of [6], $G_i = C \cdot H_i$ for all i.

Clearly $V_0H_0 = G_0$. Assume inductively that $V_{i-1}H_{i-1} = G_{i-1}$. Now $V_{i-1}/V_{i-1} \cap H_{i-1} \cong G_{i-1}/H_{i-1} \in \mathscr{F}$ and $H_{i-1}/H_i \in \mathscr{N}$, so that $V_{i-1} \cap H_i \geqslant (V_{i-1}^{\mathscr{F}})^{\mathscr{N}}$. Thus $V_iH_i = U_i(V_{i-1}^{\mathscr{F}})^{\mathscr{N}} H_i = C(V_{i-1}^{\mathscr{F}})^{\mathscr{N}} H_i = CH_i = G_i$. The result follows by induction on i.

7. Comparison with the Reducing Method

Another scheme for converging to $\mathscr{F}$-covering groups has been developed by Fischer, Mann and Graddon. Its most general setting, given by Graddon in [2], is that in which $\mathscr{F}$ is a formation locally induced by an integrated set of subgroup-closed formations $\mathscr{F}(p)$. The essence of the method (with a slight change of notation) is the following.

Say that the $\mathscr{F}$-system $\mathscr{F}(\Sigma) = \{G^{\mathscr{F}(p)} \cap \Sigma^p \mid p \in \pi\}$ reduces into the subgroup H of G in case $G^{\mathscr{F}(p)} \cap \Sigma^p \cap H = H^{\mathscr{F}(p)} \cap \Sigma^p$ for each p in π. Choose a Sylow system Σ of G, and let

$$R_G(U; \mathscr{F}) = \langle x \in G \mid \mathscr{F}(\Sigma)^x \text{ reduces into } U\rangle$$

Define subgroups E_i and R_i of G by

(1) $R_0 = G$,

(2) E_i is an $\mathscr{F}$-normalizer of R_i with respect to Σ for $i = 0, 1,\ldots$, and

(3) $R_i = R_{R_{i-1}}(E_{i-1}; \mathscr{F})$ for $i = 1, 2,\ldots$.

Graddon shows in [2] that $\mathscr{F}(\Sigma)$ reduces into each E_i and R_i, and it follows from the proof of Lemma 2.11 of [2] that Σ does, too. Moreover, he shows that if $G \in \mathscr{N}^t\mathscr{F}$, then

$$E_0 \leqslant E_1 \leqslant \cdots \leqslant E_{t-1} = R_t \leqslant \cdots \leqslant R_1 \leqslant R_0,$$

and R_t is $\mathscr{F}$-covering in G.

Proposition 11. *In the setting just described, let D_i and G_i be taken as in Theorem* 5 *using $\mathscr{F}$-normalizers for* Nor *and taking all normalizers with respect to Σ. Then $D_i \leqslant E_i$ and $R_i \leqslant G_i$ for $i = 0, 1,\ldots$.*

Proof. Clearly $R_0 = G_0$ and $D_0 = E_0$. Assume inductively that $D_i \leqslant E_i$ and $R_i \leqslant G_i$ for some i. Since E_i, as an $\mathscr{F}$-normalizer, has just one $\mathscr{F}$-system, and since (by Lemma 2.2 of [2]) some $\mathscr{F}$-system of E_i reduces into D_i,

$$\begin{aligned} R_{i+1} &= \langle x \in R_i \mid \mathscr{F}(\Sigma)^x \text{ reduces into } E_i\rangle \\ &\leqslant \langle x \in R_i \mid \mathscr{F}(\Sigma)^x \text{ reduces into } D_i\rangle. \end{aligned}$$

Since $R_{i+1} \leqslant R_i \leqslant G_i$, $R_{i+1} \leqslant R_{G_i}(D_i; \mathscr{F})$.

If $G \in \mathcal{N}\mathcal{F}$, Theorem 2.15 of [2] shows that $E_0 = R_1$, and thus $R_G(E_0; \mathcal{F})$, which is R_1, is an $\mathcal{F}$-normalizer of G. It follows from this observation, applied to $G_i/G_i^{\mathcal{N}\mathcal{F}}$ rather than to G, and from Theorem 2.8 of [2] that

$$R_{G_i}(D_i\,;\mathcal{F})\cdot G_i^{\mathcal{N}\mathcal{F}} = D_i \cdot G_i^{\mathcal{N}\mathcal{F}}.$$

Since $H_{i+1} \geqslant G_i^{\mathcal{N}\mathcal{F}}$,

$$R_{i+1} \leqslant R_{G_i}(D_i\,;\mathcal{F}) \leqslant D_i \cdot H_{i+1} = G_{i+1}\text{, as desired.}$$

By Corollary 2.1 of [6] the system Σ reduces into a unique $\mathcal{F}$-covering group of G. Hence $D_n = E_{t-1}$, and so $D_{i+1} \leqslant R_{i+1}$. Thus

$$\begin{aligned} D_{i+1} &= \textstyle\sum_\pi \cap R_{i+1} \cap \bigcap_{p\in\pi} N_G(G_{i+1}^{\mathcal{F}(p)} \cap \textstyle\sum^p) \\ &\leqslant \textstyle\sum_\pi \cap R_{i+1} \cap \bigcap_{p\in\pi} N_G(R_{i+1}^{\mathcal{F}(p)} \cap \textstyle\sum^p) \\ &= E_{i+1}\,. \end{aligned}$$

The proposition now follows by induction on i.

As a corollary, we obtain the fact that if $G \in \mathcal{N}^n$, then $D_{n-1} = E_{n-1} = R_n = G_n$. This represents a slight improvement of Graddon's estimate.

The method of Theorem 5 converges most rapidly in case the length of the chain $H_0 \geqslant H_1 \geqslant \cdots \geqslant H_n$ is the Fitting length of G. It is worth noting, though, that the chain need not itself be the upper or lower Fitting series for G in order to have the minimal length. Moreover, by taking a chief series as $H_0 \geqslant H_1 \geqslant \cdots \geqslant H_n$ we can come down by easy stages and can keep close track of the process to determine how the covering groups of G meet its chief factors. Rapid convergence may not always be desirable.

Note also that the reducing method requires subgroup-closure of the local formations $\mathcal{F}(p)$, whereas the method of Theorem 5 does not. Without such an assumption on $\mathcal{F}(p)$'s, however, the D_i's may not be increasing, as our earlier example has shown.

Even if we use $\mathcal{F}$-normalizers for Nor, we can change systems from one D_i to the next and still get $\mathcal{F}$-covering groups at the end. How much freedom of choice there is and whether anything is to be gained from such switching is not clear, though.

References

1. R. Carter and T. Hawkes, The $\mathcal{F}$-normalizers of a finite soluble group, *J. Algebra* **5** (1967), 175–202.
2. C. J. Graddon, $\mathcal{F}$-reducers in finite soluble groups, *J. Algebra* **18** (1971), 574–587.

3. M. J. PRENTICE, $\mathscr{X}$-normalizers and $\mathscr{X}$-covering subgroups, *Proc. Cambridge Phil. Soc.* **66** (1969), 215–230.
4. C. R. B. WRIGHT, On screens and $\mathscr{L}$-izers of finite solvable groups, *Math. Zeit.* **115** (1970), 273–282.
5. C. R. B. WRIGHT, On complements to normal subgroups in finite solvable groups, *Arch. der Math.* **23** (1972), 125–132.
6. TI YEN, On $\mathscr{F}$-normalizers, *Proc. Amer. Math. Soc.* **26** (1970), 49–56.

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