Strict comparison for crossed products by free minimal actions of $\mathbb{Z}^d$

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9 September 2011

Conference on C*-Algebras and Related Topics

Research Institute for Mathematical Sciences, Kyoto University

Kyoto, Japan

5–9 September 2011

Our setup

Throughout, $h \colon \mathbb{Z}^d \times X \to X$ will be a **free minimal** action of $\mathbb{Z}^d$ on a finite dimensional compact metric space X.

We write

$$h^\gamma(x) \qquad \text{for} \qquad h(\gamma, x).$$

If we are actually given a homeomorphism $h \colon X \to X$, this is the usual notation for the corresponding action of $\mathbb{Z}$. Here, however, remember that $\gamma \in \mathbb{Z}^d$.

Free means that for $\gamma \in \mathbb{Z}^d \setminus \{0\}$, we have $h^\gamma(x) \neq x$ for all $x \in X$. That is, h^γ has no fixed points.

Minimal means that the only closed $\mathbb{Z}^d$-invariant subsets of X are X and $\varnothing$. Equivalently, all orbits are dense. (This is the topological version of ergodicity.)

Results

Theorem

Let h be a free minimal action of $\mathbb{Z}^d$ on a finite dimensional compact metric space X. If the action has the topological small boundary property (defined below), then $C^*(\mathbb{Z}^d, X, h)$ has strict comparison of positive elements (defined below).

The topological small boundary property condition is probably redundant.

Strict comparison of positive elements means that the order on the Cuntz semigroup is determined by tracial states. See below.

It is probably also true (work in progress with Dawn Archey) that, under the same hypotheses, $C^*(\mathbb{Z}^d, X, h)$ has stable rank one.

Strict comparison of positive elements

Let A be a C*-algebra, and let $a, b \in A_+$. We write $a \precsim b$ if there is a sequence $(v_n)_{n \in \mathbb{Z}_{>0}}$ such that $\lim_{n \to \infty} v_n b v_n^* = a$.

(This is the order which goes into the definition of the Cuntz semigroup.)

Let $T(A)$ be the set of tracial states on A. For $\tau \in T(A)$, the corresponding dimension function d_τ is defined by $d_\tau(a) = \lim_{n \to \infty} \tau(a^{1/n})$ for $a \in (K \otimes A)_+$. (It is the "measure" of the support projection of a.)

Assume that A is simple. Strict comparison of positive elements means that if $a, b \in (K \otimes A)_+$ and $d_\tau(a) < d_\tau(b)$ for all $\tau \in T(A)$, then $a \precsim b$.

It is the analog for positive elements of having the order on projections over A be determined by traces, that is, if projections $p, q \in M_\infty(A)$ satisfy $\tau(p) < \tau(q)$ for all $\tau \in T(A)$, then $p \precsim q$ in the sense of Murray-von Neumann.

There is a more general concept, the radius of comparison $\mathrm{rc}(A)$. Strict comparison of positive elements is equivalent to $\mathrm{rc}(A) = 0$.

The topological small boundary property

Definition

A closed subset $E \subset X$ is said to be *topologically* $\mathbb{Z}^d$*-small* (here, just *topologically small*) if there is some $m \in \mathbb{Z}_{\geq 0}$ such that whenever $\gamma_0, \gamma_1, \ldots, \gamma_m$ are $m + 1$ distinct elements of $\mathbb{Z}^d$, then

$$h^{\gamma_0}(E) \cap h^{\gamma_1}(E) \cap \cdots \cap h^{\gamma_m}(E) = \varnothing.$$

We refer to m as the *topological smallness constant*.

Definition

The action $h \colon \mathbb{Z}^d \times X \to X$ has the *topological small boundary property* if whenever $K, L \subset X$ are disjoint compact sets, then there exist open sets $U, V \subset X$ such that $K \subset U$, $L \subset V$, $\overline{U} \cap \overline{V} = \varnothing$, and ∂U is topologically $\mathbb{Z}^d$-small.

The topological small boundary property (continued)

Definition

The action $h \colon \mathbb{Z}^d \times X \to X$ has the *topological small boundary property* if whenever $K, L \subset X$ are disjoint compact sets, then there exist open sets $U, V \subset X$ such that $K \subset U$, $L \subset V$, $\overline{U} \cap \overline{V} = \varnothing$, and ∂U is topologically $\mathbb{Z}^d$-small.

If X is a manifold and the action is free and smooth, it has the topological small boundary property with constant $\dim(X)$. One arranges that ∂U is a submanifold of codimension one and that the intersections

$$h^{\gamma_0}(\partial U) \cap h^{\gamma_1}(\partial U) \cap \cdots \cap h^{\gamma_m}(\partial U),$$

for arbitrary $m \in \mathbb{Z}_{>0}$ and distinct $\gamma_0, \gamma_1, \ldots, \gamma_m \in \mathbb{Z}^d$, are transverse.

It is very likely that the topological small boundary property is automatic whenever X has finite covering dimension. (The methods have been developed by Kulesza in 1995.) If $\dim(X) = \infty$, then the topological small boundary property presumably can't hold.

The topological small boundary property (continued)

Definition

A closed subset $E \subset X$ is said to be *topologically small* if there is some $m \in \mathbb{Z}_{\geq 0}$ such that whenever $\gamma_0, \gamma_1, \ldots, \gamma_m$ are $m + 1$ distinct elements of $\mathbb{Z}^d$, then

$$h^{\gamma_0}(E) \cap h^{\gamma_1}(E) \cap \cdots \cap h^{\gamma_m}(E) = \varnothing.$$

If $K \subset X$ is a topologically small compact set, then $\mu(K) = 0$ for every $\mathbb{Z}^d$-invariant Borel probability measure μ on X. The converse is false.

So the topological small boundary property is a stronger condition than the small boundary property associated with mean dimension zero.

Results (continued)

Theorem

Let h be a free minimal action of $\mathbb{Z}^d$ on a finite dimensional compact metric space X. If the action has the topological small boundary property, then $C^*(\mathbb{Z}^d, X, h)$ has strict comparison of positive elements.

Strict comparison of positive elements means that $d_\tau(a) < d_\tau(b)$ for all $\tau \in T(A)$ implies $a \precsim b$.

Corollary

Let h be a free minimal action of $\mathbb{Z}^d$ on a finite dimensional compact metric space X. If the action has the topological small boundary property, then the order on projections over $C^*(\mathbb{Z}^d, X, h)$ is determined by traces.

That is, if projections $p, q \in M_\infty(A)$ satisfy $\tau(p) < \tau(q)$ for all $\tau \in T(A)$, then $p \precsim q$.

The corollary is already known when X is the Cantor set. (The topological small boundary property is then automatic.)

Approximate outline of the rest of the talk

- "Large" subalgebras
- The machinery used to construct "large" subalgebras in crossed products: Rokhlin towers and partition valued functions.
- How "large" subalgebras are constructed for actions of $\mathbb{Z}$ and actions on the Cantor set.
- Why these constructions do not work when $d > 1$ and X is not totally disconnected.
- A sketch of some of what needs to be done to modify the construction to work in the presence of the topological small boundary property.

"Large" subalgebras: background

For context, suppose $h\colon X \to X$ is a minimal homeomorphism. The following "large" subalgebra of $C^*(\mathbb{Z}, X, h)$ has played a key role in the structure theory for this crossed product. Let $u \in C^*(\mathbb{Z}, X, h)$ be the unitary corresponding to the generator of $\mathbb{Z}$. For $Y \subset X$ closed, set

$$C^*(\mathbb{Z}, X, h)_Y = C^*\big(C(X),\, uC_0(X \setminus Y)\big) \subset C^*(\mathbb{Z}, X, h).$$

Then take Y to be a one point set.

This subalgebra is the direct limit of much more tractable subalgebras obtained by taking $\operatorname{int}(Y) \neq \varnothing$. It is not dense, but it is large enough that analyzing it gives a lot of information about $C^*(\mathbb{Z}, X, h)$. In particular, it is a tool for converting Rokhlin towers (see below) into something that can be used algebraically. It was originally introduced by Putnam.

Our approach is to generalize the method used for actions of $\mathbb{Z}^d$ on the Cantor set. This method uses an analog of $C^*(\mathbb{Z}, X, h)_Y$, but even for actions of $\mathbb{Z}^d$ on the Cantor set, no useful easy formula is known.

Definition of a "large" subalgebra

Definition

Let A be an infinite dimensional stably finite simple separable unital exact C*-algebra. A subalgebra $B \subset A$ is said to be *large* in A if:

1. B contains the identity of A.
2. B is simple.
3. The restriction map $T(A) \to T(B)$ is surjective.
4. For every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:
    1. $0 \leq g \leq 1$.
    2. For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
    3. For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j,\ c_j(1 - g) \in B$.
    4. $g \precsim y$ relative to the subalgebra B. (Cuntz subequivalence in B.)

Nothing is said about g being a projection or about g approximately commuting with anything.

Let A be as above, and let $B \subset A$. Then B is large in A if B has the same unit, the same ideals (that is, none), and the same traces, and, in addition:

For every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in A$, $\varepsilon > 0$, and $y \in B_+ \setminus \{0\}$, there are $c_1, c_2, \ldots, c_m \in A$ and $g \in B$ such that:

1. $0 \leq g \leq 1$.
2. For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
3. For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j$, $c_j(1 - g) \in B$.
4. $g \precsim y$ relative to the subalgebra B.

Theorem

Let A be an infinite dimensional stably finite simple separable unital exact C*-algebra. Let $B \subset A$ be large. Then $\mathrm{rc}(A) = \mathrm{rc}(B)$.

In particular, if B has strict comparison of positive elements, then so does A.

We can probably also prove that if B is large in A and B has stable rank one, then so does A. (Joint with Dawn Archey.)

Existence of large subalgebras for actions of $\mathbb{Z}$

Recall that if $h \colon X \to X$ is a (minimal) homeomorphism and $Y \subset X$ is closed, then

$$C^*(\mathbb{Z}, X, h)_Y = C^*\big(C(X),\, uC_0(X \setminus Y)\big) \subset C^*(\mathbb{Z}, X, h).$$

Theorem

Let X be a compact metric space, and let $h \colon X \to X$ be a homeomorphism. Let $Y \subset X$ be finite and intersect each orbit at most once. Then $C^*(\mathbb{Z}, X, h)_Y$ is large in $C^*(\mathbb{Z}, X, h)$.

In particular, Y could be a one point set.

Application of large subalgebras to actions of $\mathbb{Z}$

Theorem

Let X be a compact metric space, and let $h \colon X \to X$ be a homeomorphism. Let $x_0 \in X$. Then $C^*(\mathbb{Z}, X, h)_{\{x_0\}}$ is large in $C^*(\mathbb{Z}, X, h)$.

Recall that

$$C^*(\mathbb{Z}, X, h)_Y = C^*\big(C(X),\, uC_0(X \setminus Y)\big) \subset C^*(\mathbb{Z}, X, h).$$

If we take sets Y_n with $Y_1 \supset Y_2 \supset \cdots$, $\mathrm{int}(Y_n) \neq \varnothing$ for all n, and $\bigcap_{n=1}^{\infty} Y_n = \{x_0\}$, then

$$C^*(\mathbb{Z}, X, h)_{\{x_0\}} = \overline{\bigcup_{n=1}^{\infty} C^*(\mathbb{Z}, X, h)_{Y_n}}.$$

If $\dim(X) < \infty$, we have exhibited $C^*(\mathbb{Z}, X, h)_{\{x_0\}}$ as a direct limit, with no dimension growth, of "recursive subhomogeneous C*-algebras". By a result of Toms, $C^*(\mathbb{Z}, X, h)_{\{x_0\}}$ has strict comparison of positive elements. Since $C^*(\mathbb{Z}, X, h)_{\{x_0\}}$ is large in $C^*(\mathbb{Z}, X, h)$, a theorem stated above implies that $C^*(\mathbb{Z}, X, h)$ has strict comparison of positive elements.

Existence of large subalgebras for actions of $\mathbb{Z}^d$

The theorem on the next slide is what we actually prove to get large subalgebras for free minimal actions of $\mathbb{Z}^d$ with the topological small boundary property.

Parts of the first conditions differ from their counterparts in the definition.

The condition on approximation of cutdowns has a different requirement for smallness of the element g.

The last condition has nothing to do with being large; its purpose is to ensure that the subalgebra has strict comparison of positive elements.

Theorem

Suppose $h \colon \mathbb{Z}^d \times X \to X$ is free, minimal, and has the topological small boundary property. There is a unital $B \subset C^*(\mathbb{Z}^d, X, h)$ such that:

1. $C(X) \subset B$.
2. B is simple.
3. The restriction map $T(C^*(\mathbb{Z}^d, X, h)) \to T(B)$ is bijective.
4. For every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, \ldots, a_m \in C^*(\mathbb{Z}^d, X, h)$, and $\varepsilon > 0$, there are $c_1, c_2, \ldots, c_m \in C^*(\mathbb{Z}^d, X, h)$ and $g \in C(X)$ such that:
   1. $0 \leq g \leq 1$.
   2. For $j = 1, 2, \ldots, m$ we have $\|c_j - a_j\| < \varepsilon$.
   3. For $j = 1, 2, \ldots, m$ we have $(1 - g)c_j$, $c_j(1 - g) \in B$.
   4. $\mathrm{supp}(g)$ has a neighborhood U such that $\mu(U) < \varepsilon$ for all invariant probability measures μ.
5. B is a direct limit, with injective maps and no dimension growth, of recursive subhomogeneous algebras.

Constructing the subalgebra

In the rest of this talk, we give some ideas of the construction of the recursive subhomogeneous algebras $B_n \subset C^*(\mathbb{Z}^d, X, h)$ whose direct limit is the "large" subalgebra B.

The algebras B_n come from Rokhlin towers. There are three equivalent ways of describing the objects we need:

- Systems of Rokhlin towers in X.
- Bounded invariant partition valued functions from X to $\mathbb{Z}^d$ (following Forrest).
- Bounded open subgroupoids of the transformation group groupoid $\mathbb{Z}^d \times X$.

All of these are needed: partition valued functions for the construction, the subgroupoid picture to make the subalgebras $B_n \subset C^*(\mathbb{Z}^d, X, h)$, and the Rokhlin tower picture to prove that the closure of the union of the B_n is "large".

In this talk, I will skip the subgroupoid picture. I will keep the Rokhlin towers, because they are more intuitive than partition valued functions.

Systems of Rokhlin towers

At first, we will omit two important properties: (semi-)continuity and the Følner condition.

Definition

A *system of Rokhlin towers* for $h \colon \mathbb{Z}^d \times X \to X$ is a finite collection of pairs

$$(Y_1, F_1), (Y_2, F_2), \ldots, (Y_m, F_m)$$

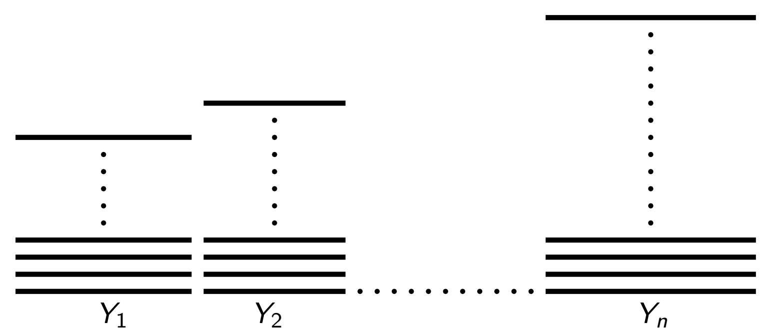
consisting of subsets $Y_j \subset X$ and finite subsets $F_j \subset \mathbb{Z}^d$ such that

$$X = \coprod_{j=1}^{m} \coprod_{n \in F_j} h^n(Y_j).$$

We really do want the sets $h^n(Y_j)$ to be exactly disjoint—no overlaps of any kind. This differs from what has been done elsewhere.

Rokhlin towers for $\mathbb{Z}$

$$X = \coprod_{j=1}^{m} \coprod_{n \in F_j} h^n(Y_j), \qquad \text{with} \qquad F_j = \{0, 1, 2, \ldots, r(j) - 1\}.$$



There are $r(1)$ levels in the first tower: $Y_1, h(Y_1), \ldots, h^{r(1)-1}(Y_1)$.

The picture for actions of $\mathbb{Z}^d$ is nowhere near as neat, but the basic idea (levels indexed by finite subsets of the group) is the same.

Partition valued functions

These are adapted from work of Forrest.

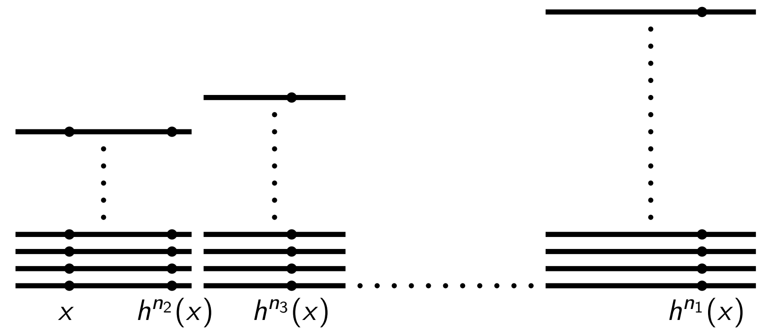
Definition

A *partition valued function* from X to $\mathbb{Z}^d$ is a function $\mathcal{P}$ from X to the set of partitions of $\mathbb{Z}^d$.

- The partition valued function $\mathcal{P}$ is *bounded* if there is a finite upper bound on the diameters of all the sets in all the $\mathcal{P}(x)$.
- The partition valued function $\mathcal{P}$ is *invariant* if for every $x \in X$ and $\gamma \in \mathbb{Z}^d$, we have $\mathcal{P}(h^{\gamma}(x)) = \mathcal{P}(x) - \gamma$. That is, the sets in $\mathcal{P}(h^{\gamma}(x))$ are exactly the sets $T - \gamma$ for $T \in \mathcal{P}(x)$.

The second condition says that if we move forward on X, by applying h^{γ}, then we translate the partition back by γ.

Rokhlin towers for $\mathbb{Z}$, with part of an orbit



In the previous notation for the heights of the towers,

$$n_1 = r(1), \quad n_2 = r(1) + r(n), \quad \text{and} \quad n_3 = r(1) + r(n) + r(1).$$

Each "traverse" of a tower gives one set in the partition associated to x. So $\mathcal{P}(x)$ contains the sets

$$\{0, 1, \ldots, n_1 - 1\}, \quad \{n_1, n_1 + 1, \ldots, n_2 - 1\}, \quad \text{and} \quad \{n_2, n_2 + 1, \ldots, n_3 - 1\}.$$

The partition for $h(x)$ is shifted back one. (This is invariance of $\mathcal{P}$.)

Comparing Rokhlin towers and partition valued functions

Roughly speaking (and with details omitted):

To get a bounded invariant partition valued function from a system of Rokhlin towers: Let $x \in X$. Each set in the partition $\mathcal{P}(x)$ of $\mathbb{Z}^d$ corresponds to a "traverse" by the orbit of x of a single tower.

To get a system of Rokhlin towers from a bounded invariant partition valued function $\mathcal{P}$: We have to single out one set to be the "base" of each of the Rokhlin towers we are supposed to have. Consider all the sets which appear in any $\mathcal{P}(x)$. If a set appears, so do all its translates. Choose one representative from each translation class, getting, say, $F_1, F_2, \ldots, F_m \subset \mathbb{Z}^d$. Define

$$Y_j = \{x \in X \colon F_j \in \mathcal{P}(x)\}.$$

Semicontinuous partition valued functions

The subalgebra $B_n \subset C^*(\mathbb{Z}^d, X, h)$ will be the C*-algebra of the subgroupoid G_n of $\mathbb{Z}^d \times X$ associated with the nth system of Rokhlin towers. In order to get a subalgebra of $C^*(\mathbb{Z}^d, X, h)$, the subset $G_n \subset \mathbb{Z}^d \times X$ must be open.

For partition valued functions, the corresponding condition turns out to be semicontinuity. This means that, given x_0, the partitions associated with points x sufficiently close to x_0 are coarser (as seen by a very large finite set $F \subset \mathbb{Z}^d$).

Definition

Let $\mathcal{P}$ be a partition valued function from X to $\mathbb{Z}^d$. Then $\mathcal{P}$ is *semicontinuous* at $x_0 \in X$ if for every finite set $F \subset \mathbb{Z}^d$ there is an open subset $U \subset X$ containing x_0 such that for every $x \in U$, the partition $\mathcal{P}(x_0) \cap F$, consisting of the nonempty sets of the form $T \cap F$ for $T \in \mathcal{P}(x_0)$, refines the partition $\mathcal{P}(x) \cap F$.

Semicontinuous partition valued functions (continued)

Definition

Let $\mathcal{P}$ be a partition valued function from X to $\mathbb{Z}^d$. Then $\mathcal{P}$ is *semicontinuous* at $x_0 \in X$ if for every finite set $F \subset \mathbb{Z}^d$ there is an open subset $U \subset X$ containing x_0 such that for every $x \in U$, the partition $\mathcal{P}(x_0) \cap F$, consisting of the nonempty sets of the form $T \cap F$ for $T \in \mathcal{P}(x_0)$, refines the partition $\mathcal{P}(x) \cap F$.

The partition valued function $\mathcal{P}$ is *continuous* at x_0 if we can get $\mathcal{P}(x_0) \cap F = \mathcal{P}(x) \cap F$ for every $x \in U$. This condition corresponds to the subgroupoid being both open and closed in $\mathbb{Z}^d \times X$.

Semicontinuity corresponds to taking the sets in the Rokhlin towers to be closed. Continuity corresponds to taking them to be both open and closed.

We can arrange continuity if X is the Cantor set, but not in general.

One of the main new points of the proof of the main theorem is arranging for, and using, semicontinuity.

The Følner condition

To prove that the subalgebra $B = \overline{\bigcup_{n=0}^{\infty} B_n}$ is "large", we will need the finite subsets $F_j \subset \mathbb{Z}^d$ that occur in the systems of Rokhlin towers

$$(Y_1, F_1),\ (Y_2, F_2),\ \ldots,\ (Y_m, F_m)$$

to be increasingly good Følner subsets of $\mathbb{Z}^d$.

Equivalently, the sets in the partitions $\mathcal{P}_n(x)$ are increasingly good Følner subsets of $\mathbb{Z}^d$, uniformly in x as $n \to \infty$.

Refinement

To get $B_1 \subset B_2$:

If $\mathcal{P}_1$ and $\mathcal{P}_2$ are the corresponding partition valued functions from X to $\mathbb{Z}^d$, then we require that $\mathcal{P}_1(x)$ refines $\mathcal{P}_2(x)$ for all $x \in X$. That is, $\mathcal{P}_2$ is coarser than $\mathcal{P}_1$.

The towers associated to $\mathcal{P}_2$ will be taller than those associated to $\mathcal{P}_1$. The condition means that when a point x traverses one of the Rokhlin towers associated with $\mathcal{P}_2$, it must traverse only whole towers (not fractions of them) associated with $\mathcal{P}_1$.

What happens for actions of $\mathbb{Z}$

Notation

For any set $Y \subset X$ and any $x \in X$, we denote by $S_Y(x)$ the set

$$S_Y(x) = \{\gamma \in \mathbb{Z}^d \colon h^{\gamma}(x) \in Y\}.$$

Suppose $d = 1$, so we are considering an action of $\mathbb{Z}$. Let $Y \subset X$ be a closed subset such that $\operatorname{int}(Y) \neq \varnothing$. For $x \in X$, we can write the set $S_Y(x)$ of $n \in \mathbb{Z}$ such that $h^n(x) \in Y$ as

$$S_Y(x) = \{\ldots, n_{-2}(x),\ n_{-1}(x),\ n_0(x),\ n_1(x),\ n_2(x),\ \ldots\},$$

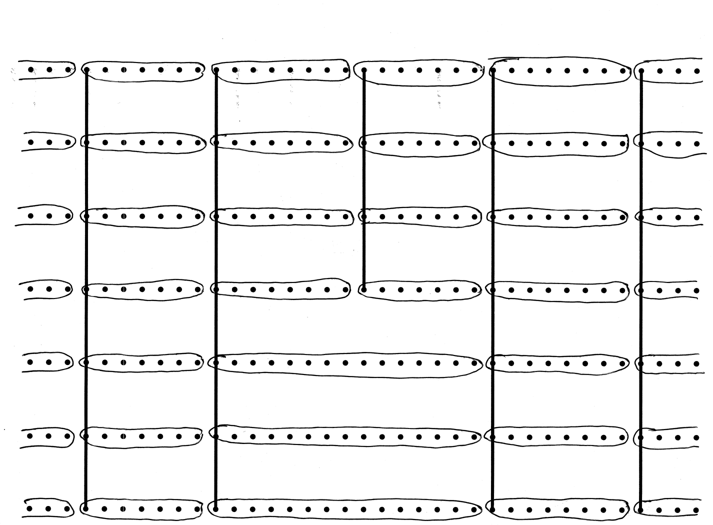
with

$$\cdots < n_{-2}(x) < n_{-1}(x) < 0 \leq n_0(x) < n_1(x) < n_2(x) < \cdots.$$

For $x \in X$, define

$$\mathcal{P}(x) = \big\{[n_{l-1}(x),\ n_l(x)) \cap \mathbb{Z} \colon l \in \mathbb{Z}\big\}.$$

Picture of $\mathcal{P}$



The solid lines are all part of Y. The homeomorphism h moves points one space to the right. For semicontinuity: The endpoint of the segment which ends in the middle is contained in Y. (This is true because Y is closed.)

Actions of $\mathbb{Z}$ (continued)

One gets semicontinuity by choosing Y to be closed.

One gets the Følner condition by choosing Y to have small diameter, so that the intervals appearing in the partitions $\mathcal{P}(x)$ are long.

To get $B = \overline{\bigcup_{n=0}^{\infty} B_n}$ to be "large", one also needs ∂Y to have a universally ε-small neighborhood, that is, a neighborhood U such that $\mu(U) < \varepsilon$ for every invariant Borel probability measure μ on X.

If Y is also open, then $\mathcal{P}$ is continuous.

Let $u \in C^*(\mathbb{Z}, X, h)$ be the standard unitary generator of $\mathbb{Z}$ in the crossed product. The C*-algebra of the corresponding groupoid turns out to the the C*-algebra

$$C^*(\mathbb{Z}, X, h)_{h^{-1}(Y)} = C^*\big(C(X),\, uC_0(X \setminus h^{-1}(Y)\big) \subset C^*(\mathbb{Z}, X, h),$$

as described near the beginning of the talk.

Choosing partition valued functions for actions of $\mathbb{Z}^d$ on the Cantor set

If we try an analogous method to construct $\mathcal{P}(x)$ for an action of $\mathbb{Z}^2$, we need an order on $\mathbb{Z}^2$. If we use the order

$$(\gamma_1, \gamma_2) \leq (\eta_1, \eta_2) \quad \text{if} \quad \gamma_1 \leq \eta_1 \quad \text{and} \quad \gamma_2 \leq \eta_2,$$

we get subsets of $\mathbb{Z}^2$ which may have long arms extending right and up, and which are not even obviously finite.

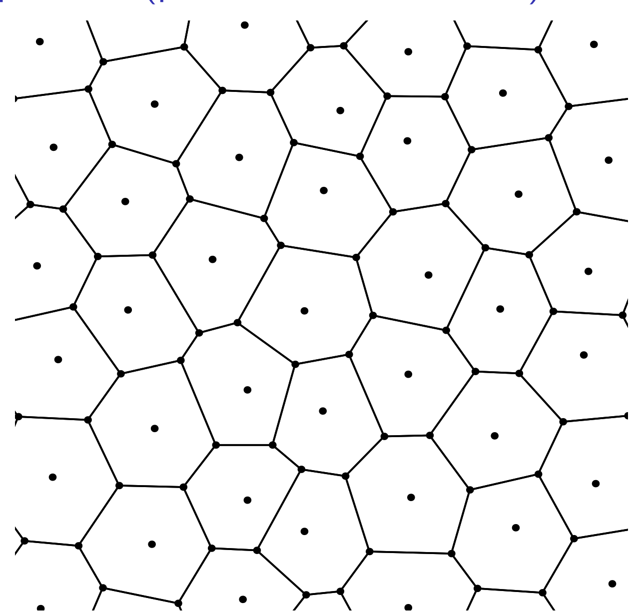
Recall

$$S_Y(x) = \{\gamma \in \mathbb{Z}^d \colon h^{\gamma}(x) \in Y\}.$$

For free minimal actions of $\mathbb{Z}^d$ on the Cantor set, Forrest obtains a partition valued function as follows. One takes the set $T \subset \mathbb{Z}^d$ corresponding to $\gamma \in S_Y(x)$ to be the set of all $\zeta \in \mathbb{Z}^d$ which are closer to γ than to any other element of $S_Y(x)$, using the usual distance from $\|\cdot\|_2$ on $\mathbb{Z}^d$.

Ties must be broken in an invariant way.

Forrest's partition (points of $\mathbb{Z}^d$ not shown)



Choosing partition valued functions for actions of $\mathbb{Z}^d$ on the Cantor set (continued)

If $\mathrm{int}(Y) \neq \varnothing$ and Y has sufficiently small diameter, then $\mathcal{P}$ will be bounded and consist of Følner sets. If Y is both closed and open, then $\mathcal{P}$ will be continuous.

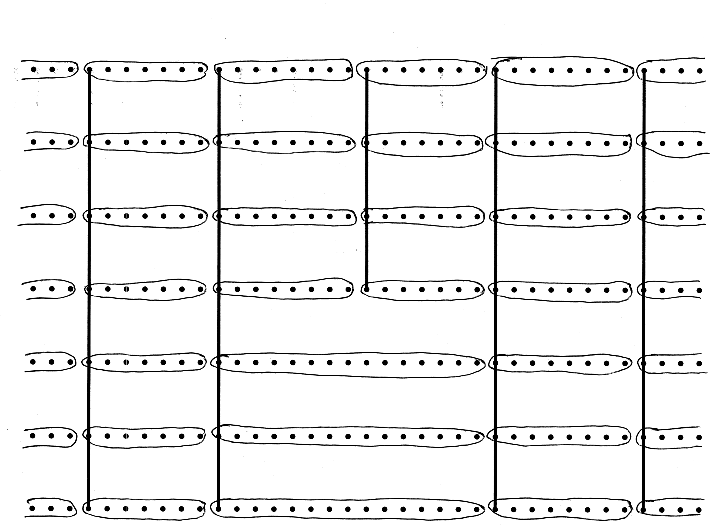
It becomes somewhat awkward to choose a second partition valued function $\mathcal{Q}$ in such a way that $\mathcal{P}$ refines it. To do so, one must group together not just individual points in $\mathbb{Z}^d$, as done above, but rather sets in $\mathcal{P}(x)$.

Unfortunately, if Y is not open, $\mathcal{P}$ will not even be semicontinuous.

In the construction for $\mathbb{Z}$, when the point x moves in such a way that $h^{\gamma}(x)$ crosses out of Y, the result is to combine two sets in the old $\mathcal{P}(x)$ to make one set in the new $\mathcal{P}(x)$.

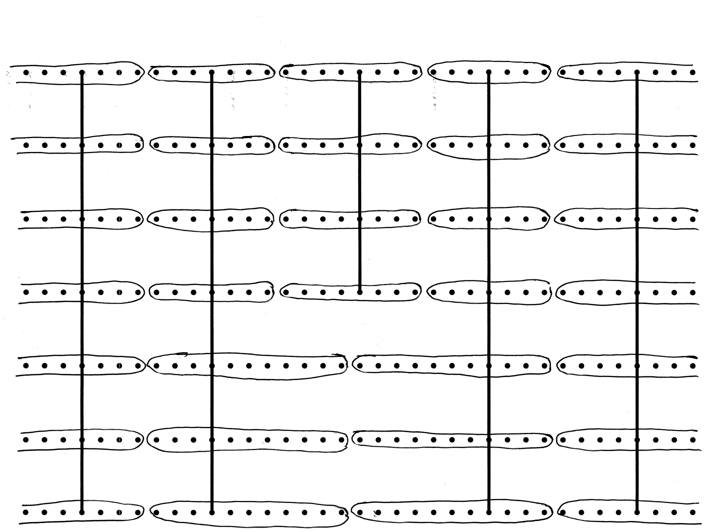
With Forrest's construction, when the set in $\mathcal{P}(x)$ corresponding to γ disappears, its points get distributed among all its neighbors.

What we did before for $\mathbb{Z}$



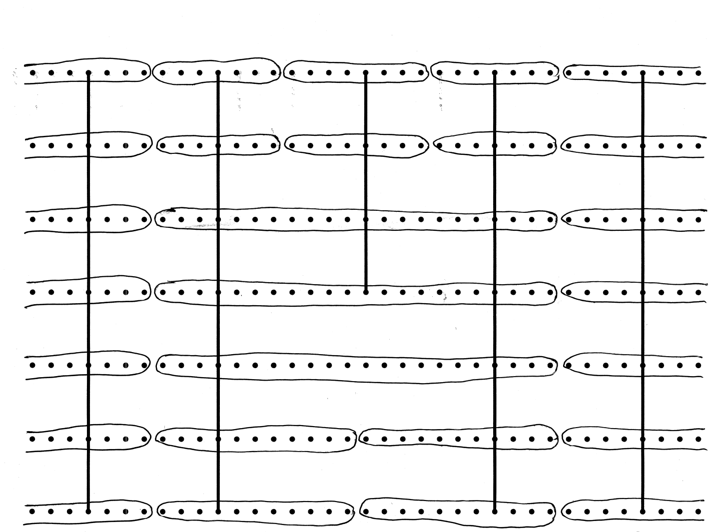
The solid lines are all part of Y. The homeomorphism h moves points one space to the right. For semicontinuity: The endpoint of the segment which ends in the middle is contained in Y. (This is true because Y is closed.)

What Forrest's construction does for $\mathbb{Z}$



You can see the failure of semicontinuity in the middle of the picture: the third partition from the bottom is supposed to be coarser than the one above it.

How to fix Forrest's construction for actions of $\mathbb{Z}$



The long intervals in the middle of the picture occur for an open set of values of x.

Choosing partition valued functions for actions of $\mathbb{Z}^d$: preliminary version

Recall

$$S_Y(x) = \{\gamma \in \mathbb{Z}^d : h^\gamma(x) \in Y\}.$$

Forrest took the set $T \subset \mathbb{Z}^d$ in the partition $\mathcal{P}(x)$ corresponding to $\gamma \in S_Y(x)$ to be the set of all $\zeta \in \mathbb{Z}^d$ which are closer to γ than to any other element of $S_Y(x)$.

We can't make it semicontinuous by choosing Y to be closed (or by choosing Y to be open).

Nevertheless, choose $Y \subset X$ to be closed, to have $\mathrm{int}(Y) \neq \varnothing$, and to have small diameter, and let $\mathcal{P}$ be the partition valued function obtained via Forrest's method. Choose a small open set U containing ∂Y. (We will say something later about how small.)

We construct a new partition valued function $\mathcal{Q}$ which is coarser than $\mathcal{P}$. At $\gamma \in \mathbb{Z}^d$ for which $h^\gamma(x) \in U$, we form a single set in $\mathcal{Q}(x)$ by combining all sets in $\mathcal{P}(x)$ which are within a distance M (chosen later) of γ.

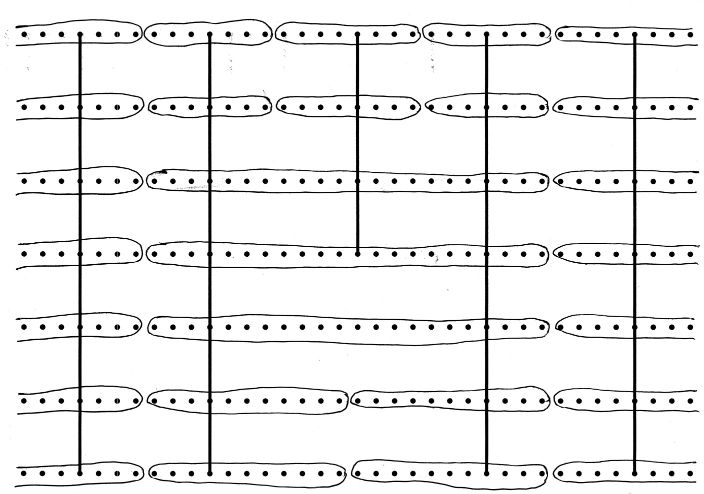
Choosing partition valued functions for actions of $\mathbb{Z}^d$: preliminary version (continued)

To get $\mathcal{Q}$, at $\gamma \in \mathbb{Z}^d$ for which $h^\gamma(x) \in U$, we formed a single set in $\mathcal{Q}(x)$ by combining all sets in $\mathcal{P}(x)$ which are within a distance M (chosen later) of γ.

As $h^\gamma(x)$ crosses ∂U from U into Y, the partition $\mathcal{P}(x)$, for $x \in \partial U$, refines $\mathcal{Q}(x)$, because we just break up the sets which were combined to form the set in $\mathcal{Q}(x)$ containing γ. This is semicontinuity. (We need to ensure $h^\eta(x) \notin \partial Y$ unless η is very far from γ.)
The same thing happens as $h^\gamma(x)$ crosses ∂U from U into $X \setminus Y$.

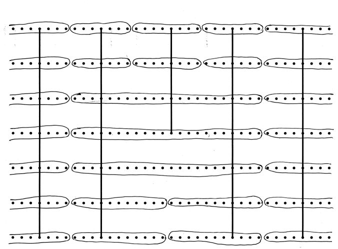
One must also arrange for continuity as $h^\gamma(x)$ moves within U. This requires additional conditions (U is small enough) which will be implicit in the following, but which I will not make explicit here.

How to fix Forrest's construction for actions of $\mathbb{Z}$



In this picture, X is implicitly one dimensional.
We need to be sure one set of long intervals does not interfere with another.

How to fix Forrest's construction for actions of $\mathbb{Z}$

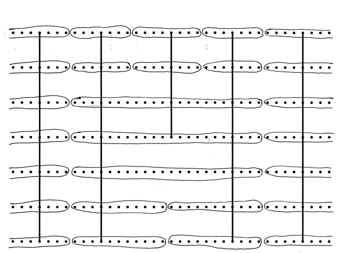


The solid vertical lines are parts of Y. The long intervals in the middle of the picture occur for a small open set U of values of x.

If $\dim(X) = 0$, we can take Y closed and open, so $U = \varnothing$, and there are no long intervals at all.

If $\dim(X) = 1$ (as shown), there would be trouble if one of the other solid lines ended at the same height as the one in the middle. But the solid lines not shown are far enough away that it does not matter if they end at the same height as the one in the middle.

How to fix Forrest's construction for actions of $\mathbb{Z}$

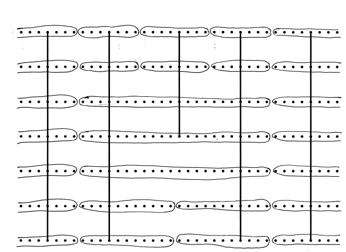


The solid vertical lines are parts of Y. The long intervals in the middle of the picture occur for a small open set U of values of x.

If $\dim(X) = 1$ (as shown), there would be trouble if one of the other solid lines ended at the same height as the one in the middle. But the solid lines not shown are far enough away that it does not matter if they end at the same height as the one in the middle.

By a small perturbation of Y, we can arrange that the ends of the solid lines are all at different heights. Think of U as the unions of vertical open intervals about the end of each solid line. Then U can be chosen small enough that for nearby solid lines, these intervals do not overlap.

How to fix Forrest's construction for actions of $\mathbb{Z}$

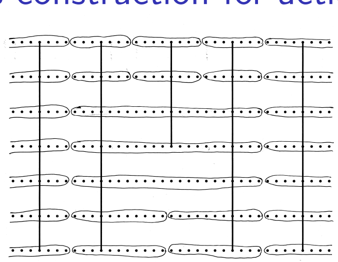


The long intervals in the middle of the picture occur for a small open set U of values of x.

By a small perturbation of Y, we can arrange that the ends of the solid lines are all at different heights. This corresponds to $h^{\gamma_0}(\partial Y) \cap h^{\gamma_1}(\partial Y) = \varnothing$ for $\gamma_0 \neq \gamma_1$.

Think of U as the unions of vertical open intervals about the end of each solid line. Then U can be chosen small enough that for nearby solid lines, these intervals do not overlap. This corresponds to $h^{\gamma_0}(U) \cap h^{\gamma_1}(U) = \varnothing$ for γ_0, γ_1 in a suitable finite set and $\gamma_0 \neq \gamma_1$.

How to fix Forrest's construction for actions of $\mathbb{Z}$



The long intervals occur for a small open set U of values of x.

By a small perturbation of Y, we can arrange that the ends of the solid lines are all at different heights. Then U can be chosen suitably. This corresponds to $h^{\gamma_0}(\partial Y) \cap h^{\gamma_1}(\partial Y) = \varnothing$ for $\gamma_0 \neq \gamma_1$. This is possible when $\dim(X) = 1$.

When $\dim(X) = 2$, the best we can hope for is that

$$h^{\gamma_0}(\partial Y) \cap h^{\gamma_1}(\partial Y) \cap h^{\gamma_2}(\partial Y) = \varnothing.$$

for distinct $\gamma_0, \gamma_1, \gamma_2$. Now M must be larger (putting more sets together) to accommodate overlaps of even the long intervals shown.

Choosing partition valued functions for actions of $\mathbb{Z}^d$: the topological small boundary property

In general, we choose Y so that, in addition, ∂Y is topologically small. That is, there is $m \in \mathbb{Z}_{\geq 0}$ such that whenever $\gamma_0, \gamma_1, \ldots, \gamma_m$ are $m + 1$ distinct elements of $\mathbb{Z}^d$, then

$$h^{\gamma_0}(\partial Y) \cap h^{\gamma_1}(\partial Y) \cap \cdots \cap h^{\gamma_m}(\partial Y) = \varnothing.$$