MATH 618 (SPRING 2024): FINAL EXAM SOLUTIONS

For some solutions, essentially no proofreading has been done.

1. (a) (10 points) State the general version of Cauchy's Theorem.

Solution. (Part of) Theorem 10.35 of Rudin: Let $\Omega \subset \mathbb{C}$ be an open set. Let Γ be a cycle in Ω, and suppose that $\mathrm{Ind}_\Gamma(z) = 0$ for all $z \in \mathbb{C} \setminus \Omega$. Let $f \colon \Omega \to \mathbb{C}$ be a holomorphic function. Then $\int_\Gamma f(\zeta)\, d\zeta = 0$. □

Rudin also includes the following two additional statements. First, the equation

$$\frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = \mathrm{Ind}_\gamma(z) \cdot f(z)$$

holds for every $z \in \Omega \setminus \mathrm{Ran}(\gamma)$. Second, if Γ_1, Γ_2 are cycles in Ω such that $\mathrm{Ind}_{\Gamma_1}(z) = \mathrm{Ind}_{\Gamma_2}(z)$ for all $z \in \mathbb{C} \setminus \Omega$, then

$$\int_{\Gamma_1} f(\zeta)\, d\zeta = \int_{\Gamma_2} f(\zeta)\, d\zeta.$$

The second additional statement is immediate from the statement given in the solution by simply taking $\Gamma = \Gamma_1 - \Gamma_2$, so I don't require it in the solution. The first additional statement should, by Rudin's terminology earlier, be called "the general version of Cauchy's Formula", and I do not expect it as part of the solution.

(b) (10 points) State the Open Mapping Theorem. (The one from complex analysis, not the one about surjective bounded linear maps.)

Solution. Stated before Lemma 10.29 of Rudin: Let $\Omega \subset \mathbb{C}$ be a region. Let $f \colon \Omega \to \mathbb{C}$ be a nonconstant holomorphic function. Then $f(\Omega)$ is a region. □

The important part is that $f(\Omega)$ is open.

The more detailed statement in Theorem 10.32 of Rudin, of which this is a corollary, is not required. The statement that $f(\Omega)$ is open, which isn't explicitly in Theorem 10.32 of Rudin, is required. The statement of Theorem 10.30 of Rudin isn't enough, because it doesn't imply that if $f'(z_0) = 0$ then $f(z_0)$ is in the interior of $f(\Omega)$.

(c) (5 points) State the Prime Number Theorem.

Solution. For $x \in (1, \infty)$, let $\pi(x)$ be the number of positive prime numbers $p \in \mathbb{Z}_{\geq 0}$ such that $p \leq x$. Then $\lim_{x \to \infty} \frac{\pi(x)}{x / \log(x)} = 1$. □

2. (35 points) Let a, b, $c \in \mathbb{C}$ be constants. Let f be the meromorphic function on $\mathbb{C}$ given by

$$f(z) = \frac{a}{z - 1} + \frac{b}{(z - 7)^2} + \frac{c}{z + 27} + e^{iz}.$$

Date: 12 June 2024.

Let $\gamma\colon [0, 2\pi] \to \mathbb{C}$ be given by $\gamma(t) = 19e^{it}$. Evaluate

$$\int_\gamma f(z)\, dz.$$

Solution. Clearly γ is a C^1 closed curve. The Residue Theorem tells us that

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz$$

is the sum of the residues of f at its poles, each multiplied by the winding number of γ about the corresponding pole. The poles of f are only at 1, 7, and -27. (Some of these might not be poles, since some of the constants a, b, and c might be zero.) By Theorem 10.11 of Rudin, $\mathrm{Ind}_\gamma(1) = \mathrm{Ind}_\gamma(7) = 1$ and $\mathrm{Ind}_\gamma(-27) = 0$. (Something must be said here.)

We need only find the residues at 1 and 7. The residue at 1 is the coefficient of $(z-1)^{-1}$ in the expression $q(z) = \sum_{k=1}^{n} d_k (z-1)^{-k}$ when n and the coefficients $d_1, d_2, \ldots, d_n$ are chosen so that $f - q$ has a removable singularity at 1. We can clearly take $q(z) = a(z-1)^{-1}$. So $\mathrm{Res}(f; 1) = a$.

Similarly, since $z \mapsto f(z) - b(z-7)^{-2}$ has a removable singularity at 7, it follows that $\mathrm{Res}(f; 7) = 0$.

So

$$\int_\gamma f(z)\, dz = 2\pi i a.$$

This completes the solution. □

3. (35 points) Set $U = \{z \in \mathbb{C}\colon |z| < 2\}$. Prove that there is no holomorphic function f on U such that for all $z \in \mathbb{C}$ with $|z| = 1$, we have $|f(z) - \frac{1}{z}| < 1$.

Solution. Let f be such a function. Then for all $z \in \mathbb{C}$ with $|z| = 1$, we have $|zf(z) - 1| < 1$. By Rouché's Theorem, $z \mapsto zf(z)$ and the constant function 1 have the same number of zeros in $D = \{z \in \mathbb{C}\colon |z| < 1\}$. Since the constant function 1 has no zeros in D, but $z \mapsto zf(z)$ vanishes at $0 \in D$, this is a contradiction. □

The following solution is adapted from one written by a student.

Solution. Let f be such a function. For $x \in U$ set $g(z) = zf(z) - 1$. Set $D = \{z \in \mathbb{C}\colon |z| < 1\}$. Since $\overline{D}$ is compact, there is $z_0 \in \overline{D}$ at which $|g|$ has a maximum on $\overline{D}$. Now $|g(0)| = |-1| = 1$. However, for $z \in \partial D$, we have

$$|g(z)| = |z| \left| f(z) - \frac{1}{z} \right| = \left| f(z) - \frac{1}{z} \right| < 1.$$

Therefore $z_0 \in D$, and the holomorphic function g has a maximum for its absolute value at z_0. By the Maximum Modulus Theorem, $g|_D$ is constant. By continuity, $g|_{\overline{D}}$ is constant. This contradicts $g(0) = -1$ and $|g(1) < 1$. So no such function f can exist. □

4. (35 points) Let F be the collection of holomorphic functions f on $B_1(0)$ for which the coefficients of the power series expansion $f(z) = \sum_{n=0}^{\infty} c_n z^n$ satisfy $\sup_{n \in \mathbb{Z}_{\geq 0}} |c_n| \leq 2024$. Prove that F is a normal family.

Solution. By Theorem 14.6 of Rudin, it is enough to prove that F is uniformly bounded on every compact set $K \subset B_1(0)$. So let $K \subset B_1(0)$ be compact. Since K is compact, there is $r < 1$ such that $K \subset B_r(0)$. Let $f \in F$, and let $f(z) = \sum_{n=0}^{\infty} c_n z^n$ be its power series expansion. For $z \in K$, we then have

$$|f(z)| \leq \sum_{n=0}^{\infty} |c_n||z|^n \leq \sum_{n=0}^{\infty} 2024\, r^n = \frac{2024}{1-r}.$$

Since this number is finite and independent of f, we have proved that F is uniformly bounded on K. □

The set F is not bounded, let alone uniformly bounded, on $B_1(0)$. For example, it contains the function

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n.$$

5. (35 points) Let $b_0, b_1, \ldots, c_0, c_1, \ldots \in \mathbb{C}$. Set

$$V = \big\{ z \in \mathbb{C} \colon |z - 3| < 1 \big\}.$$

Suppose that for every $z \in V$, the series $\sum_{n=0}^{\infty} b_n z^n$ and $\sum_{n=0}^{\infty} c_n z^n$ converge, and that the sums are equal. Prove that $b_n = c_n$ for every $n \in \mathbb{Z}_{\geq 0}$.

(Caution: $0 \notin V$, so the usual method can't be applied directly.)

Solution. Set

$$W = \big\{ z \in \mathbb{C} \colon |z| < 4 \big\}.$$

For every $r \in [0, 4)$, there is $z \in V$ such that $|z| \geq r$, so that the series $\sum_{n=0}^{\infty} b_n z^n$ and $\sum_{n=0}^{\infty} c_n z^n$ both converge. Therefore both series have radius of convergence at least 4. Accordingly, the formulas

$$f(z) = \sum_{n=0}^{\infty} b_n z^n \qquad \text{and} \qquad g(z) = \sum_{n=0}^{\infty} c_n z^n$$

define holomorphic functions on W.

The hypotheses imply that $f|_V = g|_V$. Since W is open and connected, and V has a cluster point in W, it follows that $f = g$. The usual uniqueness theorem for power series therefore implies that $b_n = c_n$ for every $n \in \mathbb{Z}_{\geq 0}$. (For example, $n! b_n = f^{(n)}(0) = g^{(n)}(0) = n! c_n$.) □

6. (35 points) Let $\Omega \subset \mathbb{C}$ be a region, let $f \colon \Omega \to \mathbb{C}$ be holomorphic and not the zero function, and set $A = \{a \in \Omega \colon f(a) = 0\}$. Suppose A is the disjoint union $A = A_1 \amalg A_2$. Prove that there are holomorphic functions $f_1, f_2 \colon \Omega \to \mathbb{C}$ such that $f_1(z) f_2(z) = f(z)$ for all $z \in \Omega$, $f_1(z) = 0$ only when $z \in A_1$, and $f_2(z) = 0$ only when $z \in A_2$.

Solution. Write $A_2 = \{a_1, a_2, \ldots\}$ with $a_1, a_2, \ldots$ distinct, or, if A_2 is finite, $A_2 = \{a_1, a_2, \ldots, a_n\}$ with $a_1, a_2, \ldots, a_n$ distinct. For $j \in \mathbb{Z}_{>0}$ or $j \in \{1, 2, \ldots, n\}$ as appropriate, let m_j be the multiplicity of a_j as a zero of f.

Since f is not the zero function, A has no limit points in Ω, so neither does A_2. By Theorem 15.11 of Rudin, there is a holomorphic function f_1 on Ω such that f_1 has a zero of multiplicity m_j at a_j for every $j \in \mathbb{Z}_{>0}$ or $j \in \{1, 2, \ldots, n\}$ as appropriate, and no other zeros.

Define a holomorphic function $g\colon \Omega \setminus A_2 \to \mathbb{C}$ by

$$g(z) = \frac{f(z)}{f_1(z)}$$

for $z \in \Omega \setminus A_2$. We claim that for every $a \in A_2$, the limit $\lim_{z\to a} g(z)$ exists and is nonzero. Given this, g has a removable singularity at every point in A_2, so extends to a holomorphic function $f_2\colon \Omega \to \mathbb{C}$, and moreover $f_2(a) \neq 0$ for all $a \in A_2$. By continuity, we have $f_1(z)f_2(z) = f(z)$ for all $z \in \Omega$. Since $f = f_1 f_2$ vanishes only on $A_1 \amalg A_2$ and f_2 does not vanish on A_2, f_2 can only vanish on A_1.

To prove the claim, since f and f_1 both have an isolated zero of multiplicity m_j at a_j, there are holomorphic functions $h, l\colon \Omega \to \mathbb{C}$ such that $h(a_j) \neq 0$, $l(a_j) \neq 0$, and $f(z) = (z - a_j)^{m_j} h(z)$ and $f_1(z) = (z - a_j)^{m_j} l(z)$ for all $z \in \Omega$. Then

$$\lim_{z\to a} g(z) = \lim_{z\to a} \frac{f(z)}{f_1(z)} = \lim_{z\to a} \frac{h(z)}{l(z)} = \frac{h(a_j)}{l(a_j)} \neq 0.$$

This proves the claim, and completes the solution. □

Extra Credit. (50 extra credit points) Define $f(x) = \exp(-x^4)$ for $x \in \mathbb{R}$. Prove carefully that there is an entire function g whose restriction to $\mathbb{R}$ is the Fourier transform $\widehat{f}$ of f. (Grading will be considerably stricter than on the regular problems.)

Solution. Define

$$g(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-izx}\, dx.$$

We need to show that g is well defined for $z \in \mathbb{C}$, and then that g is holomorphic there.

The first thing to do is to estimate the integrand.

For $z \in \mathbb{C}$, we have

$$\begin{aligned} |f(x)e^{-izx}| &= |\exp(-x^4 - izx)| = \exp(\mathrm{Re}(-x^4 - izx)) \\ &= \exp(-x^4 + x\mathrm{Im}(z)) \leq \exp(-x^4 + |x| \cdot |z|). \end{aligned}$$

For use below, we work a little more on this. We claim that

$$|x| \cdot |z| \leq \tfrac{1}{2} x^4 + |z|(2|z| + 1).$$

This claim holds because $|x| \cdot |z|$ is bounded by the first term on the right when $|x| \geq 2|z| + 1$, and by the second term on the right when $|x| \leq 2|z| + 1$. So

$$|f(x)e^{-izx}| \leq \exp\big(-\tfrac{1}{2}x^4\big) \exp\big(|z|(2|z| + 1)\big).$$

We know that $\exp\big(-\tfrac{1}{2}x^4\big)$ is integrable on $\mathbb{R}$, and $\exp\big(|z|(2|z|+1)\big)$ is a constant, so $x \mapsto |f(x)e^{-izx}|$ is integrable on $\mathbb{R}$, and $g(z)$ is defined for all z.

Now we have to prove that g is holomorphic. Possibly the simplest procedure is to combine Morera's Theorem and Fubini's Theorem. It is important to get the details right here.

First, we need to prove that g is continuous. (This is one of the hypotheses of Morera's Theorem.)

Let $z \in \mathbb{C}$, and let $(z_n)_{n\in\mathbb{Z}_{>0}}$ be a sequence in $\mathbb{C}$ such that $\lim_{n\to\infty} z_n = z$. Set $r = \sup_n |z_n|(2|z_n| + 1)$. Then $r < \infty$. For $x \in \mathbb{R}$ and $n \in \mathbb{Z}_{>0}$, set

$$h(x) = f(x)e^{-izx}, \quad h_n(x) = f(x)e^{-iz_n x}, \quad \text{and} \quad k(x) = \exp\big(-\tfrac{1}{2}x^4\big)\exp(r).$$

Then k is integrable, $|h_n| \leq k$ for all $n \in \mathbb{Z}_{>0}$, and $h_n \to h$ pointwise. So the Dominated Convergence Theorem implies that $\lim_{n\to\infty} \int_{-\infty}^{\infty} h_n(x)\,dx = \int_{-\infty}^{\infty} h(x)\,dx$, that is, $\lim_{n\to\infty} g(z_n) = g(z)$. So g is continuous.

(One must use sequences in the Dominated Convergence Theorem. It isn't true for more general kinds of limits.)

Now we verify the other hypothesis of Morera's Theorem.

Let $\gamma\colon [\alpha, \beta] \to \mathbb{C}$ be a piecewise C^1 closed curve in $\mathbb{C}$. (A triangle suffices, but this restriction doesn't help with the proof.) We prove that $\int_\gamma g(z)\,dz = 0$. Rewrite using the definition of the path integral and the definition of g:

$$\begin{aligned}\int_\gamma g(z)\,dz &= \int_\alpha^\beta \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-x^4) e^{-itx}\,dx \right) \gamma'(t)\,dt \\ &= \frac{1}{\sqrt{2\pi}} \int_\alpha^\beta \left(\int_{-\infty}^{\infty} \exp(-x^4) e^{-itx} \gamma'(t)\,dx \right) dt.\end{aligned}$$

Let $H(t,x) = \exp(-x^4)e^{-itx}\gamma'(t)$, for $(t,x) \in [\alpha,\beta] \times \mathbb{R}$. If γ is differentiable everywhere except at $t_1, \ldots, t_n$, then H is continuous on

$$([\alpha,\beta] \setminus \{t_1,\ldots,t_n\}) \times \mathbb{R},$$

and so is measurable on $[\alpha,\beta] \times \mathbb{R}$. (We can ignore the set $\{t_1,\ldots,t_n\} \times \mathbb{R}$, which has measure 0.) Therefore $|H|$ is measurable on $[\alpha,\beta] \times \mathbb{R}$. We apply Fubini for nonnegative functions (Theorem 8.8(a) of Rudin). For this, we use

$$\|\gamma\| = \sup_{t\in[\alpha,\beta]} |\gamma(t)| \qquad \text{and} \qquad \|\gamma'\| = \sup_{t\in[\alpha,\beta]} |\gamma'(t)|,$$

which are both finite because γ is assumed piecewise C^1. Using the estimate

$$|f(x)e^{-izx}| \leq \exp(-\tfrac{1}{2}x^4)\exp(|z|(2|z|+1)).$$

from above, we get

$$|H(t,x)| \leq \|\gamma'\| \exp(\|\gamma\|(2\|\gamma\|+1)) \exp(-\tfrac{1}{2}x^4).$$

Theorem 8.8(a) of Rudin allows us to estimate

$$\begin{aligned}&\int_{[\alpha,\beta]\times\mathbb{R}} |H|\,d(m\times m) \\ &\qquad \leq \int_{[\alpha,\beta]\times\mathbb{R}} \|\gamma'\| \exp(\|\gamma\|(2\|\gamma\|+1)) \exp(-\tfrac{1}{2}x^4)\,d(m\times m)(t,x) \\ &\qquad = \int_\alpha^\beta \left(\int_{-\infty}^{\infty} \|\gamma'\| \exp(\|\gamma\|(2\|\gamma\|+1)) \exp(-\tfrac{1}{2}x^4)\,dx \right) dt \\ &\qquad = (\beta-\alpha)\|\gamma'\| \exp(\|\gamma\|(2\|\gamma\|+1)) \int_{-\infty}^{\infty} \exp(-\tfrac{1}{2}x^4)\,dx < \infty.\end{aligned}$$

Therefore H is integrable, and so also is $\frac{1}{\sqrt{2\pi}}H$. Now we are allowed to apply Fubini for integrable functions (Theorem 8.8(c) of Rudin, as extended in Theorem 8.12 of

Rudin) to get

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}} \int_\alpha^\beta \left(\int_{-\infty}^\infty \exp(-x^4) e^{-itx} \gamma'(t)\, dx \right) dt \\
&\qquad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \left(\int_\alpha^\beta \exp(-x^4) e^{-itx} \gamma'(t)\, dt \right) dx \\
&\qquad = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \left(\exp(-x^4) \int_\gamma e^{-izx}\, dz \right) dx.
\end{aligned}$$

We know $\int_\gamma e^{-izx}\, dz = 0$ by Cauchy's Theorem, for all $x \in \mathbb{R}$. So we have shown that $\int_\gamma g(z)\, dz = 0$ for all γ, and Morera's Theorem implies that g is holomorphic. $\square$

It is also possible to use the Dominated Convergence Theorem to prove directly that the appropriate difference quotients converge. This proof is omitted.