MATH 618 (SPRING 2024): FINAL EXAM

Instructions: All lemmas, claims, examples, counterexamples, etc. require proof, except when explicitly stated otherwise. If you use a major theorem, be sure to cite it by name.

Closed book: No notes, books, calculators, cell phones, other electronic devices, or any outside assistance of any kind.

Please do not write anything less than 1/4 inch from any side of any page.

1. (a) (10 points) State the general version of Cauchy's Theorem.

(b) (10 points) State the Open Mapping Theorem. (The one from complex analysis, not the one about surjective bounded linear maps.)

(c) (5 points) State the Prime Number Theorem.

2. (35 points) Let a, b, $c \in \mathbb{C}$ be constants. Let f be the meromorphic function on $\mathbb{C}$ given by

$$f(z) = \frac{a}{z-1} + \frac{b}{(z-7)^2} + \frac{c}{z+27} + e^{iz}.$$

Let $\gamma\colon [0, 2\pi] \to \mathbb{C}$ be given by $\gamma(t) = 19e^{it}$. Evaluate

$$\int_\gamma f(z)\, dz.$$

3. (35 points) Set $U = \{z \in \mathbb{C}\colon |z| < 2\}$. Prove that there is no holomorphic function f on U such that for all $z \in \mathbb{C}$ with $|z| = 1$, we have $|f(z) - \frac{1}{z}| < 1$.

4. (35 points) Let F be the collection of holomorphic functions f on $B_1(0)$ for which the coefficients of the power series expansion $f(z) = \sum_{n=0}^{\infty} c_n z^n$ satisfy $\sup_{n \in \mathbb{Z}_{\geq 0}} |c_n| \leq 2024$. Prove that F is a normal family.

5. (35 points) Let $b_0, b_1, \ldots, c_0, c_1, \ldots \in \mathbb{C}$. Set

$$V = \big\{z \in \mathbb{C}\colon |z - 3| < 1\big\}.$$

Suppose that for every $z \in V$, the series $\sum_{n=0}^{\infty} b_n z^n$ and $\sum_{n=0}^{\infty} c_n z^n$ converge, and that the sums are equal. Prove that $b_n = c_n$ for every $n \in \mathbb{Z}_{\geq 0}$.

(Caution: $0 \notin V$, so the usual method can't be applied directly.)

6. (35 points) Let $\Omega \subset \mathbb{C}$ be a region, let $f\colon \Omega \to \mathbb{C}$ be holomorphic and not the zero function, and set $A = \{a \in \Omega\colon f(a) = 0\}$. Suppose A is the disjoint union $A = A_1 \amalg A_2$. Prove that there are holomorphic functions $f_1, f_2\colon \Omega \to \mathbb{C}$ such that $f_1(z) f_2(z) = f(z)$ for all $z \in \Omega$, $f_1(z) = 0$ only when $z \in A_1$, and $f_2(z) = 0$ only when $z \in A_2$.

Extra Credit. (50 extra credit points) Define $f(x) = \exp(-x^4)$ for $x \in \mathbb{R}$. Prove carefully that there is an entire function g whose restriction to $\mathbb{R}$ is the Fourier transform $\widehat{f}$ of f. (Grading will be considerably stricter than on the regular problems.)

Date: 12 June 2024.