MATH 618 (SPRING 2025, PHILLIPS): HOMEWORK 9

Problem 1 (Problem 13 in Chapter 14 of Rudin's book)**.** Let $\Omega \subset \mathbb{C}$ be a region, let $(f_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence of injective holomorphic functions on Ω, and suppose that there is a function $f \colon \Omega \to \mathbb{C}$ such that $f_n \to f$ uniformly on compact subsets of Ω. Prove that f is constant or injective. Show by example that both cases can occur.

The problem in in Chapter 14 because it is relevant to the content there, but only needs methods from Chapter 10.

The next problem counts as 1.5 ordinary problems. Rudin discusses infinite products in the chapter on Weierstrass factorization, but the results below aren't there. They are about the analog for infinite products of just convergence, rather than absolute convergence, so are not directly used in either the proof of the Prime Number Theorem or the proof of Weierstrass factorization.

Problem 2. Let $w_1, w_2, \ldots \in \mathbb{C} \setminus \{0\}$.

(1) Prove that if $\prod_{n=1}^{\infty} w_n$ converges to a nonzero value, then $\lim_{n \to \infty} w_n = 1$. Show that this can fail if $\prod_{n=1}^{\infty} w_n$ converges to 0.
(2) Prove that $\prod_{n=1}^{\infty} w_n$ converges to a nonzero value if and only if $\sum_{n=1}^{\infty} \log(w_n)$ converges.

The second part counts for about twice as much as the first.

I am sure this can be found in some textbook, but please work out the details yourself.

So as to ensure that $\sum_{n=1}^{\infty} \log(w_n)$ makes sense, take log to be defined on $\mathbb{C} \setminus \{0\}$ by $\log(re^{i\theta}) = \log(r) + i\theta$ when $r > 0$ and $\theta \in (-\pi, \pi]$, with $\log(r)$ using the usual definition of the logarithm as a function $(0, \infty) \to \mathbb{R}$.

There are two annoyances to deal with. First, we haven't formally proved that the definition $\log(re^{i\theta}) = \log(r) + i\theta$ gives a continuous function on $\mathbb{C} \setminus (-\infty, 0]$, since we never proved that $z \mapsto \arg(z)$ is continuous on any domain. (It is certainly not continuous on the domain given above.) You can prove this directly, but there are easier ways to proceed. You can use Problem 6 in Chapter 10 of Rudin's book (which was in a previous homework assignment), but this is overkill. Second, it is not generally true that $\log(ab) = \log(a) + \log(b)$, with any continuous definition on any nonempty neighborhood of 1 in $\mathbb{C}$.

The next problem counts as 2.5 ordinary problems. (I don't yet know how hard it actually is, or even for sure if the method works, and I don't have a prepared solution. Start with the extension to $\text{Re}(z) > -1$.)

Problem 3. Iterate a suitable modification of the method presented in class to extend the Riemann zeta function over $\{z \in \mathbb{C} \colon \text{Re}(z) > 0\} \setminus \{1\}$ to show that the Riemann zeta function extends to a holomorphic function on $\mathbb{C} \setminus \{1\}$.

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Recall the method. We showed that for $\mathrm{Re}(s) > 1$,

$$\zeta(s) - \frac{1}{s-1} = \sum_{n=1}^{\infty} \frac{1}{n^s} - \int_1^{\infty} \frac{1}{x^s}\, ds = \sum_{n=1}^{\infty} \int_n^{n+1} \left(\frac{1}{n^s} - \frac{1}{x^s} \right) ds,$$

and that the series in the last expression converges uniformly on compact sets in $\{z \in \mathbb{C} \colon \mathrm{Re}(z) > 0\}$ to a holomorphic function defined on this set.