MATH 618 (SPRING 2025, PHILLIPS): SOLUTIONS TO HOMEWORK 8

This assignment is due on Canvas on Wednesday 28 May 2025 at 9:00 pm. (Not Monday 26 May 2025: Monday is a holiday.)

Problems and all other items use two independent numbering sequences. This is annoying, but necessary to preserve the problem numbers in the solutions files.

Little proofreading has been done.

Some parts of problems have several different solutions.

Solutions are written to be read independently. Arguments used in more than one solution are therefore repeated in each one.

The next problem counts as two ordinary problems.

Problem 1 (Problem 8 in Chapter 10 of Rudin's book)**.** Let P and Q be polynomials such that $\deg(Q) \geq \deg(P) + 2$ and $Q(x) \neq 0$ for all $x \in \mathbb{R}$. Let R be the rational function $R(z) = P(z)/Q(z)$ for $z \in \mathbb{C}$ such that $Q(z) \neq 0$.

(1) Prove that $\int_{-\infty}^{\infty} R(x)\, dx$ is equal to $2\pi i$ times the sum of the residues of R in the upper half plane. (Replace the integral over $[-A, A]$ by the integral over a suitable semicircle, and apply the Residue Theorem.)
(2) What is the analogous statement for the lower half plane?
(3) Use this method to compute

$$\int_{-\infty}^{\infty} \frac{x^2}{1 + x^4}\, dx.$$

It is convenient to begin the solution with a lemma.

Lemma 1. Let p be a polynomial of degree n. Then there exist constants $m_p, M_p, r_p > 0$ such that for all $z \in \mathbb{C}$ with $|z| \geq r_p$, we have $m_p |z|^n \leq |p(z)| \leq M_p |z|^n$.

We give a direct proof below. But one can also derive this lemma by showing, using algebraic properties of limits, that if $p(z) = \sum_{k=0}^{n} a_k z^k$ for $z \in \mathbb{C}$, then

$$\lim_{|z| \to \infty} \frac{p(z)}{z^n} = \lim_{|z| \to \infty} \sum_{k=0}^{n} a_k z^{k-n} = a_n.$$

Proof of Lemma 1. There are $a_0, a_1, \ldots, a_n \in \mathbb{C}$, with $a_n \neq 0$, such that $p(z) = \sum_{k=0}^{n} a_k z^k$ for all $z \in \mathbb{C}$. Define

$$m_p = \frac{|a_n|}{2}, \qquad M_p = \sum_{k=0}^{n} |a_k|, \qquad \text{and} \qquad r_p = \max\left(1, \, \frac{2}{|a_n|} \sum_{k=0}^{n-1} |a_k|\right).$$

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Let $z \in \mathbb{C}$ satisfy $|z| \geq r_p$. Then, using $|z| \geq 1$ at the second step and $r_p \geq \frac{2}{|a_n|}\sum_{k=0}^{n-1}|a_k|$ at the fourth step,

$$|p(z)| \geq |a_n| \cdot |z|^n - \sum_{k=0}^{n-1} |a_k| \cdot |z|^k \geq |a_n| \cdot |z|^n - |z|^{n-1} \sum_{k=0}^{n-1} |a_k|$$
$$\geq |a_n| \cdot |z|^n - r_p^{-1}|z|^n \sum_{k=0}^{n-1} |a_k| \geq |a_n| \cdot |z|^n - \left(\frac{|a_n|}{2}\right)|z|^n = m_p|z|^n.$$

Also, using $|z| \geq 1$ at the second step,

$$|p(z)| \leq \sum_{k=0}^{n} |a_k| \cdot |z|^k \leq |z|^n \sum_{k=0}^{n} |a_k| = M_p|z|^n.$$

This completes the proof. $\square$

Solution to part (1). For $A > 0$, we define curves γ_A, ρ_A, and σ_A in $\mathbb{C}$ by $\gamma_A(t) = t$ for $t \in [-A, A]$, $\rho_A(t) = Ae^{it}$ for $t \in [0, \pi]$, and $\sigma_A(t) = Ae^{it}$ for $t \in [\pi, 2\pi]$. Then $[\gamma_A + [\rho_A]$, $[\gamma_A] - [\sigma_A]$, and $[\rho_A] + [\sigma_A]$ are cycles.

We further let Z_+ be the set of z in the upper half plane such that $Q(z) = 0$, and we let Z_- be the set of z in the lower half plane such that $Q(z) = 0$. Thus Z_+ and Z_- are finite sets. Let $m_P, M_P, r_P, m_Q, M_Q, r_Q$ be the constants of Lemma 1 for the polynomials P and Q. Also set $L = \sup_{z \in Z_+ \cup Z_-} |z|$.

We first claim that if $z \in Z_-$ and $A > L$, then $\mathrm{Ind}_{\gamma_A + \rho_A}(z) = 0$. Indeed, the path $t \mapsto z - it$, for $t \in [0, \infty)$, does not intersect $\mathrm{Ran}(\gamma_A + \rho_A)$, so z is in the unbounded component of $\mathrm{Ran}(\gamma_A + \rho_A)$.

We next claim that if $z \in Z_+$ and $A > L$, then $\mathrm{Ind}_{\gamma_A + \rho_A}(z) = 1$. Indeed, by Theorem 10.11 of Rudin, we know that $\mathrm{Ind}_{\rho_A + \sigma_A}(z) = 1$, since $\rho_A + \sigma_A$ is essentially the circle of radius A and center 0, and $|z| < A$. Moreover, consideration of the path $t \mapsto z + it$, for $t \in [0, \infty)$, which does not intersect $\mathrm{Ran}(\gamma_A - \sigma_A)$, shows that z is in the unbounded component of $\mathrm{Ran}(\gamma_A - \sigma_A)$. Thus $\mathrm{Ind}_{\gamma_A - \sigma_A}(z) = 0$. Since integration of a fixed function is additive in the chains over which one is integrating, it follows that

$$\mathrm{Ind}_{\gamma_A + \rho_A}(z) = \mathrm{Ind}_{\gamma_A - \sigma_A}(z) + \mathrm{Ind}_{\gamma_A + \rho_A}(z) = 1.$$

The claim is proved.

The Residue Theorem now implies that if $A > L$ then

$$\int_{-A}^{A} R(x)\,dx = 2\pi i \sum_{z \in Z_+} \mathrm{Res}(R; z) - \int_{\rho_A} R(z)\,dz. \tag{1}$$

We now claim that $\lim_{A \to \infty} \int_{\rho_A} R(z)\,dz = 0$. For $A \geq \max(r_P, r_Q)$, we have, using the choices of m_Q and M_P and the estimates from Lemma 1,

$$\left|\int_{\rho_A} R(z)\,dz\right| = \left|\int_0^{\pi} \frac{P(Ae^{-it}) i A e^{-it}}{Q(Ae^{-it})}\,dt\right| \leq \int_0^{\pi} \frac{|P(Ae^{-it})|A|e^{-it}|}{|Q(Ae^{-it})|}\,dt \tag{2}$$
$$\leq \int_0^{\pi} \frac{M_P A^{\deg(P)+1}}{m_Q A^{\deg(Q)}}\,dt \leq \left(\frac{\pi M_P}{m_Q}\right) A^{\deg(P) - \deg(Q) + 1}.$$

Since $\deg(P) - \deg(Q) + 1 < 0$, the claim follows.

Substituting the claim into (1), we deduce that $\lim_{A\to\infty}\int_{-A}^{A} R(x)\,dx$ exists and is equal to $2\pi i \sum_{z\in Z_+} \operatorname{Res}(R; z)$. □

It isn't sufficient to prove that $\lim_{z\to\infty} R(z) = 0$. Knowing this sets one up to use the Dominated Convergence Theorem, but one must still produce a dominating function.

It is easy to use Lemma 1 to prove directly that the function R is Lebesgue integrable on $(-\infty, \infty)$.

It is not hard to compute the relevant winding numbers using Theorem 10.37 of Rudin. But some justification *does* need to be given.

Solution to part (2) (sketch). Let the notation be the same as in the solution to part (1). Methods similar to those used there show that if $A > L$ then $\operatorname{Ind}_{\gamma_A-\sigma_A}(z) = 0$ for $z \in Z_+$, while $\operatorname{Ind}_{\gamma_A-\sigma_A}(z) = -1$ for $z \in Z_-$. So the Residue Theorem gives

$$\int_{-A}^{A} R(x)\,dx = -2\pi i \sum_{z\in Z_-} \operatorname{Res}(R; z) + \int_{\sigma_A} R(z)\,dz.$$

Using the same methods as used to get (2), one shows that $\lim_{A\to\infty}\int_{\sigma_A} R(z)\,dz = 0$. Therefore $\int_{-\infty}^{\infty} R(x)\,dx = -2\pi i \sum_{z\in Z_-} \operatorname{Res}(R; z)$. □

Instead of repeating all the work, one can reduce part (2) to part (1).

Second solution to part (2). Let the notation be the same as in the solution to part (1).

We claim that $\lim_{A\to\infty}\int_{\rho_A+\sigma_A} R(z)\,dz = 0$. For $A \geq \max(r_P, r_Q)$, we have, using the choices of m_Q and M_P and the estimates from Lemma 1,

$$\begin{aligned}\left|\int_{\rho_A+\sigma_A} R(z)\,dz\right| &= \left|\int_0^{2\pi} \frac{P(Ae^{-it})iAe^{-it}}{Q(Ae^{-it})}\,dt\right| \leq \int_0^{2\pi} \frac{|P(Ae^{-it})|A|e^{-it}|}{|Q(Ae^{-it})|}\,dt \\ &\leq \int_0^{2\pi} \frac{M_P A^{\deg(P)+1}}{m_Q A^{\deg(Q)}}\,dt \leq \left(\frac{2\pi M_P}{m_Q}\right) A^{\deg(P)-\deg(Q)+1}.\end{aligned}$$

Since $\deg(P) - \deg(Q) + 1 < 0$, the claim follows.

For $A > L$, by Theorem 10.11 of Rudin we have $\operatorname{Ind}_{\rho_A+\sigma_A}(z) = 1$ for all $z \in Z_+ \cup Z_-$. Therefore

$$\int_{\rho_A+\sigma_A} R(z)\,dz = 2\pi i \sum_{z\in Z_+\cup Z_-} \operatorname{Res}(R; z).$$

Combining this fact with the claim, we get

$$2\pi i \sum_{z\in Z_+\cup Z_-} \operatorname{Res}(R; z) = 0.$$

Therefore

$$\lim_{A\to\infty}\int_{-A}^{A} R(x)\,dx = 2\pi i \sum_{z\in Z_+} \operatorname{Res}(R; z) = -2\pi i \sum_{z\in Z_-} \operatorname{Res}(R; z).$$

This completes the proof. □

The following lemma is convenient for the computation of the residues needed in part (3). It isn't in Chapter 10 of Rudin's book, but it was proved in class this year.

Lemma 2. Let $\Omega \subset \mathbb{C}$ be an open set, let $a \in \Omega$, and let f be a holomorphic function on $\Omega \setminus \{a\}$ which has a simple pole at a. Then $\operatorname{Res}(f; a) = \lim_{z \to a} (z - a) f(z)$.

Proof. Since f has a simple pole at a, by definition there are $c \in \mathbb{C} \setminus \{0\}$ and a holomorphic function g on Ω such that

$$f(z) = g(z) + \frac{c}{z - a}$$

for all $z \in \Omega \setminus \{a\}$. Moreover, by definition, $\operatorname{Res}(f; a) = c$. Now

$$\lim_{z \to a} (z - a) f(z) = \lim_{z \to a} \big((z - a) g(z) + c \big) = 0 \cdot g(a) + c = c.$$

This completes the proof. □

Solution to part (3). Set $\omega = \exp(\pi i / 4)$. Then

$$1 + z^4 = (z - \omega)(z - \omega^3)(z - \omega^5)(z - \omega^7).$$

So the function $R(z) = \frac{z^2}{1 + z^4}$ has two poles in the upper half plane, namely simple poles at ω and at ω^3. By part (1) and Lemma 3, we therefore have, factoring out powers of ω and repeatedly using $\omega^2 = i$ at the fourth step,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{x^2}{1 + x^4} \, dx &= 2\pi i (\operatorname{Res}(R; \omega) + \operatorname{Res}(R; \omega^3)) \\
&= 2\pi i \left(\lim_{z \to \omega} (z - \omega) R(z) + \lim_{z \to \omega^3} (z - \omega^3) R(z) \right) \\
&= 2\pi i \left(\frac{\omega^2}{(\omega - \omega^3)(\omega - \omega^5)(\omega - \omega^7)} + \frac{\omega^6}{(\omega^3 - \omega)(\omega^3 - \omega^5)(\omega^3 - \omega^7)} \right) \\
&= 2\pi i \left(\frac{\omega^{-1}}{(1 - i)(1 - (-1))(1 - (-i))} + \frac{\omega^{-3}}{(1 - (-i))(1 - i)(1 - (-1))} \right) \\
&= \left(\frac{\pi i}{2} \right) (\omega^{-1} + \omega^{-3}) = \left(\frac{\pi i}{2} \right) \big(- i\sqrt{2} \big) = \frac{\pi}{\sqrt{2}}.
\end{aligned}$$

This completes the solution. □

Alternate residue computation for part (3) (sketch). The residues can be read off directly from the partial fraction decomposition

$$\frac{z^2}{1 + z^4} = \frac{1}{4} \left(\frac{\omega^3}{z - \omega} + \frac{\omega^5}{z - \omega^3} + \frac{\omega^7}{z - \omega^5} + \frac{\omega}{z - \omega^7} \right).$$

One can also use a partial fraction decomposition for $(1 + z^4)^{-1}$ and multiply it by z^2. □

Problem 2 (Problem 13 in Chapter 10 of Rudin's book)**.** Prove that

$$\int_0^{\infty} \frac{1}{1 + x^n} \, dx = \frac{\pi / n}{\sin(\pi / n)}$$

for $n \in \mathbb{Z}_{>0}$ with $n \geq 2$.

The following lemma is convenient for the computation of the residues needed here. It isn't in Chapter 10 of Rudin's book, but it was proved in class this year.

Lemma 3. Let $\Omega \subset \mathbb{C}$ be an open set, let $a \in \Omega$, and let f be a holomorphic function on $\Omega \setminus \{a\}$ which has a simple pole at a. Then $\operatorname{Res}(f; a) = \lim_{z \to a} (z - a) f(z)$.

Proof. Since f has a simple pole at a, by definition there are $c \in \mathbb{C} \setminus \{0\}$ and a holomorphic function g on Ω such that

$$f(z) = g(z) + \frac{c}{z - a}$$

for all $z \in \Omega \setminus \{a\}$. Moreover, by definition, $\mathrm{Res}(f; a) = c$. Now

$$\lim_{z \to a} (z - a) f(z) = \lim_{z \to a} \big((z - a) g(z) + c \big) = 0 \cdot g(a) + c = c.$$

This completes the proof. □

Solution. Set $\omega = \exp(\pi i/n)$. For $r \in (1, \infty)$, define paths $\rho_r, \sigma_r \colon [0, r] \to \mathbb{C}$ by $\rho_r(t) = t$ and $\sigma_r(t) = t\omega^2$ for $t \in [0, r]$. Also define $\gamma_r \colon [0, 2\pi/n] \to \mathbb{C}$ and $\beta_r \colon [2\pi/n, 2\pi] \to \mathbb{C}$ by $\gamma_r(t) = re^{it}$ for $t \in [0, 2\pi/n]$ and $\beta_r(t) = re^{it}$ for $t \in [2\pi/n, 2\pi]$. Then $[\gamma_r] + [\beta_r]$, $[\rho_r] + [\gamma_r] - [\sigma_r]$, and $[\sigma_r] + [\beta_r] - [\rho_r]$ are cycles.

The formula

$$f(z) = \frac{1}{1 + z^n}$$

defines a meromorphic function on $\mathbb{C}$, with poles at $\omega, \omega^3, \ldots, \omega^{2n-1}$.

Using $r > 1$, we get $\mathrm{Ind}_{\gamma_r + \beta_r}(\omega) = 1$ by Theorem 10.11 of Rudin. Also, the path $t \mapsto t\omega$, for $t \in [1, \infty)$, is continuous, goes to ∞ as $t \to \infty$, and has range disjoint from $\mathrm{Ran}(\sigma_r + \beta_r - \rho_r)$, so $\mathrm{Ind}_{\sigma_r + \beta_r - \rho_r}(\omega) = 0$. Therefore

$$\mathrm{Ind}_{\rho_r + \gamma_r - \sigma_r}(\omega) = \mathrm{Ind}_{\gamma_r + \beta_r}(\omega) - \mathrm{Ind}_{\sigma_r + \beta_r - \rho_r}(\omega) = 1.$$

On the other hand, for $k = 2, 3, \ldots, n$, the path $t \mapsto t\omega^k$, for $t \in [1, \infty)$, is continuous, goes to ∞ as $t \to \infty$, and has range disjoint from $\mathrm{Ran}(\rho_r + \gamma_r - \sigma_r)$. So $\mathrm{Ind}_{\rho_r + \gamma_r - \sigma_r}(\omega^k) = 0$. We can now apply the Residue Theorem using the cycle $\rho_r + \gamma_r - \sigma_r$. The condition $\mathrm{Ind}_{\rho_r + \gamma_r - \sigma_r}(z) = 0$ for $z \notin \mathbb{C}$ is vacuous, so we get

$$\int_{\rho_r + \gamma_r - \sigma_r} f(z)\, dz = 2\pi i \mathrm{Res}(f; \omega).$$

Since $n \geq 2$,

$$\lim_{r \to \infty} \int_{\rho_r} f(z)\, dz = \lim_{r \to \infty} \int_0^r \frac{1}{1 + t^n}\, dt = \int_0^\infty \frac{1}{1 + t^n}\, dt$$

exists and is finite. Similarly

$$\lim_{r \to \infty} \int_{\sigma_r} f(z)\, dz = \lim_{r \to \infty} \int_0^r \frac{1}{1 + (\omega^2 t)^n} \omega^2\, dt = \omega^2 \int_0^\infty \frac{1}{1 + t^n}\, dt.$$

We claim that

$$\lim_{r \to \infty} \int_{\gamma_r} f(z)\, dz = 0.$$

For $|z| > 2$, we have

$$\left| \frac{1}{1 + z^n} \right| \leq \frac{1}{|z|^n - 1} \leq \frac{2}{|z|^n},$$

so for $r > 2$ we have, since the length of γ_r is $2\pi r/n$,

$$\left| \int_{\gamma_r} f(z)\, dz \right| \leq \left(\frac{2\pi r}{n} \right) \left(\frac{2}{r^n} \right) = \frac{4\pi}{n r^{n-1}}.$$

Since $n \geq 2$, the claim follows. So

$$(1 - \omega^2) \int_0^\infty \frac{1}{1 + t^n}\, dt = 2\pi i \mathrm{Res}(f; \omega).$$

We next calculate $\mathrm{Res}(f;\omega)$. We use Lemma 3. We have, using $\omega^n = -1$ at the second step,

$$\begin{aligned}\mathrm{Res}(f;\omega) &= \lim_{z\to\omega}\frac{z-\omega}{z^n+1} = \lim_{z\to 1}\frac{\omega z-\omega}{(\omega z)^n+1} = -\omega\lim_{z\to 1}\frac{z-1}{z^n-1}\\ &= -\omega\lim_{z\to 1}\frac{1}{z^{n-1}+z^{n-2}+\cdots+1} = -\frac{\omega}{n}.\end{aligned}$$

We conclude

$$\begin{aligned}\int_0^\infty \frac{1}{1+t^n}\,dt &= \frac{2\pi i\mathrm{Res}(f;\omega)}{1-\omega^2} = -\frac{2\pi i\omega}{n(1-\omega^2)}\\ &= \frac{2\pi i}{n(\omega-\omega^{-1})} = \frac{\pi/n}{(\omega-\omega^{-1})/(2i)} = \frac{\pi/n}{\sin(\pi/n)}.\end{aligned}$$

This completes the proof. □

Alternate residue computation. The residues can be read off directly from the partial fraction decomposition

$$\frac{1}{1+z^n} = -\frac{1}{n}\sum_{k=1}^{n}\frac{\omega^{2k-1}}{z-\omega^{2k-1}}.$$

(This partial fraction decomposition has not been checked.) □

Problem 3 (Problem 21 in Chapter 10 of Rudin's book)**.** Let $\Omega \subset \mathbb{C}$ be an open set which contains the closed unit disk. Let f be a holomorphic function on Ω such that $|f(z)| < 1$ for all $z \in \mathbb{C}$ such that $|z| = 1$. Determine, with proof, the possible numbers of fixed points of f (that is, solutions to the equation $f(z) = z$) in the open unit disk.

Solution. For $z \in \Omega$, define $g(z) = f(z) - z$ and $h(z) = z$. We apply Rouché's Theorem (Theorem 10.43(b) of Rudin), with $\gamma(t) = \exp(it)$ for $t \in [0, 2\pi]$. Observe that, for $z \in \mathrm{Ran}(\gamma)$, we have

$$|h(z) - g(z)| = |-f(z)| < 1 = |z| = |h(z)|.$$

Moreover, by Theorem 10.11 of Rudin, $\mathrm{Ind}_\gamma(z)$ is 0 or 1 for all $z \in \mathbb{C} \setminus \mathrm{Ran}(\gamma)$, and is equal to 1 exactly on the open unit disk. Therefore Rouché's Theorem implies that g and h have the same number of zeros in the open unit disk. Since h has exactly one zero in the open unit disk, so does g. This means that f has exactly one fixed point in the open unit disk. □

Problem 4 (Problem 20 in Chapter 10 of Rudin's book)**.** Let $\Omega \subset \mathbb{C}$ be a region, let $f\colon \Omega \to \mathbb{C}$, and let $(f_n)_{n\in\mathbb{Z}_{>0}}$ be a sequence of holomorphic functions on Ω. Suppose that $f_n \to f$ uniformly on compact sets in Ω.

(1) Suppose that, for all $n \in \mathbb{Z}_{>0}$, the function f_n is never zero on Ω. Prove that either $f(z) = 0$ for all $z \in \Omega$ or $f(z) \neq 0$ for all $z \in \Omega$.
(2) If $U \subset \mathbb{C}$ is open and $f_n(\Omega) \subset U$ for all n, prove that f is constant or $f(\Omega) \subset U$.

Solution to (1). Assume that there is $z \in \Omega$ such that $f(z) \neq 0$. Let $z_0 \in \Omega$; we prove $f(z_0) \neq 0$.

First, f is holomorphic by Theorem 10.28 of Rudin. Therefore $\{z \in \Omega\colon f(z) = 0\}$ is countable, by Theorem 10.18 of Rudin. Since there are uncountably many $r > 0$

such that $\overline{B_r(z_0)} \subset \Omega$, there is $r > 0$ such that $\overline{B_r(z_0)} \subset \Omega$ and such that $f(z) \neq 0$ for all $z \in \partial B_r(z_0)$. Choose $n \in \mathbb{Z}_{>0}$ such that

$$\sup_{z \in \partial B_r(z_0)} |f_n(z) - f(z)| < \inf_{z \in \partial B_r(z_0)} |f(z)|.$$

Since f_n does not vanish on $B_r(z_0)$, it follows from Theorem 10.43(b) of Rudin that f also does not vanish on $B_r(z_0)$. In particular, $f(z_0) \neq 0$. □

Alternate solution to (1). Assume that there is $z_0 \in \Omega$ such that $f(z_0) = 0$. We prove that $f(z) = 0$ for all $z \in \Omega$.

Choose $r_0 > 0$ such that $B_{r_0}(z_0) \subset \Omega$. Let $0 < r < r_0$. For each n, apply the Maximum Modulus Theorem to $1/f_n$ (see the corollary to Theorem 10.24 of Rudin) to find $\theta_n \in [0, 2\pi]$ such that $|f_n(z_0 + re^{i\theta_n})| \leq |f_n(z_0)|$. Passing to a subsequence of $(f_n)_{n \in \mathbb{Z}_{>0}}$, we may assume that $\theta = \lim_{n \to \infty} \theta_n$ exists.

We claim that $f(z_0 + re^{i\theta}) = 0$. Let $\varepsilon > 0$. Choose N so large that $n \geq N$ implies $|f_n(z) - f(z)| < \frac{1}{3}\varepsilon$ for all $z \in \overline{B_r(z_0)}$. Since f is continuous, we may choose $\delta > 0$ such that $|z - (z_0 + re^{i\theta})| < \delta$ implies $|f(z) - f(z_0 + re^{i\theta})| < \frac{1}{3}\varepsilon$. Choose $n \geq N$ such that $|re^{i\theta_n} - re^{i\theta}| < \delta$. Then, using $|(z_0 + re^{i\theta}) - (z_0 + re^{i\theta_n})| < \delta$ at the first step, and $|f_n(z_0)| = |f_n(z_0) - f(z_0)| < \frac{1}{3}\varepsilon$ at the second step,

$$\begin{aligned} |f(z_0 + re^{i\theta})| &\leq |f(z_0 + re^{i\theta}) - f(z_0 + re^{i\theta_n})| \\ &\qquad + |f(z_0 + re^{i\theta_n}) - f_n(z_0 + re^{i\theta_n})| + |f_n(z_0 + re^{i\theta_n})| \\ &< \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + |f_n(z_0)| < \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, this shows that $f(z_0 + re^{i\theta}) = 0$.

We have shown that for every $r \in (0, r_0)$ there is $z \in \Omega$ with $|z - z_0| = r$ such that $f(z) = 0$. Thus, z_0 is a limit point of the set of zeros of f. So $f(z) = 0$ for all $z \in \Omega$. □

Solution to (2). Assume that there is $z_0 \in \Omega$ such that $f(z_0) \notin U$. Let $g_n = f_n - f_n(z_0)$ and let $g = f - f(z_0)$. Then $g_n \to g$ uniformly on compact sets in Ω, and each g_n is never zero on Ω, but $g(z_0) = 0$. The first statement of the problem implies that $g(z) = 0$ for all $z \in \Omega$. Therefore f is constant, with value $f(z_0)$. □

Remark 4. The Open Mapping Theorem does not help with the second statement. All it gives is that if f is not constant, then $f(\Omega) \subset \operatorname{int}(\overline{U})$. In general U is a proper subset of $\operatorname{int}(\overline{U})$, even for connected open subsets of $\mathbb{C}$. For example, if $U = \mathbb{C} \setminus \{0\}$ then $\operatorname{int}(\overline{U}) = \mathbb{C}$. Even requiring U to be simply connected does not help: if $U = \mathbb{C} \setminus [0, \infty)$ then still $\operatorname{int}(\overline{U}) = \mathbb{C}$.