MATH 618 (SPRING 2025, PHILLIPS): HOMEWORK 8

This assignment is due on Canvas on Wednesday 28 May 2025 at 9:00 pm. (Not Monday 26 May 2025: Monday is a holiday.)

The next problem counts as two ordinary problems.

Problem 1 (Problem 8 in Chapter 10 of Rudin's book)**.** Let P and Q be polynomials such that $\deg(Q) \geq \deg(P) + 2$ and $Q(x) \neq 0$ for all $x \in \mathbb{R}$. Let R be the rational function $R(z) = P(z)/Q(z)$ for $z \in \mathbb{C}$ such that $Q(z) \neq 0$.

(1) Prove that $\int_{-\infty}^{\infty} R(x)\,dx$ is equal to $2\pi i$ times the sum of the residues of R in the upper half plane. (Replace the integral over $[-A, A]$ by the integral over a suitable semicircle, and apply the Residue Theorem.)
(2) What is the analogous statement for the lower half plane?
(3) Use this method to compute

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx.$$

Problem 2 (Problem 13 in Chapter 10 of Rudin's book)**.** Prove that

$$\int_{0}^{\infty} \frac{1}{1+x^n}\,dx = \frac{\pi/n}{\sin(\pi/n)}$$

for $n \in \mathbb{Z}_{>0}$ with $n \geq 2$.

Problem 3 (Problem 21 in Chapter 10 of Rudin's book)**.** Let $\Omega \subset \mathbb{C}$ be an open set which contains the closed unit disk. Let f be a holomorphic function on Ω such that $|f(z)| < 1$ for all $z \in \mathbb{C}$ such that $|z| = 1$. Determine, with proof, the possible numbers of fixed points of f (that is, solutions to the equation $f(z) = z$) in the open unit disk.

Problem 4 (Problem 20 in Chapter 10 of Rudin's book)**.** Let $\Omega \subset \mathbb{C}$ be a region, let $f\colon \Omega \to \mathbb{C}$, and let $(f_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence of holomorphic functions on Ω. Suppose that $f_n \to f$ uniformly on compact sets in Ω.

(1) Suppose that, for all $n \in \mathbb{Z}_{>0}$, the function f_n is never zero on Ω. Prove that either $f(z) = 0$ for all $z \in \Omega$ or $f(z) \neq 0$ for all $z \in \Omega$.
(2) If $U \subset \mathbb{C}$ is open and $f_n(\Omega) \subset U$ for all n, prove that f is constant or $f(\Omega) \subset U$.

Date: 19 May 2025.