MATH 618 (SPRING 2025, PHILLIPS): SOLUTIONS TO HOMEWORK 7

This assignment is due on Canvas on Wednesday 19 May 2025 at 9:00 pm.

Problems and all other items use two independent numbering sequences. This is annoying, but necessary to preserve the problem numbers in the solutions files.

Little proofreading has been done.

Some parts of problems have several different solutions.

Problem 1 (Problem 19 in Chapter 10 of Rudin's book)**.** Let f and g be holomorphic functions on $B_1(0)$, suppose that $f(z) \neq 0$ and $g(z) \neq 0$ for all $z \in B_1(0)$, and suppose that

$$\frac{f'\left(\frac{1}{n}\right)}{f\left(\frac{1}{n}\right)} = \frac{g'\left(\frac{1}{n}\right)}{g\left(\frac{1}{n}\right)}$$

for all $n \in \mathbb{Z}_{>0}$ with $n > 1$. Find and prove another simple relation between f and g.

Motivation for the relation: the statement appears to say that the functions $\log \circ f$ and $\log \circ g$ have the same derivative on a set with a cluster point in $B_1(0)$, so they have the same derivative everywhere on $B_1(0)$, so they differ by a constant. To solve the problem this way requires proving that there are holomorphic branches of $\log \circ f$ and $\log \circ g$ on $B_1(0)$. This follows easily from Theorem 13.11 of Rudin (which isn't available to us at this stage), and there are proofs using convexity which are accessible now, but there is an easier way to proceed.

Solution. The relation is that there is a nonzero constant c such that $cf = g$.

Nothing more can be said. Indeed, for any holomorphic function f on $B_1(0)$ with no zeroes in $B_1(0)$, and any $c \in \mathbb{C} \setminus \{0\}$, taking $g = cf$ gives a pair of functions satisfying the condition in the problem.

Now let f and g satisfy the condition in the problem. Set $h(z) = g(z)/f(z)$ for $z \in B_1(0)$. Then for $z \in B_1(0)$ we have

$$h'(z) = \frac{g'(z)f(z) - g(z)f'(z)}{f(z)^2} = \frac{g(z)}{f(z)} \left(\frac{g'(z)}{g(z)} - \frac{f'(z)}{f(z)} \right).$$

This function vanishes on $\left\{ \frac{1}{2}, \frac{1}{3}, \ldots \right\}$, which has a cluster point in $B_1(0)$. Since $B_1(0)$ is connected, $h(z) = 0$ for all $z \in B_1(0)$.

It follows (for example, by considering the power series for h), that h is constant, that is, there is $c \in \mathbb{C}$ such that

$$\frac{g(z)}{f(z)} = h(z) = c$$

for all $z \in B_1(0)$. So $g(z) = cf(z)$ for all $z \in B_1(0)$. Since $g(0) \neq 0$, we must have $c \neq 0$. ∎

Date: 19 May 2025.

The following is a rewording (to be more careful) of Rudin, Chapter 10, Problem 28. Do this problem, but possibly with the modifications suggested afterwards. It counts as 1.5 ordinary problems.

Problem 2 (Problem 28 in Chapter 10 of Rudin's book)**.** Let Γ be a closed curve in the plane (continuous but not necessarily piecewise C^1), with parameter interval $[0, 2\pi]$. Let $\alpha \in \mathbb{C} \setminus \mathrm{Ran}(\Gamma)$. Choose a sequence $(\Gamma_n)_{n \in \mathbb{Z}_{>0}}$ of closed curves given by trigonometric polynomials which converges uniformly to Γ. Show that for all sufficiently large m and n, we have $\mathrm{Ind}_{\Gamma_m}(\alpha) = \mathrm{Ind}_{\Gamma_n}(\alpha)$. Define $\mathrm{Ind}_{\Gamma}(\alpha)$ to be this common value. Prove that it does not depend on the choice of the sequence $(\Gamma_n)_{n \in \mathbb{Z}_{>0}}$. Prove that Lemma 10.39 now holds for closed curves which are merely continuous. Use this result to prove that Theorem 10.40 holds for closed curves which are merely continuous.

The problem says to use trigonometric polynomials for the approximation, but feel free to use piecewise linear functions instead, or some other convenient approximation. Furthermore, it is probably better not to use sequences, despite the statement of the problem. (Of course, don't use Theorem 10.40 of Rudin, but you will want Lemma 10.39.)

For reference, here are the statements of Lemma 10.39 and Theorem 10.40.

Lemma 1. Let $\Gamma_0, \Gamma_1 \colon [0, 2\pi] \to \mathbb{C}$ be piecewise C^1 closed curves in $\mathbb{C}$. Let $\alpha \in \mathbb{C}$. Suppose that

$$|\Gamma_1(t) - \Gamma_0(t)| < |\alpha - \Gamma_0(t)|$$

for all $t \in [0, 2\pi]$. Then $\mathrm{Ind}_{\Gamma_0}(\alpha) = \mathrm{Ind}_{\Gamma_1}(\alpha)$.

Theorem 2. Let $\Omega \subset \mathbb{C}$ be open, and let $\Gamma_0, \Gamma_1 \colon [0, 2\pi] \to \mathbb{C}$ be piecewise C^1 closed curves in Ω which are homotopic in Ω. Let $\alpha \in \mathbb{C} \setminus \Omega$. Then $\mathrm{Ind}_{\Gamma_0}(\alpha) = \mathrm{Ind}_{\Gamma_1}(\alpha)$.

We state the steps in the solution as several lemmas. The proofs are all short.

Lemma 3. Let $\Gamma \colon [0, 2\pi] \to \mathbb{C}$ be a continuous closed curve in $\mathbb{C}$. Let $\varepsilon > 0$. Then there is a piecewise C^1 closed curve γ in $\mathbb{C}$ such that $|\gamma(t) - \Gamma(t)| < \varepsilon$ for all $t \in [0, 2\pi]$.

We omit the details of the proof. It is easy to do using approximation by trigonometric polynomials (as suggested by Rudin), piecewise linear functions (with care taken to ensure that $\gamma(2\pi) = \gamma(0)$), or by using the Stone-Weierstrass Theorem to show that the C^∞ functions from the circle to $\mathbb{C}$ are uniformly dense in the continuous functions from the circle to $\mathbb{C}$.

Lemma 4. Let $\Gamma \colon [0, 2\pi] \to \mathbb{C}$ be a continuous closed curve in $\mathbb{C}$. Let $\alpha \in \mathbb{C} \setminus \mathrm{Ran}(\Gamma)$. Let $\gamma_1, \gamma_2 \colon [0, 2\pi] \to \mathbb{C}$ be piecewise C^1 closed curves in $\mathbb{C}$ such that for all $t \in [0, 2\pi]$, we have

$$|\gamma_1(t) - \Gamma(t)| < \tfrac{1}{3}\mathrm{dist}(\alpha, \mathrm{Ran}(\Gamma)) \qquad \text{and} \qquad |\gamma_2(t) - \Gamma(t)| < \tfrac{1}{3}\mathrm{dist}(\alpha, \mathrm{Ran}(\Gamma)).$$

Then $\mathrm{Ind}_{\gamma_1}(\alpha) = \mathrm{Ind}_{\gamma_2}(\alpha)$.

It will later become clear that one can use $\mathrm{dist}(\alpha, \mathrm{Ran}(\Gamma))$ in place of $\frac{1}{3}\mathrm{dist}(\alpha, \mathrm{Ran}(\Gamma))$, but this result is easier and sufficient.

Proof of Lemma 4. The triangle inequality implies that for all $t \in [0, 2\pi]$, we have

$$|\alpha - \gamma_1(t)| > \tfrac{2}{3}\mathrm{dist}(\alpha, \mathrm{Ran}(\Gamma)) \qquad \text{and} \qquad |\gamma_2(t) - \gamma_1(t)| < \tfrac{2}{3}\mathrm{dist}(\alpha, \mathrm{Ran}(\Gamma)).$$

Since γ_1 and γ_2 are piecewise C^1 closed curves, the result now follows from Lemma 10.39 of Rudin. $\square$

It follows from Lemma 3 that the quantity $\mathrm{Ind}_\Gamma(\alpha)$ in the following definition exists, and from Lemma 4 that it is well defined. Also, it is obvious that it agrees with the original definition when Γ is already piecewise C^1, since we can take $\gamma = \Gamma$.

Definition 5. Let $\Gamma\colon [0, 2\pi] \to \mathbb{C}$ be a continuous closed curve in $\mathbb{C}$. Let $\alpha \in \mathbb{C} \setminus \mathrm{Ran}(\Gamma)$. Let $\gamma\colon [0, 2\pi] \to \mathbb{C}$ be a piecewise C^1 closed curve in $\mathbb{C}$ such that for all $t \in [0, 2\pi]$, we have

$$|\gamma(t) - \Gamma(t)| < \tfrac{1}{3}\mathrm{dist}(\alpha, \mathrm{Ran}(\Gamma))$$

We define $\mathrm{Ind}_\Gamma(\alpha) = \mathrm{Ind}_\gamma(\alpha)$.

We can now prove the generalization of Lemma 10.39 of Rudin to continuous closed curves.

Lemma 6. Let $\Gamma_0, \Gamma_1\colon [0, 2\pi] \to \mathbb{C}$ be continuous closed curves in $\mathbb{C}$. Let $\alpha \in \mathbb{C}$. Suppose that

$$|\Gamma_1(t) - \Gamma_0(t)| < |\alpha - \Gamma_0(t)|$$

for all $t \in [0, 2\pi]$. Then $\mathrm{Ind}_{\Gamma_0}(\alpha) = \mathrm{Ind}_{\Gamma_1}(\alpha)$.

Proof. Set

$$\rho = \inf_{t \in [0, 2\pi]} \big(|\alpha - \Gamma_0(t)| - |\Gamma_1(t) - \Gamma_0(t)| \big).$$

Then $\rho > 0$ since $[0, 2\pi]$ is compact. Set

$$\varepsilon = \min\left(\frac{\rho}{3}, \, \frac{1}{3}\mathrm{dist}(\alpha, \mathrm{Ran}(\Gamma_1)), \, \frac{1}{3}\mathrm{dist}(\alpha, \mathrm{Ran}(\Gamma_2)) \right).$$

Choose (Lemma 3) piecewise C^1 closed curves $\gamma_0, \gamma_1\colon [0, 2\pi] \to \mathbb{C}$ such that

$$|\gamma_0(t) - \Gamma_0(t)| < \varepsilon \qquad \text{and} \qquad |\gamma_1(t) - \Gamma_1(t)| < \varepsilon$$

for all $t \in [0, 2\pi]$. Then $\mathrm{Ind}_{\gamma_0}(\alpha) = \mathrm{Ind}_{\Gamma_0}(\alpha)$ and $\mathrm{Ind}_{\gamma_1}(\alpha) = \mathrm{Ind}_{\Gamma_1}(\alpha)$. The triangle inequality implies that for all $t \in [0, 2\pi]$, we have

$$|\gamma_1(t) - \gamma_0(t)| < \frac{2\rho}{3} + |\Gamma_1(t) - \Gamma_0(t)| < \frac{2\rho}{3}|\alpha - \Gamma_0(t)| < \frac{2\rho}{3} + \frac{\rho}{3} + |\alpha - \gamma_0(t)|.$$

So Lemma 10.39 of Rudin implies that $\mathrm{Ind}_{\gamma_0}(\alpha) = \mathrm{Ind}_{\gamma_1}(\alpha)$. $\square$

Now we can give the generalization of Theorem 10.40 of Rudin.

Theorem 7. Let $\Omega \subset \mathbb{C}$ be open, and let $\Gamma_0, \Gamma_1\colon [0, 2\pi] \to \mathbb{C}$ be continuous closed curves in Ω which are homotopic in Ω. Let $\alpha \in \mathbb{C} \setminus \Omega$. Then $\mathrm{Ind}_{\Gamma_0}(\alpha) = \mathrm{Ind}_{\Gamma_1}(\alpha)$.

Proof. Let $(s, t) \mapsto \Gamma_s(t)$, for $s \in [0, 1]$ and $t \in [0, 2\pi]$, be a homotopy as in the hypotheses, with Γ_0 and Γ_1 as already given. Let

$$K = \big\{ \Gamma_s(t)\colon s \in [0, 1] \text{ and } t \in [0, 2\pi] \big\}$$

Then $K \subset \Omega$ and K is compact, so $\varepsilon = \mathrm{dist}(K, \mathbb{C} \setminus \Omega) > 0$. Since $(s, t) \mapsto \Gamma_s(t)$ is uniformly continuous, there exists $\delta > 0$ such that, in particular, for all $s_1, s_2 \in [0, 1]$ and $t \in [0, 2\pi]$ with $|s_1 - s_2| < \delta$, we have $|\Gamma_{s_1}(t) - \Gamma_{s_2}(t)| < \varepsilon$. Choose $n \in \mathbb{Z}_{>0}$ such that $\frac{1}{n} < \delta$. For all $t \in [0, 2\pi]$ and for $j = 1, 2, \ldots, n$, we have

$$|\Gamma_{j/n}(t) - \Gamma_{(j-1)/n}(t)| < \varepsilon \leq \mathrm{dist}(K, \mathbb{C} \setminus \Omega) \leq \mathrm{dist}(K, \alpha) \leq |\alpha - \Gamma_{(j-1)/n}(t)|.$$

Applying Lemma 6 repeatedly, we get

$$\mathrm{Ind}_{\Gamma_0}(\alpha) = \mathrm{Ind}_{\Gamma_{1/n}}(\alpha) = \cdots = \mathrm{Ind}_{\Gamma_{(n-1)/n}}(\alpha) = \mathrm{Ind}_{\Gamma_1}(\alpha).$$

This completes the proof. □

The following problem counts as 1.5 ordinary problems.

Problem 3 (Problem 12 in Chapter 10 of Rudin's book)**.** For $t \in \mathbb{R}$, use the Residue Theorem to compute

$$\int_{-\infty}^{\infty} \left(\frac{\sin(x)}{x} \right)^2 e^{itx}\, dx.$$

Compare with Rudin Chapter 9 Problem 2.

Solution (with a few steps just sketched). For $t \in \mathbb{R}$ define

$$f_t(z) = \begin{cases} \left(\frac{\sin(z)}{z} \right)^2 e^{itz} & z \in \mathbb{C} \setminus \{0\} \\ 1 & z = 0. \end{cases}$$

Then f_t is an entire function. Also, f_t is integrable on $\mathbb{R}$ because $|f_t(x)| \leq x^{-2}$ when $|x| \geq 1$ and f_t is bounded on $[-1, 1]$ (since $[-1, 1]$ is compact). Further, for $s \in \mathbb{R}$ define $g_s \colon \mathbb{C} \setminus \{0\} \to \mathbb{C}$ by $g_s(z) = e^{isz}/z^2$. Using the relation

$$[\sin(z)]^2 = \left[\frac{1}{2i} (e^{iz} - e^{-iz}) \right]^2,$$

one checks that

$$f_t(z) = -\left(\frac{1}{4} \right) g_{t+2}(z) + \left(\frac{1}{2} \right) g_t(z) - \left(\frac{1}{4} \right) g_{t-2}(z)$$

for $z \in \mathbb{C} \setminus \{0\}$.

For $a > 1$ define (one of these does not depend on a):

(1) $\sigma_a(\theta) = ae^{i\theta}$ for $\theta \in [0, \pi]$
(2) $\tau_a(\theta) = ae^{i\theta}$ for $\theta \in [\pi, 2\pi]$.
(3) $\rho(\theta) = e^{i\theta}$ for $\theta \in [0, \pi]$.
(4) $\alpha_a(t) = t$ for $t \in [1, a]$.
(5) $\beta_a(t) = t$ for $t \in [-a, -1]$.
(6) $\iota_a(t) = t$ for $t \in [-a, a]$.

Define $\Gamma_a = [\alpha_a] - [\rho] + [\beta_a]$. Then

$$\Gamma_a + [\sigma_a], \qquad \Gamma_a - [\tau_a], \qquad \text{and} \qquad \Gamma_a - [\iota_a]$$

are immediately seen to be cycles.

For $a > 1$ and $s \in \mathbb{R}$, define

$$\varphi_a(s) = \int_{\Gamma_a} g_s(z)\, dz.$$

We claim that $\psi(s) = \lim_{a \to \infty} \varphi_s(a)$ exists and is given by

$$\psi(s) = \begin{cases} 0 & s > 0 \\ -2\pi s & s < 0. \end{cases}$$

We prove the claim for $s \geq 0$. By considering the negative imaginary axis, one sees that 0 is in the unbounded component of $\mathrm{Ran}(\Gamma_a) \cup \mathrm{Ran}(\sigma_a)$. Therefore

$\mathrm{Ind}_{\Gamma_a+[\sigma_a]}(0) = 0$. (Something must be said here.) Using Cauchy's Theorem at the fist step, we then get

$$\begin{aligned} \int_{\Gamma_a} g_s(z)\,dz &= -\int_{\sigma_a} g_s(z)\,dz = -\int_0^{\pi} \frac{\exp(is\sigma_a(\theta))iae^{i\theta}}{\sigma_a(\theta)^2}\sigma_a'(\theta)\,d\theta \\ &= -\int_0^{\pi} \frac{\exp(isae^{i\theta})iae^{i\theta}}{(ae^{i\theta})^2}\,d\theta = -\int_0^{\pi} a^{-1}\exp(isae^{i\theta})iae^{i\theta}\,d\theta. \end{aligned} \tag{1}$$

Since $\sin(\theta) \geq 0$ for $\theta \in [0, \pi]$, and $a, s \geq 0$, we get

$$|\exp(isae^{i\theta})| = |\exp(isa[\cos(\theta) + i\sin(\theta)])| = \exp(-as\sin(\theta)) \leq 1.$$

Therefore the integrand in the last expression in (1) convereges uniformly to 0 as $a \to \infty$. So $\psi(s) = 0$.

To prove the claim for $s < 0$, we first write

$$\int_{\Gamma_a} g_s(z)\,dz = \int_{\tau_a} g_s(z)\,dz + \int_{\Gamma_a - \tau_a} g_s(z)\,dz.$$

One checks (details omitted, but something must be said) that $\mathrm{Ind}_{\Gamma_a - \tau_a}(0) = -1$. The expansion

$$g_s(z) = \frac{1}{z^2} + \frac{is}{z} + \frac{(is)^2}{2!} + \frac{(is)^3 z}{3!} + \cdots$$

shows that $\mathrm{Res}(g_s; 0) = is$. Similar methods to the case $s \geq 0$ show that

$$\lim_{a\to\infty} \int_{\tau_a} g_s(z)\,dz = 0.$$

Therefore

$$\lim_{a\to\infty} \int_{\Gamma_a} g_s(z)\,dz = \lim_{a\to\infty} \int_{\tau_a} g_s(z)\,dz + 2\pi i \mathrm{Res}(g_s; 0) = 2\pi s,$$

as desired. The completes the proof of the claim.

With the last step justified because f_t is entire and $\Gamma_a - [\iota_a]$ is a cycle, we have

$$\begin{aligned} \int_{-\infty}^{\infty} \left(\frac{\sin(x)}{x}\right)^2 e^{itx}\,dx &= \int_{-\infty}^{\infty} f_t(x)\,dx = \lim_{a\to\infty} \int_{-a}^{a} f_t(x)\,dx \\ &= \lim_{a\to\infty} \int_{\iota_a} f_t(x)\,dx = \lim_{a\to\infty} \int_{\Gamma_a} f_t(x)\,dx \\ &= -\left(\frac{1}{4}\right)\varphi(t+2) + \left(\frac{1}{2}\right)\varphi(t) - \left(\frac{1}{4}\right)\varphi(t-2). \end{aligned}$$

For $t \geq 2$, all terms are zero. For $t \leq -2$, we get

$$\int_{-\infty}^{\infty} \left(\frac{\sin(x)}{x}\right)^2 e^{itx}\,dx = -\left(\frac{1}{4}\right)2\pi(t+2) + \left(\frac{1}{2}\right)2\pi t - \left(\frac{1}{4}\right)2\pi(t-2) = 0.$$

For $t \in [0, 2]$, we get

$$\int_{-\infty}^{\infty} \left(\frac{\sin(x)}{x}\right)^2 e^{itx}\,dx = -\left(\frac{1}{4}\right)\cdot 0 + \left(\frac{1}{2}\right)\cdot 0 - \left(\frac{1}{4}\right)2\pi(t-2) = \frac{\pi}{2}(2-t).$$

For $t \in [-2, 0]$, we get

$$\int_{-\infty}^{\infty} \left(\frac{\sin(x)}{x}\right)^2 e^{itx}\,dx = -\left(\frac{1}{4}\right)\cdot 0 + \left(\frac{1}{2}\right)2\pi t - \left(\frac{1}{4}\right)2\pi(t-2) = \frac{\pi}{2}(t+2).$$

One can put this together in one formula (not necessary):

$$\int_{-\infty}^{\infty} \left(\frac{\sin(x)}{x} \right)^2 e^{itx}\, dx = \frac{\pi}{2}(2 - |t|).$$

This completes the solution. □

Problem 4 (Problem 11 in Chapter 10 of Rudin's book)**.** Let $\alpha \in \mathbb{C}$ satisfy $|\alpha| \neq 1$. Calculate

$$\int_0^{2\pi} \frac{1}{1 - 2\alpha\cos(\theta) + \alpha^2}\, d\theta$$

by integrating $(z-\alpha)^{-1}(z - 1/\alpha)^{-1}$ around the unit circle.

We will use the following lemma to compute residues. It isn't in Chapter 10 of Rudin's book, but it was proved in class this year. For the residues needed in this problem, a different calculation is given in Remark 9.

Lemma 8. Let $\Omega \subset \mathbb{C}$ be an open set, let $a \in \Omega$, and let f be a holomorphic function on $\Omega \setminus \{a\}$ which has a simple pole at a. Then $\operatorname{Res}(f; a) = \lim_{z \to a}(z-a)f(z)$.

Proof. Since f has a simple pole at a, by definition there are $c \in \mathbb{C} \setminus \{0\}$ and a holomorphic function g on Ω such that

$$f(z) = g(z) + \frac{c}{z-a}$$

for all $z \in \Omega \setminus \{a\}$. Moreover, by definition, $\operatorname{Res}(f; a) = c$. Now

$$\lim_{z \to a}(z-a)f(z) = \lim_{z \to a}\big((z-a)g(z) + c\big) = 0 \cdot g(a) + c = c.$$

This completes the proof. □

Solution. Define a closed curve γ in $\mathbb{C}$ by $\gamma(\theta) = e^{i\theta}$ for $\theta \in [0, 2\pi]$. Define a meromorphic function f_α on $\mathbb{C}$ by

$$f_\alpha(z) = \frac{1}{(z-\alpha)\left(z - \frac{1}{\alpha}\right)}.$$

Then f_α has simple poles at α and at α^{-1}.

We have

$$\int_\gamma f(z)\, dz = \int_0^{2\pi} \frac{1}{(e^{i\theta} - \alpha)\left(e^{i\theta} - \frac{1}{\alpha}\right)} ie^{i\theta}\, d\theta$$
$$= \int_0^{2\pi} \frac{-i\alpha}{(e^{i\theta} - \alpha)(e^{-i\theta} - \alpha)}\, d\theta = \int_0^{2\pi} \frac{-i\alpha}{1 - 2\alpha\cos(\theta) + \alpha^2}\, d\theta.$$

We now compute this integral by the residue theorem.

Suppose $|\alpha| < 1$. Then $\operatorname{Ind}_\gamma(\alpha) = 1$ and $\operatorname{Ind}_\gamma(1/\alpha) = 0$ by Theorem 10.11 of Rudin. Lemma 8 gives

$$\operatorname{Res}(f_\alpha, \alpha) = \frac{1}{\alpha - \frac{1}{\alpha}} = \frac{\alpha}{\alpha^2 - 1}.$$

Therefore

$$\int_0^{2\pi} \frac{1}{1 - 2\alpha\cos(\theta) + \alpha^2}\, d\theta = \left(\frac{1}{-i\alpha}\right)\int_\gamma f_\alpha(z)\, dz$$
$$= \left(\frac{1}{-i\alpha}\right) 2\pi i \operatorname{Res}(f_\alpha, \alpha) = -\frac{2\pi}{\alpha^2 - 1}.$$

Suppose now $|\alpha| > 1$. Then, using the result for $1/\alpha$ at the second step, we have

$$\int_0^{2\pi} \frac{1}{1 - 2\alpha\cos(\theta) + \alpha^2}\, d\theta = \int_0^{2\pi} \frac{\alpha^{-2}}{1 - 2\alpha^{-1}\cos(\theta) + \alpha^{-2}}\, d\theta$$
$$= -\frac{2\pi\alpha^{-2}}{\alpha^{-2} - 1} = \frac{2\pi}{\alpha^2 - 1}.$$

This completes the solution. □

Remark 9. The residues

$$\operatorname{Res}(f_\alpha, \alpha) = \frac{\alpha}{\alpha^2 - 1} \qquad \text{and} \qquad \operatorname{Res}(f_\alpha, \alpha^{-1}) = -\frac{\alpha}{\alpha^2 - 1}$$

can be read directly off the partial fraction decomposition

$$f_\alpha(z) = \left(\frac{\alpha}{\alpha^2 - 1}\right)\left(\frac{1}{z - \alpha} - \frac{1}{z - \alpha^{-1}}\right),$$

without the need for Lemma 8.