MATH 618 (SPRING 2025, PHILLIPS): HOMEWORK 7

Problem 1 (Problem 19 in Chapter 10 of Rudin's book)**.** Let f and g be holomorphic functions on $B_1(0)$, suppose that $f(z) \neq 0$ and $g(z) \neq 0$ for all $z \in B_1(0)$, and suppose that

$$\frac{f'\left(\frac{1}{n}\right)}{f\left(\frac{1}{n}\right)} = \frac{g'\left(\frac{1}{n}\right)}{g\left(\frac{1}{n}\right)}$$

for all $n \in \mathbb{Z}_{>0}$ with $n > 1$. Find and prove another simple relation between f and g.

The following is a rewording (to be more careful) of Rudin, Chapter 10, Problem 28. Do this problem, but possibly with the modifications suggested afterwards. It counts as 1.5 ordinary problems.

Problem 2 (Problem 28 in Chapter 10 of Rudin's book)**.** Let Γ be a closed curve in the plane (continuous but not necessarily piecewise C^1), with parameter interval $[0, 2\pi]$. Let $\alpha \in \mathbb{C} \setminus \mathrm{Ran}(\Gamma)$. Choose a sequence $(\Gamma_n)_{n \in \mathbb{Z}_{>0}}$ of closed curves given by trigonometric polynomials which converges uniformly to Γ. Show that for all sufficiently large m and n, we have $\mathrm{Ind}_{\Gamma_m}(\alpha) = \mathrm{Ind}_{\Gamma_n}(\alpha)$. Define $\mathrm{Ind}_{\Gamma}(\alpha)$ to be this common value. Prove that it does not depend on the choice of the sequence $(\Gamma_n)_{n \in \mathbb{Z}_{>0}}$. Prove that Lemma 10.39 now holds for closed curves which are merely continuous. Use this result to prove that Theorem 10.40 holds for closed curves which are merely continuous.

The problem says to use trigonometric polynomials for the approximation, but feel free to use piecewise linear functions instead, or some other convenient approximation. Furthermore, it is probably better not to use sequences, despite the statement of the problem. (Of course, don't use Theorem 10.40 of Rudin, but you will want Lemma 10.39.)

For reference, here are the statements of Lemma 10.39 and Theorem 10.40.

Lemma 1. Let $\Gamma_0, \Gamma_1 \colon [0, 2\pi] \to \mathbb{C}$ be piecewise C^1 closed curves in $\mathbb{C}$. Let $\alpha \in \mathbb{C}$. Suppose that

$$|\Gamma_1(t) - \Gamma_0(t)| < |\alpha - \Gamma_0(t)|$$

for all $t \in [0, 2\pi]$. Then $\mathrm{Ind}_{\Gamma_0}(\alpha) = \mathrm{Ind}_{\Gamma_1}(\alpha)$.

Theorem 2. Let $\Omega \subset \mathbb{C}$ be open, and let $\Gamma_0, \Gamma_1 \colon [0, 2\pi] \to \mathbb{C}$ be piecewise C^1 closed curves in Ω which are homotopic in Ω. Let $\alpha \in \mathbb{C} \setminus \Omega$. Then $\mathrm{Ind}_{\Gamma_0}(\alpha) = \mathrm{Ind}_{\Gamma_1}(\alpha)$.

The following problem counts as 1.5 ordinary problems.

Problem 3 (Problem 12 in Chapter 10 of Rudin's book)**.** For $t \in \mathbb{R}$, use the Residue Theorem to compute

$$\int_{-\infty}^{\infty} \left(\frac{\sin(x)}{x}\right)^2 e^{itx}\, dx.$$

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Compare with Rudin Chapter 9 Problem 2.

Problem 4 (Problem 11 in Chapter 10 of Rudin's book)**.** Let $\alpha \in \mathbb{C}$ satisfy $|\alpha| \neq 1$. Calculate

$$\int_0^{2\pi} \frac{1}{1 - 2\alpha\cos(\theta) + \alpha^2}\, d\theta$$

by integrating $(z-\alpha)^{-1}(z - 1/\alpha)^{-1}$ around the unit circle.