MATH 618 (SPRING 2025, PHILLIPS): SOLUTIONS TO HOMEWORK 4

For some of the problems, you will need to read in the book ahead of the lectures, at least through Theorem 10.15 (Cauchy's Formula) and, depending on what you do for some of them, through Theorem 10.17 (Morera's Theorem).

Remember that Morera's Theorem applies only to *continuous* functions.

Problems and all other items use two independent numbering sequences. This is annoying, but necessary to preserve the problem numbers in the solutions files.

Little proofreading has been done.

Some parts of problems have several different solutions.

Problem 1 (Problem 2 in Chapter 10 of Rudin's book)**.** Let f be an entire function. Suppose that for every $a \in \mathbb{C}$, in the power series representation

$$f(z) = \sum_{n=0}^{\infty} c_{n,a} (z - a)^n, \tag{1}$$

there is $n \in \mathbb{Z}_{\geq 0}$ such that $c_{n,a} = 0$. Prove that f is a polynomial.

Hint: $n! c_{n,a} = f^{(n)}(a)$.

Rudin wrote (**??**) as "$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n$". Suppressing the dependence on a in the notation for the coefficients makes proper writing of both the problem and its solution awkward.

Solution. For $n \in \mathbb{Z}_{\geq 0}$, set

$$Z_n = \big\{ a \in \mathbb{C} \colon c_{n,a} = 0 \big\}.$$

By hypothesis, we have $\bigcup_{n=0}^{\infty} Z_n = \mathbb{C}$. Therefore there exists $n \in \mathbb{Z}_{\geq 0}$ such that Z_n is uncountable. Since $f^{(n)}(z) = 0$ for all $z \in Z_n$ and $\mathbb{C}$ is a region, Theorem 10.18 of Rudin implies that $Z_n = \mathbb{C}$. Thus $f^{(n)} = 0$. So $f^{(m)} = 0$ for all $m > n$. In particular, for all $m \geq n$, we have $f^{(m)}(0) = 0$. Therefore $c_{m,0} = m! f^{(m)}(0) = 0$. So $f(z) = \sum_{m=0}^{n-1} c_{n,0} z^n$ is a polynomial (of degree at most $n - 1$). $\square$

Problem 2. Let $U \subset \mathbb{C}$ be open, and set $V = \{ \overline{z} \colon z \in U \}$. Let $f \colon U \to \mathbb{C}$ be holomorphic. Define $g \colon V \to \mathbb{C}$ by $g(z) = \overline{f(\overline{z})}$ for $z \in V$. Prove that g is holomorphic.

Solution. To avoid cumbersome notation, let $c \colon \mathbb{C} \to \mathbb{C}$ be complex conjugation. Thus, $g = c \circ f \circ c$.

Let $z \in V$. In the following calculation, the first step is the definition of g, the second step uses $\lim_{h \to 0} c^{-1}(h) = 0$ to replace h with $c(h)$, the third step is algebra,

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and the fourth step is the definition of the derivative:

$$\begin{aligned}\lim_{h\to 0}\frac{g(z+h)-g(z)}{h} &= \lim_{h\to 0}\frac{(c\circ f\circ c)(z+h)-(c\circ f\circ c)(z)}{h}\\ &= \lim_{h\to 0}\frac{(c\circ f\circ c)(z+c(h))-(c\circ f\circ c)(z)}{c(h)}\\ &= \lim_{h\to 0} c\left(\frac{f(c(z)+h)-f(c(z))}{h}\right) = c(f'(c(z))).\end{aligned}$$

In particular, the limit at the beginning of the calculation exists. □

Alternate solution. Define real valued functions f_1, f_2, g_1, and g_2 on U and V, regarded as subsets of $\mathbb{R}^2$, by

$$f(x+iy) = f_1(x,y) + if_2(x,y) \qquad \text{and} \qquad g(x+iy) = g_1(x,y) + ig_2(x,y).$$

The definition of g implies that

$$g_1(x,y) = f_1(x,-y) \qquad \text{and} \qquad g_2(x,y) = -f_2(x,-y).$$

Let $c(x,y) = (x,-y)$ for $x,y\in\mathbb{R}$. Then the equations above become $g_1 = f_1\circ c$ and $g_2 = -f_2\circ c$. As a function of two variables, we have

$$Dc(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for all $x,y\in\mathbb{R}$. Using the Cauchy-Riemann equations for f and the multivariable chain rule, we get, whenever $x+iy\in V$,

$$\begin{aligned}(D_1g_1)(x,y) &= D_1(f_1\circ c)(x,y) = (D_1f_1)(c(x,y))\\ &= (D_2f_2)(c(x,y)) = -D_2(f_2\circ c)(x,y) = (D_2g_2)(x,y)\end{aligned}$$

and

$$\begin{aligned}(D_2g_1)(x,y) &= D_2(f_1\circ c)(x,y) = -(D_2f_1)(c(x,y))\\ &= (D_1f_2)(c(x,y)) = D_1(f_2\circ c)(x,y) = -(D_1g_2)(x,y).\end{aligned}$$

Thus, g satisfies the Cauchy-Riemann equations. Since f is holomorphic, f is real differentiable. Since c is real differentiable, it follows that g is real differentiable. Therefore the Cauchy-Riemann equations imply that g is holomorphic. □

In this solution, it is not enough to just verify that g satisfies the Cauchy-Riemann equations. We also need to know that the real variable derivative of g exists.

Second alternate solution. We prove that f is representable by power series on V. So let $a\in V$, and let $r>0$ satisfy $B_r(a)\subset V$. Then $B_r(\overline{a})\subset U$. Since f is holomorphic on U, there are $c_0, c_1, \ldots \in\mathbb{C}$ such that for all $z\in B_r(\overline{a})$ we have $f(z) = \sum_{n=0}^{\infty} c_n(z-\overline{a})^n$. If now $z\in B_r(a)$, then

$$g(z) = \overline{f(\overline{z})} = \overline{\sum_{n=0}^{\infty} c_n(\overline{z}-\overline{a})^n} = \sum_{n=0}^{\infty}\overline{c_n}(z-\overline{a})^n.$$

This completes the solution. □

Third alternate solution. Let $z_0 \in V$. We prove that $g'(z_0) = \overline{f'(\overline{z_0})}$ directly from the definition. So let $\varepsilon > 0$. Since $\overline{z_0} \in U$, and by the definition of $f'(\overline{z_0})$, there is $\delta > 0$ such that whenever $0 < |z - \overline{z_0}| < \delta$, then

$$\left| \frac{f(z) - f(\overline{z_0})}{z - \overline{z_0}} - f'(\overline{z_0}) \right| < \varepsilon. \tag{2}$$

Now suppose $0 < |z - z_0| < \delta$. Then $0 < |\overline{z} - \overline{z_0}| < \delta$, so, using $|\overline{w}| = |w|$ for any $w \in \mathbb{C}$ at the third step and (**??**) with $\overline{z}$ in place of z at the last step,

$$\left| \frac{g(z) - g(z_0)}{z - z_0} - \overline{f'(\overline{z_0})} \right| = \left| \frac{\overline{f(\overline{z})} - \overline{f(\overline{z_0})}}{z - z_0} - \overline{f'(\overline{z_0})} \right| = \left| \overline{\frac{f(\overline{z}) - f(\overline{z_0})}{\overline{z} - \overline{z_0}} - f'(\overline{z_0})} \right|$$
$$= \left| \frac{f(\overline{z}) - f(\overline{z_0})}{\overline{z} - \overline{z_0}} - f'(\overline{z_0}) \right| < \varepsilon.$$

This completes the solution. □

Problem 3 (Rudin, Chapter 10, Problem 5)**.** Let $\Omega \subset \mathbb{C}$ be a nonempty open set, and let $(f_n)_{n \in \mathbb{Z}_{>0}}$ be a uniformly bounded sequence of holomorphic functions on Ω. Suppose there is a function $f \colon \Omega \to \mathbb{C}$ such that $f_n(z) \to f(z)$ pointwise. Prove that the convergence is uniform on every compact subset of Ω.

Hint: Apply the Dominated Convergence Theorem to the Cauchy formula for $f_n - f_m$.

Remark 1. It is not immediate from the statement that f is holomorphic. In particular, we do not know, without work, that

$$\mathrm{Ind}_\gamma(z) f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, d\zeta$$

for suitable closed curves γ.

Remark 2. It would make things easier to know that for every open subset $\Omega \subset \mathbb{C}$ and every compact subset $K \subset \Omega$, there is a cycle Γ such that $\mathrm{Ind}_\gamma(z) = 0$ for all $z \in \mathbb{C} \setminus \Omega$ and $\mathrm{Ind}_\gamma(z) = 1$ for all $z \in K$. (There is no hope of getting a closed curve in place of a cycle.) This is actually true, and will eventually be needed, but proving this fact requires considerably more work than doing Problem **??** without it.

First solution. It suffices to prove that for every $z_0 \in \Omega$ there is $r > 0$ such that the convergence is uniform on $B_r(z_0)$. (If $K \subset \Omega$ is compact, it can be covered by finitely many such balls.)

Choose $r > 0$ such that $B_{3r}(z_0) \subset \Omega$. Define a closed curve $\gamma \colon [0, 2\pi] \to \Omega$ by $\gamma(t) = z_0 + 2r \exp(it)$. Then for $z \in B_r(z_0)$ and $t \in [0, 2\pi]$ we have

$$\left| \frac{\gamma'(t)}{\gamma(t) - z} \right| < 2.$$

For $n \in \mathbb{Z}_{>0}$ define $g_n \colon [0, 2\pi] \to \mathbb{C}$ by $g_n(t) = f_n(\gamma(t))$, and define $g \colon [0, 2\pi] \to \mathbb{C}$ by $g(t) = f(\gamma(t))$. Then $g_n \to g$ pointwise. Choose M such that $|f_n(z)| \leq M$ for all $n \in \mathbb{Z}_{>0}$ and $z \in \Omega$. We apply the Dominated Convergence Theorem on $[0, 2\pi]$ with dominating function the constant function $2M$ to conclude that

$$\lim_{n \to \infty} \|g_n - g\|_1 = 0.$$

Moreover, for every $z \in B_r(z_0)$, we can use the Dominated Convergence Theorem on $[0, 2\pi]$ with dominating function the constant function $2M$ at the first step, and Cauchy's Formula at the second step, to conclude that

$$\frac{1}{2\pi i}\int_0^{2\pi} \frac{g(t)\gamma'(t)}{\gamma(t)-z}\,dt = \frac{1}{2\pi i}\lim_{n\to\infty}\int_0^{2\pi} \frac{g_n(t)\gamma'(t)}{\gamma(t)-z}\,dt = \lim_{n\to\infty} f_n(z) = f(z).$$

Now let $\varepsilon > 0$. Choose $N \in \mathbb{Z}_{>0}$ so large that for $n \geq N$ we have

$$\|g_n - g\|_1 < \frac{\varepsilon}{2}.$$

Let $n \geq N$. Then for all $z \in B_r(z_0)$, we have

$$\begin{aligned} |f_n(z) - f(z)| &= \left| \frac{1}{2\pi i}\int_0^{2\pi} \frac{g_n(t)\gamma'(t)}{\gamma(t)-z}\,dt - \frac{1}{2\pi i}\int_0^{2\pi} \frac{g(t)\gamma'(t)}{\gamma(t)-z}\,dt \right| \\ &\leq \sup_{t\in[0,2\pi]}\left(\left|\frac{\gamma'(t)}{\gamma(t)-z}\right|\right)\frac{1}{2\pi}\int_0^{2\pi} |g_n(t)-g(t)|\,dt \leq 2\cdot\|g_n - g\|_1 < \varepsilon. \end{aligned}$$

This completes the proof. □

Second solution. We will prove that for every compact set $K \subset \Omega$, the sequence $(f_n|_K)_{n\in\mathbb{Z}_{>0}}$ is uniformly Cauchy. It suffices to prove that for every $z_0 \in \Omega$ there is $r > 0$ such that $(f_n|_{B_r(z_0)})_{n\in\mathbb{Z}_{>0}}$ is uniformly Cauchy. (If $K \subset \Omega$ is compact, it can be covered by finitely many such balls.)

Let $z_0 \in \Omega$. Choose $r > 0$ such that $B_{3r}(z_0) \subset \Omega$. Define a closed curve $\gamma\colon [0, 2\pi] \to \Omega$ by $\gamma(t) = z_0 + 2r\exp(it)$. Then for $z \in B_r(z_0)$ and $t \in [0, 2\pi]$ we have

$$\left|\frac{\gamma'(t)}{\gamma(t)-z}\right| < 2. \tag{3}$$

Further choose M such that $|f_n(z)| \leq M$ for all $n \in \mathbb{Z}_{>0}$ and $z \in \Omega$. For $n \in \mathbb{Z}_{>0}$ define $g_n\colon [0, 2\pi] \to \mathbb{C}$ by $g_n(t) = f_n(\gamma(t))$.

We claim that for every $\varepsilon > 0$ there is $N \in \mathbb{Z}_{>0}$ such that whenever $m, n > N$ and $z \in B_r(z_0)$, then

$$\int_0^{2\pi} |g_m(t) - g_n(t)|\,dt < \varepsilon.$$

We want prove the claim by applying the Dominated Convergence Theorem over the index set $\mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$.

Suppose the claim is false. Then there are $\varepsilon > 0$ and $m_1, n_1, m_2, n_2, \ldots \in \mathbb{Z}_{>0}$ such that $m_1 < n_1 < m_2 < n_2 < \cdots$ and

$$\int_0^{2\pi} |g_{m_k}(t) - g_{n_k}(t)|\,dt \geq \varepsilon \tag{4}$$

for all $k \in \mathbb{Z}_{>0}$. Using

$$\begin{aligned} |g_{m_k}(t) - g_{n_k}(t)| &= |f_{m_k}(\gamma(t)) - f_{n_k}(\gamma(t))| \\ &\leq |f_{m_k}(\gamma(t)) - f(\gamma(t))| + |f(\gamma(t)) - f_{n_k}(\gamma(t))| \end{aligned}$$

for all $k \in \mathbb{Z}_{>0}$ and $t \in [0,\, 2\pi]$, and $f_l \to f$ pointwise, we get

$$\lim_{k\to\infty} |g_{m_k}(t) - g_{n_k}(t)| = 0$$

for all $t \in [0,\, 2\pi]$. We further have

$$|g_{m_k}(t) - g_{n_k}(t)| \leq |g_{m_k}(t)| + |g_{n_k}(t)| \leq 2M$$

for all $t \in [0, 2\pi]$, and $2M$ is an integrable function on $[0, 2\pi]$. The Dominated Convergence Theorem therefore implies that

$$\lim_{k\to\infty} \int_0^{2\pi} |g_{m_k}(t) - g_{n_k}(t)|\, dt = 0,$$

which contradicts (**??**). The claim is proved.

We now prove that $(f_n|_{B_r(z_0)})_{n\in\mathbb{Z}_{>0}}$ is uniformly Cauchy. Let $\varepsilon > 0$. Choose N as in the claim with $\pi\varepsilon$ in place of ε. Let $m, n \in \mathbb{Z}_{>0}$ satisfy $m, n > N$. Let $z \in B_r(z_0)$. Then, using Cauchy's formula on a convex set at the first step and (**??**) at the third step, we have

$$\begin{aligned} |f_m(z) - f_n(z)| &= \left| \frac{1}{2\pi i} \int_\gamma \frac{f_m(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_\gamma \frac{f_n(\zeta)}{\zeta - z}\, d\zeta \right| \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} \frac{|g_m(t) - g_n(t)| \cdot |\gamma'(t)|}{|\gamma(t) - z|}\, dt \\ &\leq \frac{1}{2\pi} \int_0^{2\pi} 2|g_m(t) - g_n(t)|\, dt < \left(\frac{1}{2\pi}\right) \cdot 2 \cdot \pi\varepsilon = \varepsilon. \end{aligned}$$

This completes the proof that $(f_n|_{B_r(z_0)})_{n\in\mathbb{Z}_{>0}}$ is uniformly Cauchy, and hence the proof that for every compact set $K \subset \Omega$, the sequence $(f_n|_K)_{n\in\mathbb{Z}_{>0}}$ is uniformly Cauchy.

Now let $K \subset \Omega$ be compact. Then $C(K)$ is complete. Therefore there is a function $h_K \in C(K)$ such that $f_n|_K \to h_K$ uniformly. Since $f_n|_K \to f|_K$ pointwise, we must have $h_K = f|_K$. Therefore $f_n|_K \to f|_K$ uniformly. $\square$

Problem 4 (Rudin, Chapter 10, Problem 7)**.** Let $\Omega \subset \mathbb{C}$ be open, and let f be a holomorphic function on Ω. Under certain conditions on z and Γ, the Cauchy formula for the derivatives of f,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta$$

for $n \in \mathbb{Z}_{>0}$, is valid. State the conditions, and prove the formula.

Conditions. The formula is valid whenever Cauchy's Formula for $f(z)$ holds and $\operatorname{Ind}_\Gamma(z) = 1$. The most general version, based on Theorem 10.35 of Rudin, is thus: Γ is a cycle in Ω such that $\operatorname{Ind}_\Gamma(w) = 0$ for every $w \in \mathbb{C} \setminus \Omega$, and $\operatorname{Ind}_\Gamma(z) = 1$.

Given what we have done so far, I am also happy to simply require that Ω be convex, that Γ be a closed path in Ω, and that $\operatorname{Ind}_\Gamma(z) = 1$. These conditions come from 10.15 of Rudin. $\square$

Remark 3. It is not correct to require $\operatorname{Ind}_\Gamma(w) = 1$ for all $w \in \Omega \setminus \operatorname{Ran}(\Gamma)$. This condition can never be satisfied. If $\Omega = \mathbb{C}$ then any $w \in \Omega$ with $|w| > \sup_{y\in\operatorname{Ran}(\Gamma)} |y|$ satisfies $\operatorname{Ind}_\Gamma(w) = 0$. If $\Omega \neq \mathbb{C}$, choose $w_0 \in \partial\Omega$, and choose $r > 0$ such that $B_r(w_0) \cap \operatorname{Ran}(\Gamma) = \varnothing$. Then there exists $w \in B_r(w_0) \cap \Omega$. Because we assume $\operatorname{Ind}_\Gamma(w_0) = 0$, it follows that also $\operatorname{Ind}_\Gamma(w) = 0$.

We give five proofs. All are related to approaches that have been used by students in the past.

Proof 1. We prove this by induction on n. The case $n = 0$ is Cauchy's Formula, Theorem 10.35 of Rudin or 10.15 of Rudin.

Assume now that the result is known for $n-1$. Define

$$g(\zeta) = \frac{f(\zeta)}{(\zeta - z)^n}$$

for $\zeta \in \Omega \setminus \{z\}$. Then

$$g'(\zeta) = \frac{f'(\zeta)}{(\zeta - z)^n} - \frac{n f(\zeta)}{(\zeta - z)^{n+1}}.$$

Using Theorem 10.12 of Rudin (and the discussion of 10.34 of Rudin if we are allowing general cycles), we get

$$\int_{\Gamma} g'(\zeta)\, d\zeta = 0.$$

Therefore, applying the induction hypothesis to f' at the first step, and the previous two equations at the second step,

$$\begin{aligned} f^{(n)}(z) &= \frac{(n-1)!}{2\pi i} \int_{\Gamma} \frac{f'(\zeta)}{(\zeta - z)^n}\, d\zeta \\ &= \frac{(n-1)!}{2\pi i} \int_{\Gamma} \frac{n f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta, \end{aligned}$$

as desired. □

I believe the following is probably the intended proof.

Proof 2. For a complex measure on a measure space X, a measurable function $\varphi\colon X \to \mathbb{C}$, and $a \in \mathbb{C}$ and $r > 0$ such that $B_r(a) \cap \varphi(X) = \varnothing$, the proof of Theorem 10.7 of Rudin shows that, with

$$c_n = \int_X \frac{1}{(\varphi(x) - a)^{n+1}}\, d\mu(x)$$

for $n \in \mathbb{Z}_{\geq 0}$, we have, for $z \in B_r(a)$,

$$\int_X \frac{1}{\varphi(x) - z}\, d\mu(x) = \sum_{n=0}^{\infty} c_n (z-a)^n.$$

In particular, with

$$f(z) = \int_X \frac{1}{\varphi(x) - z}\, d\mu(x),$$

it follows by differentiating the power series n times term by term that

$$f^{(n)}(a) = n! c_n = n! \int_X \frac{1}{(\varphi(x) - a)^{n+1}}\, d\mu(x).$$

Simply apply this fact to Cauchy's formula,

$$f(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

to get the result. □

The following proof uses differentiation of Cauchy's Formula under the integration sign.

Proof 3. The proof is by induction on n. The case $n = 0$ is Cauchy's Formula, Theorem 10.35 of Rudin or 10.15 of Rudin.

We will give the rest of the proof for the case of a cycle; the version corresponding to 10.15 of Rudin is essentially the same but notationally simpler.

Assume now that the result is known for $n-1$. That is,

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^n} \, d\zeta$$

for all $z \in \Omega$ with $\mathrm{Ind}_\Gamma(z) = 1$. (The set U of such z is open.) Write Γ as a formal integer combination $\sum_{j=1}^m n_j \cdot \gamma_j$, with $\gamma_j \colon [\alpha_j, \beta_j] \to \Omega$ piecewise C^1. By further breaking up the paths, we may assume we have a formal integer combination of C^1 paths. Set

$$g_j(z) = \int_{\gamma_j} \frac{f(\zeta)}{(\zeta - z)^n} \, d\zeta$$

for $z \in \Omega \setminus \mathrm{Ran}(\Gamma)$.

Now we differentiate under the integral sign. Rewrite

$$g_j(z) = \int_{\alpha_j}^{\beta_j} \frac{f(\gamma(t))}{(\gamma(t) - z)^n} \, \gamma'(t) \, dt.$$

Set

$$q(z, t) = \frac{f(\gamma(t))}{(\gamma(t) - z)^n} \, \gamma'(t)$$

for $z \in U$ and $t \in [\alpha_j, \beta_j]$. Let $D_1 q(z,t)$ be the partial derivative of q in the first variable (in the complex sense). Then

$$D_1 q(z, t) = \frac{n f(\gamma(t))}{(\gamma(t) - z)^{n+1}} \, \gamma'(t),$$

which is jointly continuous on $U \times [\alpha_j, \beta_j]$. Define

$$h_j(z) = \int_{\alpha_j}^{\beta_j} D_1 q(z, t) \, dt = \int_{\gamma_j} \frac{n f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta.$$

Choose $r > 0$ such that $\overline{B_r(z)} \subset U$. Let $(z_n)_{n \in \mathbb{Z}_{>0}}$ be a sequence in $\overline{B_r(z)} \setminus \{z\}$ with $\lim_{n \to \infty} z_n = z$. Set

$$M = \sup\left(\left\{ |D_1 q(z, t)| \colon (z, t) \in \overline{B_r(z)} \times [\alpha_j, \beta_j] \right\} \right).$$

Using the (real) vector valued mean value theorem, and the fact that the straight line path from z to z_n is contained in $\overline{B_r(z)}$, we get

$$|q(z_n, t) - q(z, t)| \leq M |z_n - z|$$

for all n and t. Therefore the functions

$$t \mapsto \frac{q(z_n, t) - q(z, t)}{z_n - z}$$

are bounded by M and converge pointwise to $D_1 q(z, t)$ as $n \to \infty$. It follows from the Dominated Convergence Theorem that

$$\lim_{n \to \infty} \frac{g_j(z_n) - g_j(z)}{z_n - z} = h_j(z).$$

Since the sequence $(z_n)_{n \in \mathbb{Z}_{>0}}$ in $\overline{B_r(z)} \setminus \{z\}$ is arbitrary, we have proved that $g_j'(z) = h_j(z)$.

Adding up the results of the previous paragraph with suitable coefficients, we see that we can differentiate the right hand side of the equation

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^n} \, d\zeta$$

under the integral sign. The result is Cauchy's Formula for $f^{(n)}(z)$. □

Remark 4. It is true that the usual method of proof of differentiation under the integral sign in elementary analysis can be adapted to justify the calculation of $g_j'(z)$ in the proof above. However, the hypotheses of that theorem do not hold, because z runs over an open set in $\mathbb{C}$ rather than $\mathbb{R}$. Thus, one may not simply quote that theorem as if it applied.

Proof 4. This proof is similar to the previous proof, but substitutes explicit calculations for the general theory of differentiation under the integral sign. As in the previous proof, and following the notation there, we assume that

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^n} \, d\zeta$$

for all $z \in \Omega$ with $\mathrm{Ind}_\Gamma(z) = 1$. Let $a \in \Omega$ satisfy $\mathrm{Ind}_\Gamma(a) = 1$. Applying the identity

$$x^n - y^n = (x-y)\big(x^{n-1} + x^{n-2}y + \cdots + y^{n-1}\big)$$

with

$$x = \frac{1}{\zeta - z} \qquad \text{and} \qquad y = \frac{1}{\zeta - a},$$

we get the first step in the following calculation:

$$\begin{aligned}
\frac{1}{(\zeta - z)^n} - \frac{1}{(\zeta - a)^n} &= \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - a} \right) \sum_{k=1}^{n} \frac{1}{(\zeta - z)^{k-1}(\zeta - a)^{n-k}} \\
&= \left(\frac{z - a}{(\zeta - z)(\zeta - a)} \right) \sum_{k=1}^{n} \frac{1}{(\zeta - z)^{k-1}(\zeta - a)^{n-k}} \\
&= (z - a) \sum_{k=1}^{n} \frac{1}{(\zeta - z)^{k}(\zeta - a)^{n-k+1}}.
\end{aligned}$$

Choose $\varepsilon > 0$ such that $\mathrm{Ind}_\Gamma(z) = 1$ for all $z \in B_{2\varepsilon}(a)$. Using the induction hypothesis at the first step, and the previous calculation at the second step,

$$\begin{aligned}
\frac{f^{(n-1)}(z) - f^{(n-1)}(a)}{z - a} &= \frac{(n-1)!}{2\pi i (z-a)} \int_\Gamma \left(\frac{f(\zeta)}{(\zeta - z)^n} - \frac{f(\zeta)}{(\zeta - a)^n} \right) d\zeta \\
&= \frac{(n-1)!}{2\pi i} \sum_{k=1}^{n} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{k}(\zeta - a)^{n-k+1}} \, d\zeta \\
&= \frac{(n-1)!}{2\pi i} \sum_{j=1}^{m} n_j \sum_{k=1}^{n} \int_{\alpha_j}^{\beta_j} \frac{f(\gamma_j(t))\gamma_j'(t)}{(\gamma_j(t) - z)^{k}(\gamma_j(t) - a)^{n-k+1}} \, dt.
\end{aligned}$$

Let $z \in B_\varepsilon(a)$. Then $|\gamma_j(t) - a| \geq 2\varepsilon$ and $|\gamma_j(t) - z| \geq \varepsilon$. Therefore the integrand is bounded by

$$\frac{1}{2^{n-k+1}\varepsilon^{n+1}} \left(\sup_{\zeta \in \mathrm{Ran}(\gamma)} |f(\zeta)| \right) \left(\sup_{t \in [\alpha_j, \beta_j]} |\gamma_j'(t)| \right),$$

which is independent of z.

Now let $(z_n)_{n\in\mathbb{Z}_{>0}}$ be any sequence in $B_\varepsilon(a)$ such that $\lim_{n\to\infty} z_n = a$. Since we are integrating over sets of finite measure, we can apply the Dominated Convergence Theorem to get

$$\begin{aligned}
\lim_{n\to\infty} & \frac{f^{(n-1)}(z_n) - f^{(n-1)}(a)}{z_n - a} \\
&= \frac{(n-1)!}{2\pi i} \sum_{j=1}^{m} n_j \sum_{k=1}^{n} \int_{\alpha_j}^{\beta_j} \lim_{n\to\infty} \left(\frac{f(\gamma_j(t))\gamma_j'(t)}{(\gamma_j(t) - z_n)^k (\gamma_j(t) - a)^{n-k+1}} \right) dt \\
&= \frac{(n-1)!}{2\pi i} \sum_{j=1}^{m} n_j \sum_{k=1}^{n} \int_{\alpha_j}^{\beta_j} \frac{f(\gamma_j(t))\gamma_j'(t)}{(\gamma_j(t) - z)^{n+1}}\, dt \\
&= \frac{n!}{2\pi i} \sum_{j=1}^{m} n_j \int_{\alpha_j}^{\beta_j} \frac{f(\gamma_j(t))\gamma_j'(t)}{(\gamma_j(t) - z)^{n+1}}\, dt \\
&= \frac{n!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta.
\end{aligned}$$

Since the sequence is arbitrary, it follows that

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta,$$

as was to be proved. □

Proof 5. Since f is holomorphic on Ω, there are $r > 0$ such that for all $\zeta \in B_r(z)$ we have $f(\zeta) = \sum_{k=0}^{\infty} \frac{1}{k!} f^{(k)}(z)(\zeta - z)^k$. Then the series

$$\sum_{k=n+1}^{\infty} \frac{1}{k!} f^{(k)}(z)(\zeta - z)^{k-n-1}$$

converges everywhere on $B_r(z)$, and the function to which it converges must be holomorphic. Therefore there is a holomorphic function $h\colon \Omega \to \mathbb{C}$ such that

$$h(\zeta) = \sum_{k=n+1}^{\infty} \frac{1}{k!} f^{(k)}(z)(\zeta - z)^{k-n-1}$$

for $\zeta \in B_r(z)$ and

$$h(\zeta) = \frac{1}{(\zeta - z)^{n+1}} \left(f(\zeta) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(z)(\zeta - z)^k \right)$$

for $\zeta \in \Omega \setminus \{z\}$.

Using Cauchy's Theorem at the first step, we get

$$0 = \int_\Gamma h(\zeta)\, d\zeta = \int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(z) \int_\Gamma \frac{1}{(\zeta - z)^{n+1-k}}\, d\zeta.$$

In the sum, the integrand in every term with $k \neq n$ has an antiderivative on $\Omega \setminus \{z\}$. By Theorem 10.12 of Rudin (and the discussion of 10.34 of Rudin if we are allowing general cycles), these terms are all zero. Also, by definition

$$\int_\Gamma \frac{1}{\zeta - z}\, d\zeta = 2\pi i \cdot \mathrm{Ind}_\Gamma(z) = 2\pi i.$$

So

$$\int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta = \frac{2\pi i}{n!} f^{(n)}(z),$$

from which the desired formula is immediate. □

One can also get the function h from general theory. Define

$$g(\zeta) = f(\zeta) - \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(z)(\zeta - z)^k.$$

By differentiating, we check that g has a zero of order $n+1$ at z. Therefore there exists a holomorphic function h on Ω such that $g(\zeta) = h(\zeta)(\zeta - z)^{n+1}$ for all $\zeta \in \Omega$.

Problem 5 (Rudin, Chapter 10, Problem 16)**.** Let $(X, \mathcal{B})$ be a measurable space, and let μ be a complex measure on $(X, \mathcal{B})$. Let $\Omega \subset \mathbb{C}$ be open, and let $\varphi\colon \Omega \times X \to \mathbb{C}$ be a bounded function such that for every $x \in X$ the function $z \mapsto \varphi(z, x)$ is holomorphic on Ω and for every $z \in \Omega$ the function $x \mapsto \varphi(z, x)$ is measurable. Define $f\colon \Omega \to \mathbb{C}$ by

$$f(z) = \int_X \varphi(z, x)\, d\mu(x).$$

Prove that f is holomorphic on Ω.

Hint: Prove that for every compact subset $K \subset \Omega$ there is a constant M such that for $x \in X$ and all distinct $z_1, z_2 \in K$, we have

$$\left| \frac{\varphi(z_1, x) - \varphi(z_2, x)}{z_1 - z_2} \right| < M.$$

We give two solutions. The second solution uses Morera's Theorem and Fubini's Theorem. The idea looks attractive, but verifying the measurability requirement for applying Fubini's Theorem turns out to be rather ugly. Several lemmas are required.

The following lemma is related to the hint given in the problem. It will be used in both solutions. In the first solution, its role is to give a dominating function for an application of the Dominated Convergence Theorem. In the second solution, it is part of the proof that φ is measurable with respect to the product σ-algebra.

Lemma 5. Let $(X, \mathcal{B})$, Ω, and φ be as in the statement of the problem. Set

$$M = \sup_{x \in X} \sup_{z \in \Omega} |\varphi(z, x)|.$$

Let $a \in \Omega$ and let $\varepsilon > 0$. Suppose that $\overline{B_{2\varepsilon}(a)} \subset \Omega$. Then for any distinct $z_1, z_2 \in B_\varepsilon(a)$ and any $x \in X$, we have

$$\left| \frac{\varphi(z_1, x) - \varphi(z_2, x)}{z_1 - z_2} \right| \leq \frac{M}{\varepsilon}.$$

Proof. Let $\gamma(t) = a + 2\varepsilon \exp(it)$ for $t \in [0, 2\pi]$. Then, by Cauchy's Formula, for $z \in B_{2\varepsilon}(a)$ and $x \in X$ we have

$$\varphi(z, x) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{\varphi(\gamma(t),\, x)\gamma'(t)}{\gamma(t) - z}\, dt.$$

If $z \in B_\varepsilon(a)$ then $|\gamma(t) - z| > \varepsilon$ for all t. Therefore, for distinct $z_1, z_2 \in B_\varepsilon(a)$ and any $x \in X$, Cauchy's Formula gives

$$\begin{aligned}
&\left| \frac{\varphi(z_1, x) - \varphi(z_2, x)}{z_1 - z_2} \right| \\
&\qquad = \frac{1}{|z_1 - z_2|} \left| \frac{1}{2\pi i} \int_0^{2\pi} \frac{\varphi(\gamma(t),\, x)\gamma'(t)}{\gamma(t) - z_1}\, dt - \frac{1}{2\pi i} \int_0^{2\pi} \frac{\varphi(\gamma(t),\, x)\gamma'(t)}{\gamma(t) - z_2}\, dt \right| \\
&\qquad \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|z_1 - z_2|} \left| \frac{1}{\gamma(t) - z_1} - \frac{1}{\gamma(t) - z_2} \right| |\varphi(\gamma(t),\, x)\gamma'(t)|\, dt \\
&\qquad = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|\gamma(t) - z_1| \cdot |\gamma(t) - z_2|} |\varphi(\gamma(t),\, x)\gamma'(t)|\, dt \\
&\qquad \leq \frac{M\varepsilon}{\varepsilon^2} = \frac{M}{\varepsilon}.
\end{aligned}$$

This completes the proof. □

First solution. Let $a \in \Omega$. We prove that $f'(a)$ exists, and in fact is equal to

$$\int_X D_1\varphi(z, x)\, d\mu(x).$$

Choose M such that $|\varphi(z, x)| \leq M$ for all $(z, x) \in \Omega \times X$. Choose $\varepsilon > 0$ such that $\overline{B_{2\varepsilon}(a)} \subset \Omega$. Let $(z_n)_{n \in \mathbb{Z}_{>0}}$ be any sequence in $B_\varepsilon(a) \setminus \{a\}$ such that $\lim_{n\to\infty} z_n = a$. Let h be the Radon-Nikodym derivative of μ with respect to $|\mu|$. Then we can use the Dominated Convergence Theorem at the second step, with the dominating function being M/ε (obtained from Lemma **??**), to get

$$\begin{aligned}
\lim_{n\to\infty} \frac{f(z_n) - f(a)}{z_n - a} &= \lim_{n\to\infty} \int_X \left(\frac{\varphi(z_n, x) - \varphi(a, x)}{z_n - a} \right) h(x)\, d|\mu|(x) \\
&= \int_X \left(\lim_{n\to\infty} \frac{\varphi(z_n, x) - \varphi(a, x)}{z_n - a} \right) h(x)\, d|\mu|(x) \\
&= \int_X D_1\varphi(a, x) h(x)\, d|\mu|(x) \\
&= \int_X D_1\varphi(a, x)\, d\mu(x).
\end{aligned}$$

In particular, since the sequence is arbitrary, $f'(a)$ exists. □

The second solution uses two further lemmas.

Lemma 6. Under the hypotheses of the problem, for every $\varepsilon > 0$ and every compact set $K \subset \Omega$, there is $\delta > 0$ such that for every $z_1, z_2 \in K$ with $|z_1 - z_2| < \delta$, and for every $x \in X$, we have $|\varphi(z_1, x) - \varphi(z_2, x)| < \varepsilon$.

This is the statement that continuity of $z \mapsto \varphi(z, x)$ is uniform in $x \in X$ and $z \in K$.

Proof of Lemma **??**. Since K is compact, we have $\operatorname{dist}(K, \mathbb{C} \setminus \Omega) > 0$. Choose $\rho > 0$ such that $\rho < \frac{1}{2}\operatorname{dist}(K, \mathbb{C} \setminus \Omega)$. Choose M such that $|\varphi(z, x)| \leq M$ for all $(z, x) \in \Omega \times X$. Choose $\delta > 0$ such that $\delta \leq \rho$ and $\delta < \rho\varepsilon/M$. Let $x \in X$, and let $z_1, z_2 \in K$ satisfy $|z_1 - z_2| < \delta$. We show that $|\varphi(z_1, x) - \varphi(z_2, x)| < \varepsilon$. We may clearly assume that $z_1 \neq z_2$.

The choice of ρ implies that $\overline{B_{2\rho}(z_1)} \subset \Omega$. Lemma **??**, with $a = z_1$ and ρ in place of ε, implies that

$$\left| \frac{\varphi(z_1, x) - \varphi(z_2, x)}{z_1 - z_2} \right| \leq \frac{M}{\rho}.$$

Therefore

$$|\varphi(z_1, x) - \varphi(z_2, x)| \leq M\rho^{-1}|z_1 - z_2| < M\rho^{-1}\delta \leq \varepsilon.$$

This completes the proof. □

Lemma 7. Let T be a second countable metric space, let $(X, \mathcal{B})$ be a measurable space, and let $\varphi \colon T \times X \to \mathbb{C}$ be a function. Suppose that for every $t \in T$ the function $x \mapsto \varphi(t, x)$ is measurable. Suppose further that the functions $t \mapsto \varphi(t, x)$, for $x \in X$, are uniformly equicontinuous. Then φ is measurable for the product σ-algebra of the σ-algebra of Borel sets in T with $\mathcal{B}$.

The uniform equicontinuity hypothesis means that for every $\varepsilon > 0$ there is $\delta > 0$ such that for every $s_1, s_2 \in T$ with $d(s_1, s_2) < \delta$, and for every $x \in X$, we have $|\varphi(s_1, x) - \varphi(s_2, x)| < \varepsilon$.

Proof of Lemma **??**. It is enough to prove this when φ has real values.

Let $\{t_n \colon n \in \mathbb{Z}_{>0}\}$ be a countable dense subset of T. For each $\beta \in \mathbb{Q}$ and $n \in \mathbb{Z}_{>0}$ let $E_{\beta,n} \subset X$ be

$$E_{\beta,n} = \{x \in X \colon \varphi(t_n, x) > \beta\}.$$

Let $\mathcal{C}$ be the collection of all sets

$$B_r(t_n) \times E_{\beta,n} \subset T \times X$$

for $n \in \mathbb{Z}_{>0}$, $r \in \mathbb{Q} \cap (0, \infty)$, and $\beta \in \mathbb{Q}$. There are countably many such sets, and they are all measurable rectangles, so it suffices to show that for every $\alpha > 0$, every $t \in T$, and every $x \in X$, if $\varphi(t, x) > \alpha$ then there is a set $R \in \mathcal{C}$ such that

$$(t, x) \in R \subset \{(s, y) \in T \times X \colon \varphi(s, y) > \alpha\}.$$

So let $\alpha \in \mathbb{R}$, and suppose $(t, x) \in T \times X$ and $\varphi(t, x) > \alpha$. Choose $\beta \in \mathbb{Q}$ with $\varphi(t, x) > \beta > \alpha$. Set

$$\varepsilon = \min(\beta - \alpha,\, \varphi(t, x) - \beta).$$

Choose $\delta > 0$ such that for every $s_1, s_2 \in T$ with $d(s_1, s_2) < \delta$, and for every $x \in X$, we have

$$|\varphi(s_1, x) - \varphi(s_2, x)| < \varepsilon.$$

Choose $r > 0$ such that $r \in \mathbb{Q}$ and $3r < \delta$. Choose $n \in \mathbb{Z}_{>0}$ such that $d(t_n, t) < r$. Then $B_{2r}(t_n) \times E_{\beta,n} \in \mathcal{C}$. We claim that

$$(t, x) \in B_{2r}(t_n) \times E_{\beta,n} \subset \{(s, y) \in T \times X \colon \varphi(s, y) > \alpha\}.$$

That $t \in B_{2r}(t_n)$ is clear. To see that $x \in E_{\beta,n}$, we observe that

$$\varphi(t_n, x) > \varphi(t, x) - \varepsilon \geq \beta.$$

Now let $s \in B_{2r}(t_n)$ and $y \in E_{\beta,n}$. Then $\varphi(t_n, y) > \beta$, so

$$\varphi(s, y) > \varphi(t_n, y) - \varepsilon \geq \beta - (\beta - \alpha) = \alpha.$$

This completes the proof of the claim, and of the lemma. □

Second solution. It is sufficient to prove that f is holomorphic on every open ball $U \subset \Omega$.

We verify the hypotheses of Morera's Theorem on U. Choose M such that $|\varphi(z,x)| \leq M$ for all $x \in X$ and $z \in \Omega$. Let g be the Radon-Nikodym derivative of μ with respect to $|\mu|$. Then $|g(x)| = 1$ for almost every $x \in X$ (with respect to $|\mu|$), so we may as well assume $|g(x)| = 1$ for every $x \in X$. We can then rewrite the definition of f as

$$f(z) = \int_X \varphi(z,x)g(x)\, d|\mu|(x).$$

We first show that f is continuous. Suppose $z_n \to z$. Then

$$\lim_{n\to\infty} \varphi(z_n,x)g(x) = \varphi(z,x)g(x)$$

for all $x \in X$. We may apply the Dominated Convergence Theorem, with the dominating function being the constant M, to get

$$f(z_n) = \int_X \varphi(z_n,x)g(x)\, d|\mu|(x) \to \int_X \varphi(z,x)g(x)\, d|\mu|(x) = f(z).$$

Thus f is continuous.

Now let Δ be any (oriented) triangle in U, and let $\gamma\colon [\alpha,\beta] \to U$ be the closed curve obtained from its boundary, as in the book. Then

$$\int_\gamma \varphi(z,x)\, dz = 0$$

for every $x \in X$, because U is convex.

Lemmas **??** and **??** imply that for every compact set $K \subset \Omega$, the function $\varphi|_{K\times X}$ is measurable. Therefore φ is measurable, and it follows that $(t,x) \mapsto \varphi(\gamma(t),\, x)$ is measurable. So the function

$$\psi(t,x) = \varphi(\gamma(t),\, x)g(x)\gamma'(t)$$

is measurable on $[\alpha,\beta] \times X$. It is bounded by

$$M \sup_{t\in[\alpha,\beta]} |\gamma'(t)|.$$

Therefore ψ is integrable. Using Fubini's Theorem at the third step and Cauchy's Theorem at the fifth step, we get:

$$\begin{aligned}
\int_\gamma f(z)\, dz &= \int_\alpha^\beta f(\gamma(t))\gamma'(t)\, dt \\
&= \int_\alpha^\beta \left(\int_X \varphi(\gamma(t),\, x)g(x)\gamma'(t)\, d\mu(x) \right) dt \\
&= \int_X \left(\int_\alpha^\beta \varphi(\gamma(t),\, x)g(x)\gamma'(t)\, dt \right) d\mu(x) \\
&= \int_X g(x) \left(\int_\gamma \varphi(z,\, x)\, dz \right) d\mu(x) = 0.
\end{aligned}$$

It now follows from Morera's Theorem that f is holomorphic on U. □