MATH 618 (SPRING 2025, PHILLIPS): HOMEWORK 4

For some of the problems, you will need to read in the book ahead of the lectures, at least through Theorem 10.15 (Cauchy's Formula) and, depending on what you do for some of them, through Theorem 10.17 (Morera's Theorem).

Remember that Morera's Theorem applies only to *continuous* functions.

Problem 1 (Problem 2 in Chapter 10 of Rudin's book)**.** Let f be an entire function. Suppose that for every $a \in \mathbb{C}$, in the power series representation

$$f(z) = \sum_{n=0}^{\infty} c_{n,a}(z-a)^n, \tag{1}$$

there is $n \in \mathbb{Z}_{\geq 0}$ such that $c_{n,a} = 0$. Prove that f is a polynomial.

Hint: $n! c_{n,a} = f^{(n)}(a)$.

Rudin wrote (1) as "$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$". Suppressing the dependence on a in the notation for the coefficients makes proper writing of both the problem and its solution awkward.

Problem 2. Let $U \subset \mathbb{C}$ be open, and set $V = \{\overline{z} \colon z \in U\}$. Let $f \colon U \to \mathbb{C}$ be holomorphic. Define $g \colon V \to \mathbb{C}$ by $g(z) = \overline{f(\overline{z})}$ for $z \in V$. Prove that g is holomorphic.

Problem 3 (Rudin, Chapter 10, Problem 5)**.** Let $\Omega \subset \mathbb{C}$ be a nonempty open set, and let $(f_n)_{n \in \mathbb{Z}_{>0}}$ be a uniformly bounded sequence of holomorphic functions on Ω. Suppose there is a function $f \colon \Omega \to \mathbb{C}$ such that $f_n(z) \to f(z)$ pointwise. Prove that the convergence is uniform on every compact subset of Ω.

Hint: Apply the Dominated Convergence Theorem to the Cauchy formula for $f_n - f_m$.

Problem 4 (Rudin, Chapter 10, Problem 7)**.** Let $\Omega \subset \mathbb{C}$ be open, and let f be a holomorphic function on Ω. Under certain conditions on z and Γ, the Cauchy formula for the derivatives of f,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta$$

for $n \in \mathbb{Z}_{>0}$, is valid. State the conditions, and prove the formula.

Problem 5 (Rudin, Chapter 10, Problem 16)**.** Let $(X, \mathcal{B})$ be a measurable space, and let μ be a complex measure on $(X, \mathcal{B})$. Let $\Omega \subset \mathbb{C}$ be open, and let $\varphi \colon \Omega \times X \to \mathbb{C}$ be a bounded function such that for every $x \in X$ the function $z \mapsto \varphi(z, x)$ is holomorphic on Ω and for every $z \in \Omega$ the function $x \mapsto \varphi(z, x)$ is measurable. Define $f \colon \Omega \to \mathbb{C}$ by

$$f(z) = \int_X \varphi(z, x) \, d\mu(x).$$

Prove that f is holomorphic on Ω.

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Hint: Prove that for every compact subset $K \subset \Omega$ there is a constant M such that for $x \in X$ and all distinct $z_1, z_2 \in K$, we have

$$\left| \frac{\varphi(z_1, x) - \varphi(z_2, x)}{z_1 - z_2} \right| < M.$$