MATH 618 (SPRING 2025, PHILLIPS): SOLUTIONS TO HOMEWORK 3

Conventions on measures: m is ordinary Lebesgue measure, $\overline{m} = (2\pi)^{-1/2} m$, and in expressions of the form $\int_{\mathbb{R}} f(x)\, dx$, ordinary Lebesgue measure is assumed.

Problems and all other items use two independent numbering sequences. This is annoying, but necessary to preserve the problem numbers in the solutions files.

Little proofreading has been done.

Some parts of problems have several different solutions.

Problem 1. Let X be a locally compact σ-compact Hausdorff space. Prove that there is a sequence $(g_n)_{n \in \mathbb{Z}_{>0}}$ in $C_0(X)$ consisting of functions with compact support and with values in $[0, 1]$ such that for every $f \in C_0(X)$ we have $\lim_{n \to \infty} \|g_n f - f\|_\infty = 0$.

If X is not σ-compact, one needs a net instead of a sequence. You will need to prove that there are compact subsets $K_1, K_2, \ldots \subset X$ such that

$$K_1 \subset \operatorname{int}(K_2) \subset K_2 \subset \operatorname{int}(K_3) \subset K_3 \subset \cdots \quad \text{and} \quad \bigcup_{n=1}^{\infty} K_n = X.$$

(I didn't find this explicitly in Rudin's book, but maybe I didn't look in the right place.)

As suggested above, the solution uses the following lemma, which doesn't seem to be explicitly in Rudin's book.

Lemma 1. Let X be a locally compact σ-compact Hausdorff space. Then there are compact subsets $K_1, K_2, \ldots \subset X$ such that

$$K_1 \subset \operatorname{int}(K_2) \subset K_2 \subset \operatorname{int}(K_3) \subset K_3 \subset \cdots \quad \text{and} \quad \bigcup_{n=1}^{\infty} K_n = X.$$

Proof. By hypothesis, there are compact subsets $L_1, L_2, \ldots \subset X$ such that $\bigcup_{n=1}^{\infty} L_n = X$.

We construct compact sets K_n satisfying

$$K_{n-1} \subset \operatorname{int}(K_n), \quad \text{and} \quad L_1, L_2, \ldots, L_n \subset K_n$$

for $n \in \mathbb{Z}_{>0}$ (with $K_0 = \varnothing$). The construction is by induction. Define $K_1 = L_1$. Given K_n, define

$$M = K_n \cup \bigcup_{k=1}^{n} L_k,$$

which is a compact subset of X. Use Theorem 2.7 of Rudin to choose an open set V with compact closure such that $M \subset V$. Then take $K_{n+1} = \overline{V}$. This completes the induction step, and the proof. $\square$

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Solution. Choose compact subsets $K_1, K_2, \ldots \subset X$ as in Lemma 1. For $n \in \mathbb{Z}_{>0}$ use Urysohn's Lemma for locally compact spaces to choose a continuous function $g_n \colon X \to [0,1]$ with compact support and such that $g_n(x) = 1$ for all $x \in K_n$. Now let $f \in C_0(X)$ and let $\varepsilon > 0$. By definition, there is a compact subset $L \subset X$ such that $|f(x)| < \frac{\varepsilon}{3}$ for all $x \in X \setminus L$. It is clear from the properties of the sets K_n We have

$$X \supset \bigcup_{n=1}^{\infty} \operatorname{int}(K_n) \supset \bigcup_{n=2}^{\infty} K_{n-1} = X.$$

Since $\operatorname{int}(K_1) \subset \operatorname{int}(K_2) \subset \operatorname{int}(K_3) \cdots$ and L is compact, there exists $N \in \mathbb{Z}_{>0}$ such that $L \subset \operatorname{int}(K_N)$. Now let $n \in \mathbb{Z}_{>0}$ satisfy $n \geq N$. For $x \in K_n$ we have $g_n(x) = 1$, so $|g_n(x)f(x) - f(x)| = 0 < \frac{2\varepsilon}{3}$. For $x \in X \setminus K_n$, we have $|f(x)| < \frac{\varepsilon}{3}$ and $0 \leq g_n(x) \leq 1$, so

$$|g_n(x)f(x) - f(x)| \leq |g_n(x)||f(x)| + |f(x)| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \frac{2\varepsilon}{3}.$$

Therefore

$$\|g_n f - f\|_\infty = \sup_{x \in X} |g_n(x)f(x) - f(x)| \leq \frac{2\varepsilon}{3} < \varepsilon.$$

This completes the proof. □

In the situation in the proof, one often wants $\operatorname{supp}(g_n) \subset \operatorname{int}(K_{n+1})$, but we don't need that here.

Alternate solution (sketch). Choose g_n as in the first solution. Prove that if $f \in C_{\mathrm{c}}(X)$ then $g_n f = f$ for all sufficiently large n. Now use density of $C_{\mathrm{c}}(X)$ in $C_0(X)$ and an $\frac{\varepsilon}{3}$-argument to get the result. □

Problem 2. Give a "direct" proof of the following part of Theorem 9.6 of Rudin's book: if $f \in L^1$ then $\widehat{f} \in C_0(\mathbb{R})$. That is, prove this first when f is the characteristic function of a bounded interval, use this result and approximation to prove $\widehat{f} \in C_0(\mathbb{R})$ when $f \in C_{\mathrm{c}}(\mathbb{R})$, and then use approximation to prove $\widehat{f} \in C_0(\mathbb{R})$ for general $f \in L^1$.

You will need $\left|\widehat{f}(t)\right| \leq \|f\|_1$ for all $t \in \mathbb{R}$. This proof takes longer than Rudin's proof, but the methods are useful much more generally, and the first step explains why the result is even true.

Solution. In principle, before doing this problem we are not even supposed to know that $\widehat{f}$ is measurable. The solution is therefore written so as to avoid implicit references to $L^\infty(\mathbb{R})$.

For $a, b \in \mathbb{R}$ with $a < b$, a computation shows that

$$\widehat{\chi_{(a,b]}}(t) = \begin{cases} \frac{i}{\sqrt{2\pi}\cdot t}\left(e^{-ibt} - e^{-iat}\right) & t \neq 0 \\ \frac{1}{\sqrt{2\pi}}(b-a) & t = 0. \end{cases}$$

This function is obviously continuous on $\mathbb{R} \setminus \{0\}$, and it is easy to check (directly from the definition of the derivative of the function $t \mapsto e^{-ibt} - e^{-iat}$, or using L'Hopital's Rule) that it is continuous at 0. Also clearly $\widehat{\chi_{(a,b]}}$ vanishes at infinity.

Now we show that if $f \in C_{\mathrm{c}}(\mathbb{R})$ then $\widehat{f} \in C_0(\mathbb{R})$. Choose $N \in \mathbb{Z}_{>0}$ such that $\operatorname{supp}(f) \subset [-N, N]$. For $n \in \mathbb{Z}_{>0}$ define

$$\delta_n = \sup\left(\left\{|f(x) - f(y)| \colon x, y \in \mathbb{R} \text{ and } |x - y \leq \tfrac{1}{n}\right\},\right.$$

and further define $I_{n,k} = \left(\frac{k}{n}, \frac{k+1}{n}\right]$ for $k = -nN, -nN+1, \ldots, nN-1$. Then define

$$f_n = \sum_{k=-nN}^{nN-1} f\left(\tfrac{k}{n}\right)\chi_{I_{n,k}}.$$

Then

$$\sup\left(\left\{|f_n(x) - f(x)| \colon x \in \mathbb{R}\right\}\right) \le \delta,$$

so

$$\|f_n - f\|_1 \le \int_{-N}^{N} |f_n(x) - f(x)|\, d\overline{m}(x) \le \frac{2N\delta_n}{\sqrt{2\pi}}.$$

We have $\lim_{n\to\infty} \delta_n = 0$ because f is uniformly continuous. So $\lim_{n\to\infty} \|f_n - f\|_1 = 0$. Therefore

$$\lim_{n\to\infty} \sup\left(\left\{\left|\widehat{f_n}(t) - \widehat{f}(t)\right| \colon t \in \mathbb{R}\right\}\right) = 0,$$

that is, $\widehat{f_n} \to \widehat{f}$ uniformly. For $n \in \mathbb{Z}_{>0}$, f_n is a linear combination of characteristic functions of intervals of the form $(a, b]$, so linearity of the Fourier transform implies that $\widehat{f_n} \in C_0(\mathbb{R})$. Therefore $\widehat{f} \in C_0(\mathbb{R})$.

Now let $f \in L^1(\mathbb{R})$ be arbitrary. Choose a sequence $(f_n)_{n\in\mathbb{Z}_{>0}}$ in $C_{\mathrm{c}}(\mathbb{R})$ such that $\lim_{n\to\infty} \|f_n - f\|_1 = 0$. Then

$$\lim_{n\to\infty} \sup\left(\left\{\left|\widehat{f_n}(t) - \widehat{f}(t)\right| \colon t \in \mathbb{R}\right\}\right) = 0,$$

that is, $\widehat{f_n} \to \widehat{f}$ uniformly. We have seen that $\widehat{f_n} \in C_0(\mathbb{R})$ for all $n \in \mathbb{Z}_{>0}$, so $\widehat{f} \in C_0(\mathbb{R})$. □

Problem 3 counts as two ordinary problems.

Problem 3 (Rudin, Chapter 9, Problem 13abc)**.** For $c \in (0, \infty)$ define $f_c \colon \mathbb{R} \to \mathbb{C}$ by $f_c(x) = \exp(-cx^2)$ for $x \in \mathbb{R}$.

(1) Compute $\widehat{f_c}$.
(2) Prove that there exists a unique $c \in (0, \infty)$ such that $\widehat{f_c} = f_c$.
(3) Let $a, b \in (0, \infty)$. Prove that there exist γ and c such that $f_a * f_b = \gamma f_c$, and find explicit formulas for γ and c in terms of a and b.

You may take as known the result that

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}. \tag{1}$$

(This is proved by writing the square of the integral as $\int_{\mathbb{R}^2} e^{-x^2-y^2}\, dx dy$ and computing it using polar coordinates in $\mathbb{R}^2$.)

Hint for part (1). One method (not the only possible method) is to let $g = \widehat{f_c}$, and use integration by parts to get $2cg'(t) + tg(t) = 0$ for all $t \in \mathbb{R}$. If you use this method, you will need to prove (directly, or by citing theorems) that this equation, together with other information you have, determines g uniquely.

Remark 2. A change of variables in (1) shows that f_c actually is in L^1 for all $c \in (0, \infty)$.

Solution to (1). We have

$$g(t) = \widehat{f}_c(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itx} \exp(-cx^2)\, dx.$$

For every t, the imaginary part of the integrand, $x \mapsto \sin(-tx)\exp(-cx^2)$, is an odd function. Therefore $g(t)$ is real for all t. Now, one checks directly that $g(-t) = g(t)$ for all t. Also, using a change of variables,

$$g(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-(c^{1/2}x)^2)\, dx = \frac{1}{\sqrt{2\pi c}} \int_{-\infty}^{\infty} \exp(-y^2)\, dy = \frac{1}{\sqrt{2c}}.$$

According to Theorem 9.2(f) of Rudin, we have

$$g'(t) = \widehat{f}_c(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itx}(-ix\exp(-cx^2))\, dx.$$

Integrating the right hand side by parts, we get

$$g'(t) = \lim_{a\to\infty} \frac{1}{\sqrt{2\pi}} \left(e^{-ita}\left(\tfrac{i}{2c}\right)\exp(-ca^2) - e^{-it(-a)}\left(\tfrac{i}{2c}\right)\exp(-c(-a)^2)\right)$$
$$- \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-ite^{-itx})\left(\tfrac{i}{2c}\right)\exp(-cx^2)\, dx.$$

The limit is zero, and rearranging the other term we get

$$g'(t) = -\tfrac{t}{2c} g(t).$$

This is valid for all t.

Let $\alpha = \inf\big(\{t > 0\colon g(t) = 0\}\big)$, which might possibly be ∞. For $s \in [0, \alpha)$ we have

$$\frac{g'(s)}{g(s)} = -\frac{s}{2c}.$$

Let $t \in [0, \alpha)$, and integrate the equation above from 0 to t. Thus, there is a constant r such that

$$\log(g(t)) = -\frac{t^2}{4c} + r$$

for all $t \in [0, \alpha)$.

If $\alpha < \infty$, then

$$\lim_{t\to\alpha^-} \log(g(t)) = -\infty \quad \text{and} \quad \lim_{t\to\alpha^-}\left(-\frac{t^2}{4c} + r\right) = -\frac{\alpha^2}{4c} + r \neq -\infty.$$

This contradiction shows that $\alpha = \infty$, so $\log(g(t)) = -\frac{t^2}{4c} + r$ for all $t \geq 0$. Therefore $g(t) = \exp(r)\exp\left(-\frac{t^2}{4c}\right)$ for all $t \geq 0$. We already know $g(0) = \frac{1}{\sqrt{2c}}$ and that g is an even function, so it follows that

$$g(t) = \frac{1}{\sqrt{2c}} \exp\left(-\frac{t^2}{4c}\right)$$

for all $t \in \mathbb{R}$. □

Remark 3. In the past, some people have proceeded as follows. After some computation, they obtained

$$\widehat{f}_c(t) = \frac{e^{-t^2/(4c)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-c\left(x + \tfrac{it}{2c}\right)^2\right)\, dx.$$

Then they wanted to make the change of variable $y = x + \frac{it}{2c}$.

This procedure is not legitimate, because it is not covered by any change of variables theorem we have seen. Thus, even if correct, it is not allowed in the proof. In fact, in general it is not even correct. Set

$$h(x) = \frac{e^{-(x-i)^2}}{x - i}$$

for $x \in \mathbb{C} \setminus \{i\}$. It is possible to show (using methods from Chapter 10 of Rudin) that $\int_{-\infty}^{\infty} h(x)\, dx = i\pi$. If a change of variables as above were legitimate, one could change x to $x + 2i$, getting

$$\int_{-\infty}^{\infty} \frac{e^{-(x+i)^2}}{x + i}\, dx.$$

Again using methods from Chapter 10 of Rudin, this integral, however, is equal to $-i\pi$.

Remark 4. One can also finish the proof using the theorem on local existence and uniqueness of solutions of differential equations of the form $y'(x) = \varphi(x, y(x))$ when φ satisfies a suitable Lipschitz condition. We outline the method.

In our case, the differential equation is

$$y'(t) = -\frac{t}{2c} \cdot y(t) \quad \text{and} \quad y(0) = \frac{1}{\sqrt{2c}},$$

so

$$\varphi(x, y) = \frac{xy}{2c}.$$

It is easy to check that the required Lipschitz condition is satisfied. Moreover, the function

$$h(t) = \frac{1}{\sqrt{2c}} \exp\left(-\frac{t^2}{4c}\right)$$

satisfies the equation for all $t \in \mathbb{R}$ and $h(0)$ has the correct value.

We claim that $h(t) = g(t)$ for all $t \in \mathbb{R}$. We consider only $t > 0$; the proof for $t < 0$ is similar. Suppose not. Let $\alpha = \inf\left(\{t > 0 \colon g(t) \neq h(t)\}\right)$. By continuity, we have $g(\alpha) = h(\alpha)$. Apply the existence and uniqueness theorem with the initial condition $y(\alpha) = g(\alpha)$. The conclusion is that there is $\delta > 0$ such that the equation

$$y'(t) = \varphi(t, y(t)) \quad \text{and} \quad y(\alpha) = g(\alpha)$$

has a unique solution on the interval $(\alpha - \delta,\, \alpha + \delta)$. Since the restrictions of both g and h to this interval satisfy the equation, we get $g(t) = h(t)$ for all $t \in (\alpha - \delta,\, \alpha + \delta)$, contradicting the definition of α.

Remark 5. There are at least two other ways to calculate $\widehat{f}_c$, but both depend on the methods of Chapter 10. We give outlines.

The first step in both is to prove that the formula

$$g(z) = \widehat{f}_c(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-izx} \exp(-cx^2)\, dx$$

defines a function which is holomorphic on the entire complex plane. We also require the formula

$$g(z) = \frac{e^{-z^2/(4c)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-c\left(x + \tfrac{iz}{2c}\right)^2\right)\, dx,$$

obtained by combining the exponentials and completing the square in the exponent.

Method 1: For $b \in \mathbb{R}$, we calculate

$$g(ib) = \frac{e^{b^2/(4c)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-c\left(x - \tfrac{b}{2c}\right)^2\right) dx.$$

In this expression, the change of variables $y = x - \frac{b}{2c}$ is legitimate, and gives $g(ib) = e^{b^2/(4c)} g(0)$. Substituting the value of $g(0)$ found in the solution above, we find that

$$g(z) = \frac{1}{\sqrt{2c}} \exp\left(-\frac{z^2}{4c}\right)$$

for all $z \in i\mathbb{R}$. Since $i\mathbb{R}$ is a subset of $\mathbb{C}$ which contains a limit point of itself, and since both sides of this equation are holomorphic functions of z, the corollary to Theorem 10.18 of Rudin shows that the formula for $g(z)$ holds for all $z \in \mathbb{C}$. Take z real to get $\widehat{f_c}$.

Method 2: We justify, in this case, the change of variables in Remark 3. Take $t > 0$; the case $t < 0$ is done the same way, or follows from the fact that $\widehat{f_c}$ is an even function. For $r > 0$ let γ_r be the piecewise C^1 closed curve consisting of four straight line segments: from $-r$ to r to $r + \frac{it}{2c}$ to $-r + \frac{it}{2c}$ and back to $-r$. Since g is entire, we have $\int_{\gamma_r} g(z)\, dz = 0$ by Cauchy's Theorem. Now explicitly write $\int_{\gamma_r} g(z)\, dz$ as the sum of path integrals along the four straight line segments, and let $r \to \infty$. One can easily prove that the integrals along the two vertical line segments approach zero. The integral along the line segment from $-r$ to r approaches

$$\frac{e^{-t^2/(4c)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-cx^2\right) dx,$$

and the integral the line segment from $r + \frac{it}{2c}$ to $-r + \frac{it}{2c}$ approaches

$$-\frac{e^{-t^2/(4c)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-c\left(x + \tfrac{it}{2c}\right)^2\right) dx.$$

Since the limit of the entire sum is zero, it follows that

$$\frac{e^{-t^2/(4c)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-c\left(x + \tfrac{it}{2c}\right)^2\right) dx = \frac{e^{-t^2/(4c)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-cx^2\right) dx,$$

as desired.

This method does not apply to the counterexample in Remark 3, because if one tries it one finds that the function has a pole inside the relevant closed path. (This is how I constructed the example.) One could also imagine an example in which the change of variables fails because the integrals along the vertical segments don't converge to zero, but such an example seems tricky to construct.

Solution to (2). Part (1) gave

$$\widehat{f_c}(t) = \frac{1}{\sqrt{2c}} e^{-t^2/(4c)}.$$

For $c = \frac{1}{2}$ one checks directly that $\widehat{f_c} = f_c$. On the other hand, if $\widehat{f_c} = f_c$ then

$$1 = f_c(0) = \widehat{f_c}(0) = \frac{1}{\sqrt{2c}},$$

whence $c = \frac{1}{2}$. $\square$

Solution to (3). Using Part (1), we write

$$(f_a * f_b)^\wedge(t) = \widehat{f}_a(t)\widehat{f}_b(t) = \left(\frac{1}{\sqrt{2a}}e^{-t^2/(4a)}\right)\left(\frac{1}{\sqrt{2b}}e^{-t^2/(4b)}\right)$$
$$= \frac{1}{\sqrt{4ab}}\exp\left(-\frac{t^2}{2}\left(\frac{1}{a}+\frac{1}{b}\right)\right).$$

Set

$$\gamma = \frac{1}{\sqrt{2(a+b)}} \quad \text{and} \quad c = \frac{ab}{a+b}.$$

A direct substitution in the result of Part (1) gives $\gamma\widehat{f}_c(t) = (f_a * f_b)^\wedge(t)$ for all t. It follows from injectivity of the Fourier transform that $f_a * f_b = \gamma f_c$ as elements of L^1, so that

$$(f_a * f_b)(x) = \gamma f_c(x) \tag{2}$$

for almost all $x \in \mathbb{R}$.

We finish by proving that (2) actually holds for all $x \in \mathbb{R}$. Since the right hand side is a continuous function of x, and since the measure of any nonempty open set is nonzero, it suffices to show that the left hand side of (2) is defined for all $x \in \mathbb{R}$, and is a continuous function of x.

We saw in Remark 2 that $f_d \in L^1$ for all $d \in (0,\infty)$. (This is also easy to prove directly without actually evaluating the integral.) Since f_d is bounded, it follows that $f_d \in L^2$. For $\alpha \in \mathbb{R}$, define $\tau_\alpha \colon L^2 \to L^2$ by $\tau_\alpha(x)(x) = f(x-\alpha)$ for $x \in \mathbb{R}$ and $f \in L^2$, and define $\sigma \colon L^2 \to L^2$ by $\sigma(f)(x) = f(-x)$ for $x \in \mathbb{R}$ and $f \in L^2$. Then, since f_b is real, we can write

$$(f_a * f_b)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_a(x-y)f_b(y)\,dy = \langle \sigma(\tau_x(f_a)),\, f_b\rangle.$$

So $(f_a * f_b)(x)$ is defined for all $x \in \mathbb{R}$. Theorem 9.5 of Rudin implies that $y \mapsto \tau_y(f_a)$ is continuous from $\mathbb{R}$ to $L^2(\mathbb{R})$, and σ is continuous, so we also conclude that $f_a * f_b$ is continuous. □

Alternate solution. In the following computation, we complete the square at the second step (algebra omitted), change the variable y to $y - \frac{ax}{a+b}$ at the third step, and change y to $y\sqrt{a+b}$ and use $\int_{-\infty}^{\infty} e^{-y^2}\,dy = \sqrt{\pi}$ at the fourth step:

$$(f_a * f_b)(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-a(y-x)^2}e^{-by^2}\,dy$$
$$= \frac{1}{\sqrt{2\pi}}\exp\left(\tfrac{abx^2}{a+b}\right)\int_{-\infty}^{\infty}\exp\left(-(a+b)\left(y-\tfrac{ax}{a+b}\right)^2\right)dy$$
$$= \frac{1}{\sqrt{2\pi}}\exp\left(-\tfrac{abx^2}{a+b}\right)\int_{-\infty}^{\infty} e^{-(a+b)y^2}\,dy$$
$$= \frac{1}{\sqrt{2\pi}}\exp\left(\tfrac{abx^2}{a+b}\right)\sqrt{\tfrac{\pi}{a+b}}$$
$$= \frac{1}{\sqrt{2(a+b)}}\exp\left(-\tfrac{abx^2}{a+b}\right).$$

The last expression is $\gamma f_c(x)$ with

$$\gamma = \frac{1}{\sqrt{2(a+b)}} \quad \text{and} \quad c = \frac{ab}{a+b}.$$

This completes the proof. □

Problem 4 (Rudin, Chapter 10, Problem 1)**.** Let (X, ρ) be a metric space, let $K \subset X$ be compact, and let $E \subset X$ be closed. Suppose $K \cap E = \varnothing$. Prove that there is $\delta > 0$ such that $\rho(x, y) \geq \delta$ for all $x \in K$ and $y \in E$.

Show by example that the conclusion fails if K is only assumed closed, even with $X = \mathbb{C}$ and its usual metric.

(The example isn't part of Rudin's problem.)

The positive statement will be frequently used with $X = \mathbb{C}$.

Solution for the positive statement. For $x \in X$ define $d(x) = \inf_{y \in E} \rho(x, y)$, which is the distance from x to E.

We claim that for $x_1, x_2 \in X$ we have $|d(x_1) - d(x_2)| \leq \rho(x_1, x_2)$. To prove the claim, first, for $y \in E$ we have $\rho(x_1, y) \leq \rho(x_2, y) + \rho(x_1, x_2)$. Taking the infimum over $y \in E$, we get $d(x_1) \leq (x_2) + \rho(x_1, x_2)$. Exchanging x_1 and x_2 and using symmetry of ρ, we get $d(x_2) \leq (x_1) + \rho(x_1, x_2)$. The claim follows.

The claim immediately implies that that d is continuous. Also, if $d(x) = 0$, then clearly $x \in \overline{E}$. Since E is closed, this means $x \in E$. Therefore $d(x) > 0$ for all $x \in K$. Since K is compact, it follows that $\inf_{x \in K} d(x) > 0$, as desired. □

Alternate solution for the positive statement. Suppose the conclusion is false. For $n \in \mathbb{Z}_{>0}$ choose $x_n \in K$ and $y_n \in E$ such that $\rho(x_n, y_n) < \frac{1}{n}$. Since K is compact, there is a subsequence $(x_{k(n)})_{n \in \mathbb{Z}_{>0}}$ of $(x_n)_{n \in \mathbb{Z}_{>0}}$ which converges to some $x \in K$. Now

$$\rho(y_n, x) \leq \rho(y_n, x_n) + \rho(x_n, x) < \frac{1}{n} + \rho(x_n, x),$$

so $\lim_{n \to \infty} y_n = x$. We have $x \in E$ since E is closed. This contradicts the assumption $K \cap E = \varnothing$. □

Second alternate solution for the positive statement. Let $d \colon X \to [0, \infty)$ be as in the first solution.

We claim that for $\delta > 0$, the set $U_\delta = \big\{ x \in X \colon d(x) > \delta \big\}$ is open. To prove the claim, let $x \in U_\delta$. Set $\varepsilon = \frac{1}{2}(d(x) - \delta)$. Suppose $\rho(x, y) < \varepsilon$. Then for all $z \in E$ we have

$$\rho(y, z) \geq \rho(x, z) - \rho(x, y) > d(x) - \varepsilon = d(x) - \frac{1}{2}\big(d(x) - \delta\big) = \frac{1}{2}\big(d(x) + \delta\big).$$

Therefore

$$d(y) = \inf_{z \in E} \rho(x, z) \geq \frac{1}{2}\big(d(x) + \delta\big) > \delta,$$

so that $y \in U_\delta$. The claim is proved.

We have $U_{\delta_1} \subset U_{\delta_2}$ whenever $\delta_1 \geq \delta_2$, and $K \subset \bigcup_{\delta > 0} U_\delta$. Since K is compact, it follows that there is $\delta > 0$ such that $K \subset U_\delta$. □

Counterexample when K is not compact. We write the example in terms of $\mathbb{R}^2$ instead of $\mathbb{C}$. Set

$$K = \mathbb{R} \times \{0\} \subset \mathbb{R}^2 \quad \text{and} \quad E = \big\{ (x, y) \in \mathbb{R}^2 \colon xy = 1 \big\}.$$

It is obvious that K is closed and $K \cap E = \varnothing$. Also, E is closed because the function $f \colon \mathbb{R}^2 \to \mathbb{R}$, given by $f(x, y) = xy$, is continuous, and $E = f^{-1}(\{1\})$.

For $n \in \mathbb{Z}_{>0}$, we have

$$(n, 0) \in K, \qquad \big(n, \tfrac{1}{n}\big) \in E, \quad \text{and} \quad \rho\big((n, 0), \, \big(n, \tfrac{1}{n}\big)\big) = \tfrac{1}{n}.$$

Therefore $\inf_{x \in K, \, y \in E} \rho(x, y) = 0$. □

Of course, many other examples are possible.