MATH 618 (SPRING 2025, PHILLIPS): HOMEWORK 2

Conventions on measures: m is ordinary Lebesgue measure, $\overline{m} = (2\pi)^{-1/2}m$, and in expressions of the form $\int_{\mathbb{R}} f(x)\,dx$, ordinary Lebesgue measure is assumed.

Problem 1 (Rudin, Chapter 9, Problem 4)**.** Give an explicit example of a function $f \in L^2(\mathbb{R})$ such that $f \notin L^1(\mathbb{R})$ but $\widehat{f} \in L^1(\mathbb{R})$. Under what circumstances can this happen?

Problem 2 (Rudin, Chapter 9, Problem 5)**.** Let $f \in L^1(\mathbb{R})$, and suppose that

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left| t\widehat{f}(t) \right| dt$$

is finite. Prove that there exists a function $g \colon \mathbb{R} \to \mathbb{C}$ such that $f(x) = g(x)$ for almost all $x \in \mathbb{R}$ and such that for all $x \in \mathbb{R}$ we have

$$g'(x) = \frac{i}{\sqrt{2\pi}} \int_{\mathbb{R}} t\widehat{f}(t) e^{ixt}\,dt.$$

Problem 3 (Rudin, Chapter 9, Problem 7)**.** Let S be the set of all C^∞ functions $f \colon \mathbb{R} \to \mathbb{C}$ such that for all $m, n \in \mathbb{Z}_{\geq 0}$ we have

$$\sup_{x \in \mathbb{R}} \left| x^n f^{(m)}(x) \right| < \infty. \tag{1}$$

Prove that $f \mapsto \widehat{f}$ is a bijection from S to S. Give examples of nonzero elements of S.

Comments: The space S is a topological vector space with topology given by the seminorms implicit in (1) for $m, n \in \mathbb{Z}_{\geq 0}$, and the map $f \mapsto \widehat{f}$ is a homeomorphism. Also, one gets the same topology with different choices of seminorms. For example, one could use the family of seminorms given by

$$\|f\|_{m,n} = \left(\int_{\mathbb{R}} (1 + x^{2n})^{1/2} |f^{(m)}(x)|^2 \, d\overline{m}(x) \right)^{1/2}$$

for $m, n \in \mathbb{Z}_{\geq 0}$, or an L^p version of these seminorms for any $p \in [1, \infty)$. The reason is that arbitrarily large powers of x appear. For example, if f is continuous and $x \mapsto x^2 f(x)$ is bounded, then f is an L^1 function on $\mathbb{R}$.

Problem 4 (Rudin, Chapter 9, Problem 9)**.** Let $p \in [1, \infty)$, let $f \in L^p(\mathbb{R})$, and define $g \colon \mathbb{R} \to \mathbb{C}$ by

$$g(x) = \int_x^{x+1} f(t)\,dt.$$

Prove that $g \in C_0(\mathbb{R})$. What can you say if $f \in L^\infty(\mathbb{R})$?

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