MATH 618 (SPRING 2025, PHILLIPS): SOLUTIONS TO HOMEWORK 1

Conventions on measures: m is ordinary Lebesgue measure, $\overline{m} = (2\pi)^{-1/2}m$, and in expressions of the form $\int_{\mathbb{R}} f(x)\,dx$, ordinary Lebesgue measure is assumed.

Little proofreading has been done.

Some parts of problems have several different solutions (as many as four).

Problem 1 (Rudin, Chapter 9, Problem 1)**.** Let $f \in L^1(\mathbb{R}, \overline{m})$, and suppose that $f \neq 0$ and $f(x) \geq 0$ for all $x \in \mathbb{R}$. Prove that $\big|\widehat{f}(y)\big| < \widehat{f}(0)$ for all $y \in \mathbb{R} \setminus \{0\}$.

The problem in Rudin is not clearly stated. It is likely to be interpreted as assuming the stronger hypothesis $f(x) > 0$ for all $x \in \mathbb{R}$. The stronger assumption doesn't help with the proof.

Solution. We have

$$\widehat{f}(y) = \int_{\mathbb{R}} e^{-iyx} f(x)\,d\overline{m}(x)$$

for $y \in \mathbb{R}$. When $y = 0$, the integrand is $f(x) \geq 0$, so $\widehat{f}(0) \geq 0$, and the desired inequality at least makes sense. Moreover, for all $y \in \mathbb{R}$, we have

$$\big|\widehat{f}(y)\big| \leq \int_{\mathbb{R}} |e^{-iyx} f(x)|\,d\overline{m}(x) = \int_{\mathbb{R}} f(x)\,d\overline{m}(x) = \widehat{f}(0).$$

We need therefore only prove that the inequality is strict when $y \neq 0$.

Let $y \in \mathbb{R} \setminus \{0\}$. Choose $\theta \in \mathbb{R}$ such that $e^{i\theta}\widehat{f}(y) = \big|\widehat{f}(y)\big|$. Then

$$e^{i\theta}\widehat{f}(y) = \mathrm{Re}\big(e^{i\theta}\widehat{f}(y)\big) = \int_{\mathbb{R}} \mathrm{Re}(e^{i(\theta - yx)}) f(x)\,d\overline{m}(x) = \int_{\mathbb{R}} \cos(\theta - yx) f(x)\,d\overline{m}(x).$$

Therefore

$$\widehat{f}(0) - \big|\widehat{f}(y)\big| = \int_{\mathbb{R}} [1 - \cos(\theta - yx)] f(x)\,d\overline{m}(x).$$

Set

$$E = \{x \in \mathbb{R} \colon f(x) > 0\} \quad \text{and} \quad S = \left\{ \frac{2\pi n + \theta}{y} \colon n \in \mathbb{Z} \right\}.$$

Then $\overline{m}(E) > 0$ and S is countable, so $\overline{m}\big(E \cap (\mathbb{R} \setminus S)\big) > 0$. We have $1 - \cos(\theta - yx) > 0$ for all $x \in \mathbb{R} \setminus S$, so

$$[1 - \cos(\theta - yx)] f(x) > 0$$

for all $x \in E \cap (\mathbb{R} \setminus S)$. Since also $[1 - \cos(\theta - yx)] f(x) \geq 0$ for all $x \in \mathbb{R}$, Theorem 1.39(a) of Rudin now implies that

$$\int_{\mathbb{R}} [1 - \cos(\theta - yx)] f(x)\,d\overline{m}(x) > 0.$$

So $\widehat{f}(0) - \big|\widehat{f}(y)\big| > 0$. □

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Alternate solution. We prove $\left|\widehat{f}(y)\right| \leq \widehat{f}(0)$ as in the first solution.

Now assume that $y \in \mathbb{R} \setminus \{0\}$ and $\left|\widehat{f}(y)\right| = \widehat{f}(0)$. Define $g(x) = e^{-iyx} f(x)$ for $x \in \mathbb{R}$. Then for all $x \in \mathbb{R}$, we have $|g(x)| = f(x)$, so

$$\int_{\mathbb{R}} |g| \, d\overline{m} = \int_{\mathbb{R}} f \, d\overline{m} = \widehat{f}(0) = \left|\widehat{f}(y)\right| = \left|\int_{\mathbb{R}} g \, d\overline{m}\right|.$$

Theorem 1.39(c) of Rudin's book gives a constant α such that $\alpha e^{-iyx} f(x) = f(x)$ for almost all $x \in \mathbb{R}$. Since $y \neq 0$, the set $E = \{x \in \mathbb{R} \colon \alpha e^{-iyx} = 1\}$ is (at most) countable. So $\overline{m}(E) = 0$. The function f vanishes off E, so f is the zero element of $L^1(\mathbb{R}, \overline{m})$. □

I have restated the next problem in labelled parts for convenience. It counts as three ordinary problems.

Problem 2 (Rudin, Chapter 9, Problem 2)**.**

(1) Compute the Fourier transform of the characteristic function of an interval.
(2) For $n \in \mathbb{Z}_{>0}$ let g_n be the characteristic function of $[-n, n]$, and let h be the characteristic function of $[-1, 1]$. Compute $g_n * h$ explicitly. (It is piecewise linear.)
(3) For $x \in \mathbb{R} \setminus \{0\}$ and $n \in \mathbb{Z}_{>0}$, set
$$f_n(x) = \frac{\sin(x)\sin(nx)}{x^2}.$$
Prove that there is a constant c such that $g_n * h$ is the Fourier transform of $c f_n$.
(4) Let f_n be as in part (3). Prove that $\lim_{n\to\infty} \|f_n\|_1 = \infty$.
(5) Conclude that $\left\{\widehat{f} \colon f \in L^1(\mathbb{R})\right\}$ is a *proper* subset of $C_0(\mathbb{R})$.
(6) Prove that $\left\{\widehat{f} \colon f \in L^1(\mathbb{R})\right\}$ is dense $C_0(\mathbb{R})$.

Solution to part (1). This is just a computation. (Reminder: we are using $\overline{m} = \left(\frac{1}{\sqrt{2\pi}}\right) m$ in the definition of convolution as well as in the definition of the Fourier transform.) The result is

$$\widehat{\chi_{[a,b]}}(t) = \begin{cases} \frac{i}{\sqrt{2\pi}\cdot t}\left(e^{-ibt} - e^{-iat}\right) & t \neq 0 \\ \left(\frac{1}{\sqrt{2\pi}}\right)(b-a) & t = 0. \end{cases}$$

Of course, $\widehat{\chi_{[a,b)}}$, $\widehat{\chi_{(a,b]}}$, and $\widehat{\chi_{(a,b)}}$ are the same. □

Solution to (2). This is also a computation.

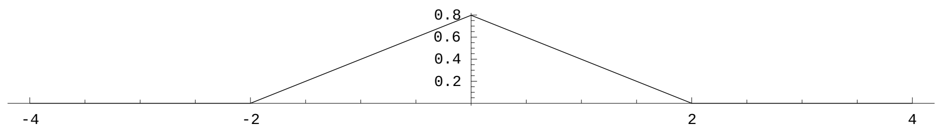
Set $p_n(x) = m([x-n, \, x+n] \cap [-1, \, 1])$, which is given by the formula

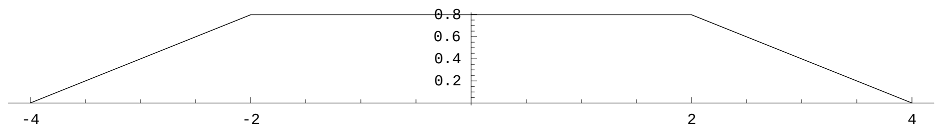
$$p_n(x) = \begin{cases} 0 & x \leq -n-1 \\ x+n+1 & -n-1 \leq x \leq -n+1 \\ 2 & -n+1 \leq x \leq n-1 \\ -x+n+1 & n-1 \leq x \leq n+1 \\ 0 & n+1 \leq x. \end{cases}$$

(For $x \in [-n-1, \, n+1]$, but not for x not in this interval, the formula can be written as $\min(1, \, x+n) - \max(-1, \, x-n)$.)

The outcome of the computation is $(g_n * h) = \frac{1}{\sqrt{2\pi}} p_n(x)$.

Here are graphs for $n = 1$ and for $n = 3$:





(These are not required as part of the solution.) ☐

Solution to part (3). For $t \neq 0$, Theorem 9.2(c) of Rudin gives the first step in the following calculation, and part (a) of this problem gives the second step:

$$\widehat{(g_n * h)}(t) = \widehat{g_n}(t)\widehat{h}(t) = \left(\frac{i}{\sqrt{2\pi} \cdot t}\left(e^{-int} - e^{int}\right)\right)\left(\frac{i}{\sqrt{2\pi} \cdot t}\left(e^{-it} - e^{it}\right)\right).$$

Using the identity $\sin(\theta) = \left(e^{i\theta} - e^{-i\theta}\right)/(2i)$, one can rewrite the last expression as $\frac{2}{\pi} f_n(t)$.

Unfortunately, this isn't quite what we want. We will get what we do want using the Fourier inversion theorem and a bit of trickery.

Clearly $g_n, h \in L^1(\mathbb{R})$, so $g_n * h \in L^1(\mathbb{R})$.

We next show that $f_n \in L^1(\mathbb{R})$. Define

$$b_n(x) = \begin{cases} x^{-2} & |x| \geq \frac{1}{\sqrt{n}} \\ n & |x| < \frac{1}{\sqrt{n}}. \end{cases}$$

It is clear that $|f_n(x)| \leq x^{-2}$ for all $x \neq 0$. Also, writing

$$f_n(x) = n\left(\frac{\sin(x)}{x}\right)\left(\frac{\sin(nx)}{nx}\right)$$

and using $|\sin(y)| \leq |y|$ for all $y \in \mathbb{R}$, we get $|f_n(x)| \leq n$ for all $x \neq 0$. We may as well take $f_n(0) = n$. (This makes f_n continuous at 0.) Then the inequalities above hold for all x, and imply that $|f_n(x)| \leq b_n(x)$ for all x. Clearly $b_n \in L^1(\mathbb{R})$, so $f_n \in L^1(\mathbb{R})$.

The Fourier Inversion Theorem therefore gives the first step in the following calculation. At the second step, we use the fact that f_n is an even function, and at the third step we change variables, replacing t with $-t$. We get:

$$\begin{aligned}(g_n * h)(x) &= \left(\frac{2}{\pi}\right)\left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty} e^{itx} f_n(t)\, dt \\ &= \left(\frac{2}{\pi}\right)\left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty} e^{itx} f_n(-t)\, dt \\ &= \left(\frac{2}{\pi}\right)\left(\frac{1}{\sqrt{2\pi}}\right)\int_{-\infty}^{\infty} e^{-itx} f_n(t)\, dt = \left(\frac{2}{\pi}\right)\widehat{f_n}(x).\end{aligned}$$

This is what is wanted. ☐

Alternate solution to part (3) (sketch). Directly compute

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{itx}(g_n * h)(t)\, dt.$$

One will have to compute three different integrals, corresponding to three different formulas for $(g_n * h)(t)$ on parts of its domain where it is nonzero. The most complicated term has the form $\int_r^s te^{itx}\,dt$, which can be done by integration by parts. Details are omitted. The result is $\frac{2}{\pi}f_n$.

One checks that $f_n \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, as in first solution. Using Theorem 9.13 of Rudin, it follows that $\frac{2}{\pi}\widehat{f_n} = g_n * h$. $\square$

Second alternate solution to part (3) (sketch). (This solution is not recommended.) Imitate the arguments in Chapter 9 of Rudin, but exchanging $\exp(itx)$ and $\exp(-itx)$ everywhere. Going as far as the Fourier inversion theorem, one gets the result that if $f \in L^1(\mathbb{R})$ and the function

$$f^\vee(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} f(x)\,dx$$

is also in $L^1(\mathbb{R})$, then for almost all x we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} f^\vee(x)\,dt.$$

Either use the analog of Theorem 9.2(c) of Rudin, or directly compute, to get $(g_n * h)^\vee = \frac{2}{\pi}f_n$, and apply the formula above to deduce that $\frac{2}{\pi}\widehat{f_n} = g_n * h$. $\square$

Third alternate solution to part (3) (sketch). Use the same method as in the second alternate solution. However, instead of repeating all the work in Chapter 9 of Rudin, deduce the results needed from the ones already there. For the Fourier inversion theorem, this is done as follows. For $f \in L^1(\mathbb{R})$, define $R(f)(x) = f(-x)$. Then $f^\vee$ is by definition $R(\widehat{f})$, so $R(f^\vee) = \widehat{f}$. Since $f \in L^1(\mathbb{R})$ if and only if $R(f) \in L^1(\mathbb{R})$, we see that if f and $f^\vee$ are both in $L^1(R)$, then for almost every x we have, changing the variable from t to $-t$ at the first step,

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-itx} f^\vee(t)\,dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} R(f^\vee)(t)\,dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{itx} \widehat{f}(t)\,dt = f(t).$$

The formula for $(g_n * h)^\vee$ can be obtained in a similar way. $\square$

Solution to part (4). Since $\lim_{x\to 0} x^{-1}\sin(x) = 0$, there exists $r > 0$ such that $x^{-1}\sin(x) > \frac{1}{2}$ whenever $|x| \leq r$.

Let $k \in \mathbb{Z}_{>0}$ satisfy $k\pi/n \leq r$. Then for $x \in [(k-1)\pi/n,\, k\pi/n]$ we have

$$|f_n(x)| \geq \left(\frac{1}{2}\right)\left(\frac{|\sin(nx)|}{k\pi/n}\right) = \frac{n|\sin(nx)|}{2k\pi},$$

whence

$$\int_{(k-1)\pi/n}^{k\pi/n} |f_n(x)|\,dx \geq \frac{n}{2k\pi} \int_{(k-1)\pi/n}^{k\pi/n} |\sin(nx)|\,dx = \frac{1}{k\pi}.$$

Accordingly,

$$\|f_n\|_1 \geq \frac{1}{\pi} \sum_{1\leq k\leq nr/\pi} \frac{1}{k}.$$

Since $r > 0$,

$$\lim_{n\to\infty} \|f_n\|_1 \geq \frac{1}{\pi} \sum_{k=1}^{\infty} \frac{1}{k} = \infty,$$

as desired. $\square$

Solution to part (5). Let $F\colon L^1(\mathbb{R}) \to C_0(\mathbb{R})$ be the Fourier transform. Then F is linear, bounded, and injective. If it were surjective, the Open Mapping Theorem would imply that its inverse F^{-1} would also be bounded. But we have seen that $\|g_n * h\|_\infty = \sqrt{2/\pi}$ for all n, while $\lim_{n\to\infty} \|F^{-1}(g_n * h)\|_1 = \infty$. So F^{-1} can't be bounded, and F is not surjective. □

Solution to part (6). We already know that $C_{\mathrm{c}}(\mathbb{R})$ is dense in $C_0(\mathbb{R})$.

Next, let E_n be the set of all continuous functions l on $\mathbb{R}$ with compact support such that l is linear on every interval $[r/n,\ (r+1)/n]$ with $r \in \mathbb{Z}$. Such a function is clearly determined by its values $l(r/n)$ for $r \in \mathbb{Z}$. Indeed, if we define

$$l_0(x) = \begin{cases} 0 & x \leq -1 \\ x+1 & -1 \leq x \leq 0 \\ -x+1 & 0 \leq x \leq 1 \\ 0 & n+1 \leq x, \end{cases}$$

then

$$l(x) = \sum_{r\in\mathbb{Z}} l(r/n) l_0(nx - r).$$

Note that $l(r/n) \neq 0$ for only finitely many $r \in \mathbb{Z}$.

Using uniform continuity (details omitted), one checks that $\bigcup_{n\in\mathbb{Z}_{>0}} E_n$ is dense in $C_{\mathrm{c}}(\mathbb{R})$. Also, we saw above that $g_1 * h$ is in the range of the Fourier transform. Using the formula for it derived above, and Theorems 9.2(a) and 9.2(e) of Rudin, one sees that the functions $x \mapsto l_0(nx - r)$ are all in the range of the Fourier transform. It follows that $\bigcup_{n\in\mathbb{Z}_{>0}} E_n$ is in the range of the Fourier transform. Therefore the range of the Fourier transform is dense $C_0(\mathbb{R})$. □

Alternate solution to part (6). Set $A = \{\widehat{f}\colon f \in L^1(\mathbb{R})\}$. Then A is a vector subspace of $C_0(\mathbb{R})$. It is closed under complex conjugation, by Theorem 9.2(d) of Rudin. It is closed under multiplication, by Theorem 9.2(c) of Rudin and because $L^1(\mathbb{R})$ is closed under convolution. It separates the points, that is, if $x, y \in \mathbb{R}$, then there is $g \in A$ such that $g(x) \neq g(y)$. Indeed, we can take $g = \widehat{\chi_{[a,b]}}$ for suitable a and b. Using the same functions, we see that for every $x \in \mathbb{R}$ there is $g \in A$ such that $g(x) \neq 0$. The version of the Stone-Weierstrass Theorem for locally compact spaces now implies that A is dense in $C_0(\mathbb{R})$. (To get this from the statement for compact spaces, extend elements of $C_0(\mathbb{R})$ to be continuous functions on the one point compactification of $\mathbb{R}$, and apply the version for compact spaces to the linear span of A and the constant functions.) □

Second alternate solution to (6). (This solution assumes Rudin's Problem 9.7.)

The Schwartz space (the space S of Problem 9.7) is dense in $C_0(\mathbb{R})$, because the set of C^∞ functions with compact support is dense in $C_0(\mathbb{R})$. (This needs proof, but is a standard result from another part of analysis.) Every function in S is the Fourier transform of a function in S, by Problem 9.7, and S is easily seen to be contained in $L^1(\mathbb{R})$. □

Problem 3 (Rudin, Chapter 9, Problem 8)**.** Let $p \in [1, \infty]$, and let $q \in [1, \infty]$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. Prove that if $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, then $f * g$ is uniformly continuous. If $1 < p < \infty$, prove that $f * g \in C_0(\mathbb{R})$. Show by example that $f * g$ need not be in $C_0(\mathbb{R})$ when $p = 1$.

In the solution, it will be convenient to use the notation $\tau_t(f)(x) = f(x-t)$ and $\sigma(f)(x) = f(-x)$ for f in the vector space of all measurable functions from $\mathbb{R}$ to $\mathbb{C}$ and for $x, t \in \mathbb{R}$. The maps τ and σ are linear, and the invariance properties of Lebesgue measure imply that $\|\sigma(f)\|_p = \|f\|_p$ for all $p \in [1, \infty]$ and all $f \in L^p(\mathbb{R})$, and that $\|\tau_t(f)\|_p = \|f\|_p$ for all $p \in [1, \infty]$, all $t \in \mathbb{R}$, and all $f \in L^p(\mathbb{R})$.

Also, recall that in this problem convolution is supposed to be defined using Lebesgue measure as normalized for use with the Fourier transform:

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y) g(y) \, dy.$$

However, exactly the same statements are true without this normalization.

We break the solution into several parts.

Proposition 1. Let $p \in [1, \infty]$, and let $q \in [1, \infty]$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, then $f * g$ is uniformly continuous.

Proof. We assume $p < \infty$. (Otherwise, exchange f and g and use $f * g = g * f$.) Let $\omega \colon L^q(\mathbb{R}) \to \mathbb{C}$ be the continuous linear functional given by

$$\omega(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f g \, dm.$$

Then

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y) g(y) \, dy = \omega(\sigma(\tau_x(f))).$$

We know that $x \mapsto \tau_x(f)$ is uniformly continuous by Theorem 9.5 of Rudin's book, and σ and ω are uniformly continuous because they are continuous and linear. So $f * g$ is the composite of uniformly continuous functions and hence uniformly continuous, as desired. $\square$

Alternate proof of Proposition 1. We assume $p < \infty$. (Otherwise, exchange f and g and use $f * g = g * f$.)

We first claim that for $t \in \mathbb{R}$ we have $\tau_t(f * g) = \tau_t(f) * g$. To prove this, let $x \in \mathbb{R}$. Then

$$\begin{aligned} \tau_t(f * g)(x) = (f * g)(x-t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-t-y) g(y) \, dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tau_t(f)(x-y) g(y) \, dy = [\tau_t(f) * g](x). \end{aligned}$$

The claim is proved.

Next we claim that for every $h \in L^p(\mathbb{R})$, $k \in L^q(\mathbb{R})$, and $x \in \mathbb{R}$, we have $|(h*k)(x)| \leq \|h\|_p \|k\|_q$. This is essentially immediate from Hölder's inequality; the calculation is carried out in the proof of Lemma 2. (This part of the proof doesn't use Proposition 1.)

Now let $\varepsilon > 0$. Use Theorem 9.5 of Rudin to choose $\delta > 0$ such that for all $t \in \mathbb{R}$ with $|t| < \delta$, we have $\|\tau_t(f) - f\|_p < \varepsilon/(\|g\|_q + 1)$. For $|t| < \delta$ and $x \in \mathbb{R}$, we then have, using the first claim at the second step and the second claim at the third step,

$$\begin{aligned} |(f * g)(x-t) - (f * g)(x)| = |\tau_t(f * g)(x) - (f * g)(x)| &= |([\tau_t(f) - f] * g)(x)| \\ &\leq \|\tau_t(f) - f\|_p \|g\|_q < \left(\frac{\varepsilon}{\|g\|_q + 1} \right) \|g\|_q < \varepsilon. \end{aligned}$$

This calculation shows that $f * g$ is uniformly continuous. $\square$

The following proof has also been used. In effect, it incorporates the proof of Theorem 9.5 of Rudin's book.

Second alternate proof of Proposition 1. We assume $p < \infty$. (Otherwise, exchange f and g and use $f * g = g * f$.)

We first claim that for every $h \in L^p(\mathbb{R})$, $k \in L^q(\mathbb{R})$, and $x \in \mathbb{R}$, we have $|(h * k)(x)| \leq \|h\|_p \|k\|_q$. This is essentially immediate from Hölder's inequality; the calculation is carried out in the proof of Lemma 2. (This part of the proof doesn't use Proposition 1.)

Now let $\varepsilon > 0$. Choose $f_0 \in C_{\mathrm{c}}(\mathbb{R})$ such that

$$\|f_0 - f\|_p < \frac{\varepsilon}{4(\|g\|_g + 1)}.$$

Choose $r \geq 0$ such that $\operatorname{supp}(f_0) \subset [-r, r]$. Define

$$\rho = \frac{\varepsilon}{2\|g\|_q + 1} \left(\frac{\sqrt{2\pi}}{2r + 1} \right)^{1/p}.$$

Since f_0 is uniformly continuous, we can choose $\delta > 0$ such that whenever $x_1, x_2 \in \mathbb{R}$ satisfy $|x_1 - x_2| < \delta$, then $|f_0(x_1) - f_0(x_2)| < \rho$.

Now suppose that $x_1, x_2 \in \mathbb{R}$ satisfy $|x_1 - x_2| < \min(1, \delta)$; we prove that

$$|(f * g)(x_1) - (f * g)(x_2)| < \varepsilon.$$

Without loss of generality $x_1 \leq x_2$. Define functions $h_1, h_2 \colon \mathbb{R} \to \mathbb{C}$ by $h_1(y) = f_0(x_1 - y)$ and $h_2(y) = f_0(x_2 - y)$ for $y \in \mathbb{R}$. (These appear in the integrands when we consider $(f * g)(x_1)$ and $(f * g)(x_2)$.) If $h_j(y) \neq 0$ then $y \in [x_j - r,\, x_j + r]$, so that (since $x_1 \leq x_2$) for every $x \notin [x_1 - r,\, x_2 + r]$ we have $h(x_1) - h(x_2) = 0$. Meanwhile, if $x \in [x_1 - r,\, x_2 + r]$ then $|h_1(x) - h_2(x)| < \rho$. Since $x_2 < x_1 + 1$, it therefore follows that

$$\|h_1 - h_2\|_p^p \leq \rho^p \left(\frac{1}{\sqrt{2\pi}} \right) m([x_1 - r,\, x_2 + r]) < \rho^p \left(\frac{1}{\sqrt{2\pi}} \right) (2r + 1) = \left(\frac{\varepsilon}{2} \right)^p,$$

whence

$$\|h_1 - h_2\|_p < \frac{\varepsilon}{2\|g\|_q + 1}.$$

Now, using the claim at the beginning of the proof several times,

$$\begin{aligned}
&|(f * g)(x_1) - (f * g)(x_2)| \\
&\qquad \leq |([f - f_0] * g)(x_1)| + |([f - f_0] * g)(x_2)| + |(f_0 * g)(x_1) - (f_0 * g)(x_2)| \\
&\qquad \leq 2\|f - f_0\|_p \|g\|_q + \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [f_0(x_1 - y) - f_0(x_2 - y)] g(y)\, dy \right| \\
&\qquad = 2\|f - f_0\|_p \|g\|_q + \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [h_1(x) - h_2(x)] g(y)\, dy \right| \\
&\qquad \leq 2\|f - f_0\|_p \|g\|_q + \|h_1 - h_2\|_p \|g\|_q \\
&\qquad < \left(\frac{2\varepsilon}{4(\|g\|_g + 1)} + \frac{\varepsilon}{2\|g\|_q + 1} \right) \|g\|_q < \varepsilon.
\end{aligned}$$

This completes the proof. $\square$

Lemma 2. Let $p \in [1, \infty]$, and let $q \in [1, \infty]$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, then $f * g \in L^\infty(\mathbb{R})$ and $\|f * g\|_\infty \leq \|f\|_p \|g\|_q$.

Proof. For every $x \in \mathbb{R}$ we have, using Hölder's inequality at the third step,

$$|(f * g)(x)| = \left| \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - y) g(y) \, dy \right| \leq \|(\sigma \circ \tau_x)(f)\|_p \|g\|_q = \|f\|_p \|g\|_q.$$

Therefore $f * g$ is bounded by $\|f\|_p \|g\|_q$. That $f * g$ is measurable follows immediately from Proposition 1. □

Proposition 3. Let $p \in (1, \infty)$, and let $q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, then $f * g \in C_0(\mathbb{R})$.

Proof. Lemma 2 implies that $(f, g) \to f * g$ is a (jointly) continuous map from $L^p(\mathbb{R}) \times L^q(\mathbb{R})$ to $L^\infty(\mathbb{R})$. It follows from Theorem 3.17 of Rudin that $C_0(\mathbb{R})$ is a closed subspace of $L^\infty(\mathbb{R})$. (Complete subsets of metric spaces are necessarily closed.) Therefore is suffices to find dense subsets $S \subset L^p(\mathbb{R})$ and $T \subset L^q(\mathbb{R})$ such that $f * g \in C_0(\mathbb{R})$ whenever $f \in S$ and $g \in T$.

We take $S = T = C_{\mathrm{c}}(\mathbb{R})$. Density follows from Theorem 3.14 of Rudin. Let $f, g \in C_{\mathrm{c}}(\mathbb{R})$. Proposition 1 implies that $f * g$ is continuous. Choose M such that $\mathrm{supp}(f), \mathrm{supp}(g) \subset [-M, M]$. If $|x| > 2M$, then for every $y \in \mathbb{R}$ at least one of y and $x - y$ must be in $\mathbb{R} \setminus [-M, M]$, so $f(x - y) g(y) = 0$. It follows that $(f * g)(x) = 0$. We have shown that $f * g$ has compact support, so $f * g \in C_{\mathrm{c}}(\mathbb{R}) \subset C_0(\mathbb{R})$. □

Alternate proof. We prove directly that $f * g$ vanishes at infinity. Let $\varepsilon > 0$.

Since $|f|^p$ and $|g|^q$ are integrable, there are $M, N \in [0, \infty)$ such that

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R} \setminus [-M, M]} |f(x)|^p \, dx < \left(\frac{\varepsilon}{2\|g\|_q + 1} \right)^p$$

and

$$\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R} \setminus [-N, N]} |g(x)|^q \, dx < \left(\frac{\varepsilon}{2\|f\|_p + 1} \right)^q.$$

We claim that $|(f * g)(x)| < \varepsilon$ whenever $|x| > M + N$.

To prove this, set

$$f_0 = \chi_{[-M, M]} f \quad \text{and} \quad g_0 = \chi_{[-N, N]} g.$$

Then for every $x \in \mathbb{R}$ we have

$$|(f * g)(x)| \leq \big|[f * (g - g_0)](x)\big| + \big|[(f - f_0) * g_0](x)\big| + |(f_0 * g_0)(x)|.$$

Whenever $|x| > M + N$, we have $(f_0 * g_0)(x) = 0$, because in the integrand in its definition is always zero. Using Lemma 2, for every $x \in \mathbb{R}$ we have

$$\begin{aligned} \big|[f * (g - g_0)](x)\big| &\leq \|f\|_p \|g - g_0\|_q \\ &= \|f\|_p \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R} \setminus [-N, N]} |g(x)|^q \, dx \right)^{1/q} \leq \frac{\varepsilon \|f\|_p}{2\|f\|_p + 1} < \frac{\varepsilon}{2} \end{aligned}$$

and (since clearly $\|g_0\|_q \leq \|g\|_q$)

$$\begin{aligned} \big|[(f - f_0) * g](x)\big| &\leq \|f - f_0\|_p \|g_0\|_q \\ &= \|g_0\|_q \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R} \setminus [-M, M]} |f(x)|^p \, dx \right)^{1/p} \leq \frac{\varepsilon \|g_0\|_q}{2\|g\|_q + 1} < \frac{\varepsilon}{2}. \end{aligned}$$

So whenever $|x| > M + N$ we have $|(f * g)(x)| < \varepsilon$. □

Proposition 4. There exists $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$ such that $f * g \notin C_0(\mathbb{R})$.

Proof. Take $f = \chi_{[-1,\,1]}$ and $g = 1$. Then for every $x \in \mathbb{R}$ we have

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)g(y)\,dy = \left(\frac{1}{\sqrt{2\pi}}\right) m([x-1,\, x+1]) = \sqrt{\frac{2}{\pi}}.$$

Thus $f * g$ is a nonzero constant function, so does not vanish at ∞. □