MATH 618 (SPRING 2025): FINAL EXAM SOLUTIONS

For some solutions, essentially no proofreading has been done.

1. (a) (10 points) State Rouché's Theorem.

Solution. Theorem 10.43(b) of Rudin: Let $\Omega \subset \mathbb{C}$ be open, let γ be a closed path in Ω such that $\mathrm{Ind}_\gamma(z) = 0$ for all $z \in \mathbb{C} \setminus \Omega$ and $\mathrm{Ind}_\gamma(z) \in \{0, 1\}$ for all $z \in \mathbb{C} \setminus \mathrm{Ran}(\gamma)$, and let $f, g \colon \Omega \to \mathbb{C}$ be holomorphic functions. Suppose that $|f(z) - g(z)| < |f(z)|$ for all $z \in \mathrm{Ran}(\gamma)$. Then f and g have the same number of zeroes, counting multiplicity, in the set $\big\{ z \in \mathbb{C} \colon \mathrm{Ind}_\gamma(z) = 1 \big\}$. □

The hypothesis $\mathrm{Ind}_\gamma(z) \in \{0, 1\}$ for all $z \in \mathbb{C} \setminus \mathrm{Ran}(\gamma)$ is essential. Also, "counting multiplicity" is essential.

The version stated in class was for a cycle in Ω, which is more general. This version is perfectly acceptable.

It is not required to give the statement of Theorem 10.43(a) of Rudin.

(b) (10 points) State the Residue Theorem.

Solution. Theorem 10.42 of Rudin: Let $\Omega \subset \mathbb{C}$ be open, and let f be a meromorphic function on Ω. Let A be the set of points in Ω at which f has poles. Let Γ be a cycle in $\Omega \setminus A$ such that $\mathrm{Ind}_\Gamma(z) = 0$ for all $z \in \mathbb{C} \setminus \Omega$. Then

$$\frac{1}{2\pi i} \int_\Gamma f(z)\, dz = \sum_{a \in A} \mathrm{Ind}_\Gamma(a) \mathrm{Res}(f; a).$$

□

(c) (10 points) State the Maximum Modulus Theorem.

Solution. Theorem 10.24 of Rudin: Let $\Omega \subset \mathbb{C}$ be a region, and let $f \colon \Omega \to \mathbb{C}$ be holomorphic. Let $a \in \Omega$, and let $r > 0$ satisfy

$$\big\{ z \in \mathbb{C} \colon |z - a| \leq r \big\} \subset \Omega.$$

Then

$$f(a) \leq \sup_{\theta \in [0, 2\pi]} |f(a + re^{i\theta})|.$$

If equality holds, then f is constant. □

This is Rudin's statement. I will also give most of the credit for the more traditional statement (which is formally weaker): Let $\Omega \subset \mathbb{C}$ be a region, and let $f \colon \Omega \to \mathbb{C}$ be holomorphic. If $|f|$ has a local maximum in Ω, then f is constant.

2. (25 points) Let $f, g \in L^1(\mathbb{R})$. Assume that $\mathrm{supp}\big(\widehat{f}\big) \subset (0, \infty)$ and that $\mathrm{supp}\big(\widehat{g}\big) \subset (-\infty, 0)$. Prove that $(f * g)(x) = 0$ for almost every $x \in \mathbb{R}$.

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Solution. Set $h = f * g$. Then $h \in L^1(\mathbb{R})$ and $\widehat{h} = \widehat{f} \cdot \widehat{g}$. This product is zero because $\mathrm{supp}(\widehat{f}) \cap \mathrm{supp}(\widehat{g}) = \varnothing$. Since $\widehat{h} = 0$, the function h must be the zero element of $L^1(\mathbb{R})$. □

3. (40 points) Suppose $f \colon D \to \mathbb{C}$ is a holomorphic function on the open unit disk $D = \{z \in \mathbb{C} \colon |z| < 1\}$. If f is injective on $D \setminus \{0\}$, prove that f is injective on D.

Solution. Suppose f is not injective on D. Then there exists $z_0 \in D \setminus \{0\}$ such that $f(z_0) = f(0)$. Choose disjoint open subsets $V, W \subset D$ such that $0 \in V$ and $z_0 \in W$. Clearly f is not constant, so f is an open mapping, and $f(V)$ and $f(W)$ are open. Now $f(0) \in f(V) \cap f(W)$, and $\{f(0)\}$ is not open, so there is some $w \in f(V) \cap f(W)$ with $w \neq f(0)$. There are $a \in V$ and $b \in W$ such that $f(a) = f(b) = w$. Clearly $a \neq b$ and neither a nor b is zero. So f is not injective on $D \setminus \{0\}$. □

Alternate solution. If f' is constant, then either f is constant or there are $a, b \in \mathbb{C}$ such that $f(z) = az + b$ for all $z \in D$. In either case, the statement of the problem is clear. So we can assume f' is not constant. Replacing f by $f - f(0)$, we can further assume that $f(0) = 0$.

Suppose f is not injective on D. Then there exists $z_0 \in D \setminus \{0\}$ such that $f(z_0) = 0$. Choose r such that $|z_0| < r < 1$. Set $C = \frac{1}{2} \inf_{|z|=r} |f(z)|$. Since f is injective on $D \setminus \{0\}$ and $f(z_0) = 0$, it follows that $f(z) \neq 0$ when $|z| = r$, so $C > 0$. Choose $\delta > 0$ such that $\delta < r$ and such that $|z| < \delta$ implies $|f(z)| < C$. Since f' is not constant, its zeros are isolated, and there is $w \in D$ such that $0 < |w| < \delta$ and $f'(w) \neq 0$. Therefore the function $g(z) = f(z) - f(w)$ has a simple zero at w. Since $w \neq 0$, the injectivity hypothesis implies that g has no other zeros in D.

When $|z| = r$, we have

$$|g(z) - f(z)| = |f(w)| < C < |f(z)|.$$

So Rouché's Theorem implies that f and g have the same number of zeros in $\{z \in \mathbb{C} \colon |z| < r\}$, counting multiplicity. But we saw that g has only one zero, which has multiplicity 1, while f has at least two zeros. This contradiction shows that f is injective on D. □

Second alternate solution (sketch). In the alternate solution, instead of using f' not constant to choose w, choose w arbitrarily and use the fact that if $f - f(w)$ has a zero at w of multiplicity more than 1, then f is not injective on any neighborhood of w. □

4. (30 points) Let f be a bounded holomorphic function on $\mathbb{C} \setminus \{i, -i\}$. Prove that f is constant.

Solution. The function f is bounded on a neighborhood of i, so has a removable singularity at i. Similarly, f has a removable singularity at $-i$. Thus there exists an entire function g such that $g|_{\mathbb{C} \setminus \{i, -i\}} = f$. Clearly g is bounded. Therefore g is constant, by Liouville's Theorem. It follows that f is constant. □

5. (35 points) Set $\Omega = \{z \in \mathbb{C} \colon \mathrm{Re}(z) > -2\}$. Let f be a holomorphic function on Ω such that $f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right)$ for $n \in \mathbb{Z}_{>0}$. Prove that there exists an entire function g such that $g|_{\Omega} = f$.

Solution. Set $\Omega_0 = \{z \in \mathbb{C} \colon \mathrm{Re}(z) < 2\}$, and define $h \colon \Omega_0 \to \mathbb{C}$ by $h(z) = f(-z)$ for $z \in \Omega_0$. Then h is holomorphic. Moreover, $h|_{\Omega \cap \Omega_0}$ and $f|_{\Omega \cap \Omega_0}$ are two holomorphic functions on $\Omega \cap \Omega_0$ which agree on the set

$$B = \left\{ \tfrac{1}{n} \colon n \in \mathbb{Z}_{>0} \right\}.$$

The set $\Omega \cap \Omega_0$ is a connected open subset of $\mathbb{C}$, and B has a cluster point in Ω, so $h|_{\Omega \cap \Omega_0} = f|_{\Omega \cap \Omega_0}$. Therefore there is a holomorphic function g on $\Omega \cup \Omega_0 = \mathbb{C}$ such that $g|_{\Omega} = f$ and $g|_{\Omega_0} = h$. □

6. (40 points) Let f be an entire function. Suppose that there are constants C and M such that $|f(z)| \leq C + M|z|$ whenever $\mathrm{Im}(z) \geq 0$, and further suppose that $\lim_{r \to \infty} f(rz)$ exists whenever $\mathrm{Im}(z) > 0$. Prove that

$$\lim_{r \to \infty} \int_{-r}^{r} \frac{f(x)}{1 + x^2} \, dx$$

exists.

Solution. For $r > 0$, define paths $\gamma_r \colon [-r, r] \to \mathbb{C}$, $\rho_r \colon [0, \pi] \to \mathbb{C}$ and $\sigma_r \colon [\pi, 2\pi] \to \mathbb{C}$ by $\gamma_r(t) = t$ for $t \in [-r, r]$, $\rho_r(t) = e^{it}$ for $t \in [0, \pi]$, and $\sigma_r(t) = e^{it}$ for $t \in [\pi, 2\pi]$. Then $[\gamma_r] + [\rho_r]$, $[\rho_r] + [\sigma_r]$, and $[\gamma_r] - [\sigma_r]$ are cycles.

The function

$$g(z) = \frac{f(z)}{1 + z^2}$$

is meromorphic on $\mathbb{C}$, with (possibly removable) singularities at i and $-i$. We have $\mathrm{Ind}_{[\gamma_r]+[\rho_r]}(-i) = 0$, because the lower half plane is an unbounded set which contains $-i$ and is disjoint from $\mathrm{Ran}([\gamma_r] + [\rho_r])$. Similarly $\mathrm{Ind}_{[\gamma_r]-[\sigma_r]}(i) = 0$. For $r > 1$, Theorem 10.11 of Rudin implies that $\mathrm{Ind}_{[\rho_r]+[\sigma_r]}(i) = 1$, so

$$\mathrm{Ind}_{[\gamma_r]+[\rho_r]}(i) = \mathrm{Ind}_{[\gamma_r]-[\sigma_r]}(i) + \mathrm{Ind}_{[\rho_r]+[\sigma_r]}(i) = 1.$$

It follows from the Residue Theorem that the function

$$r \mapsto \int_{[\gamma_r]+[\rho_r]} \frac{f(z)}{1 + z^2} \, dz$$

is constant on $(1, \infty)$ (with value $2\pi i \mathrm{Res}(g; i)$). Therefore it suffices to prove that

$$\lim_{r \to \infty} \int_{\rho_r} \frac{f(z)}{1 + z^2} \, dz$$

exists.

We have

$$\int_{\rho_r} \frac{f(z)}{1 + z^2} \, dz = \int_0^{\pi} \frac{f(re^{it}) i r e^{it}}{1 + (re^{it})^2} \, dt.$$

Set

$$h_r(t) = \frac{f(re^{it}) i r e^{it}}{1 + (re^{it})^2}$$

for $t \in [0, \pi]$ and $r \in (1, \infty)$. For $t \in (0, \pi)$, we have $\mathrm{Im}(e^{it}) > 0$. Therefore $\lim_{r \to \infty} f(re^{it})$ exists, and it follows that $\lim_{r \to \infty} h_r(t) = 0$. Thus $h_r(t) \to 0$ pointwise almost everywhere on $[0, \pi]$. Also, for $r > 2$ we have $\left|1 + (re^{it})^2\right| \geq r^2 - 1 > \frac{1}{2} r^2$, so

$$|h_r(t)| < \frac{2|f(re^{it})| \cdot r}{r^2} \leq \frac{2(C + Mr)}{r} \leq C + 2M.$$

Since the constant function $t \mapsto C + 2M$ is integrable on $[0, 2\pi]$, we can apply the Dominated Convergence Theorem to conclude that, for every sequence $(r_n)_{n \in \mathbb{Z}_{>0}}$ in $(2, \infty)$ with $\lim_{n\to\infty} r_n = 0$, we have

$$\lim_{n\to\infty} \int_{\rho_{r_n}} \frac{f(z)}{1+z^2}\, dz = 0.$$

Therefore

$$\lim_{r\to\infty} \int_{\rho_r} \frac{f(z)}{1+z^2}\, dz = 0.$$

This completes the proof. □

One must use sequences in the Dominated Convergence Theorem, since the Dominated Convergence Theorem does not work for more general nets.

One can use what in lecture was called the "path changing lemma" (Theorem 10.37 of Rudin's book) to prove that $\mathrm{Ind}_{[\gamma_r]+[\rho_r]}(i) = 1$, but the method described is simpler.

(There really are functions f satisfying the hypotheses, such as $f(z) = 1$, $f(z) = e^{iz}$, and $f(z) = ze^{iz}$. The function $f(z) = z$ does not satisfy the hypotheses.)

Extra Credit. (30 extra credit points.) Let $\Omega \subset \mathbb{C}$ be a bounded region such that $0 \in \Omega$. Let μ be the restriction to Ω of planar Lebesgue measure on $\mathbb{C} = \mathbb{R}^2$. Prove that, among all holomorphic functions $f\colon \Omega \to \mathbb{C}$ such that $f(0) = 0$ and $|f(z)| < 1$ for all $z \in \Omega$, there is one which maximizes the value of $\int_\Omega |f|\, d\mu$.

Solution. Let F be the set of all holomorphic functions $f\colon \Omega \to \mathbb{C}$ such that $f(0) = 0$ and $|f(z)| < 1$ for all $z \in \Omega$. Then F is uniformly bounded, hence in particular uniformly bounded on every compact set in Ω. Therefore F is a normal family, by Theorem 14.6 of Rudin. Obviously $F \neq \varnothing$, since the zero function is in F.

Set

$$\beta = \sup_{f\in F} \int_\Omega |f|\, d\mu.$$

Then $\beta \le \mu(\Omega) < \infty$. Choose a sequence $(f_n)_{n\in\mathbb{Z}_{>0}}$ in F such that

$$\lim_{n\to\infty} \int_\Omega |f_n|\, d\mu = \beta.$$

Since F is a normal family, by passing to a subsequence, there is a holomorphic function $f\colon \Omega \to \mathbb{C}$ such that $f_n \to f$ uniformly on compact subsets of Ω. Obviously $|f(z)| \le 1$ for all $z \in \Omega$, and $f(0) = 0$. Combining these two facts with the Maximum Modulus Theorem, we get $|f(z)| < 1$ for all $z \in \Omega$. Therefore $f \in F$.

Since $f_n \to f$ pointwise and χ_Ω is integrable, the Dominated Convergence Theorem implies that

$$\lim_{n\to\infty} \int_\Omega |f_n|\, d\mu = \int_\Omega |f|\, d\mu.$$

Therefore $\int_\Omega |f|\, d\mu = \beta$. □