# MATH 618 (SPRING 2025): FINAL EXAM

Instructions: All lemmas, claims, examples, counterexamples, etc. require proof, except when explicitly stated otherwise. If you use a major theorem, be sure to cite it by name.

Closed book. In particular, no notes, books, calculators, cell phones, other electronic devices, or any outside assistance of any kind.

Please do not write anything less than 1/4 inch from any side of any page.

1. (a) (10 points) State Rouché's Theorem.
(b) (10 points) State the Residue Theorem.
(c) (10 points) State the Maximum Modulus Theorem.

2. (25 points) Let $f, g \in L^1(\mathbb{R})$. Assume that $\operatorname{supp}(\widehat{f}) \subset (0, \infty)$ and that $\operatorname{supp}(\widehat{g}) \subset (-\infty, 0)$. Prove that $(f * g)(x) = 0$ for almost every $x \in \mathbb{R}$.

3. (40 points) Suppose $f \colon D \to \mathbb{C}$ is a holomorphic function on the open unit disk $D = \{z \in \mathbb{C} \colon |z| < 1\}$. If $f$ is injective on $D \setminus \{0\}$, prove that $f$ is injective on $D$.

4. (30 points) Let $f$ be a bounded holomorphic function on $\mathbb{C} \setminus \{i, -i\}$. Prove that $f$ is constant.

5. (35 points) Set $\Omega = \{z \in \mathbb{C} \colon \operatorname{Re}(z) > -2\}$. Let $f$ be a holomorphic function on $\Omega$ such that $f\left(\frac{1}{n}\right) = f\left(-\frac{1}{n}\right)$ for $n \in \mathbb{Z}_{>0}$. Prove that there exists an entire function $g$ such that $g|_{\Omega} = f$.

6. (40 points) Let $f$ be an entire function. Suppose that there are constants $C$ and $M$ such that $|f(z)| \leq C + M|z|$ whenever $\operatorname{Im}(z) \geq 0$, and further suppose that $\lim_{r \to \infty} f(rz)$ exists whenever $\operatorname{Im}(z) > 0$. Prove that

$$\lim_{r \to \infty} \int_{-r}^{r} \frac{f(x)}{1 + x^2} \, dx$$

exists.

Extra Credit. (30 extra credit points.) Let $\Omega \subset \mathbb{C}$ be a bounded region such that $0 \in \Omega$. Let $\mu$ be the restriction to $\Omega$ of planar Lebesgue measure on $\mathbb{C} = \mathbb{R}^2$. Prove that, among all holomorphic functions $f \colon \Omega \to \mathbb{C}$ such that $f(0) = 0$ and $|f(z)| < 1$ for all $z \in \Omega$, there is one which maximizes the value of $\int_{\Omega} |f| \, d\mu$.

*Date*: 12 June 2025.