

## MATH 251: SUMMARY OF TOPICS

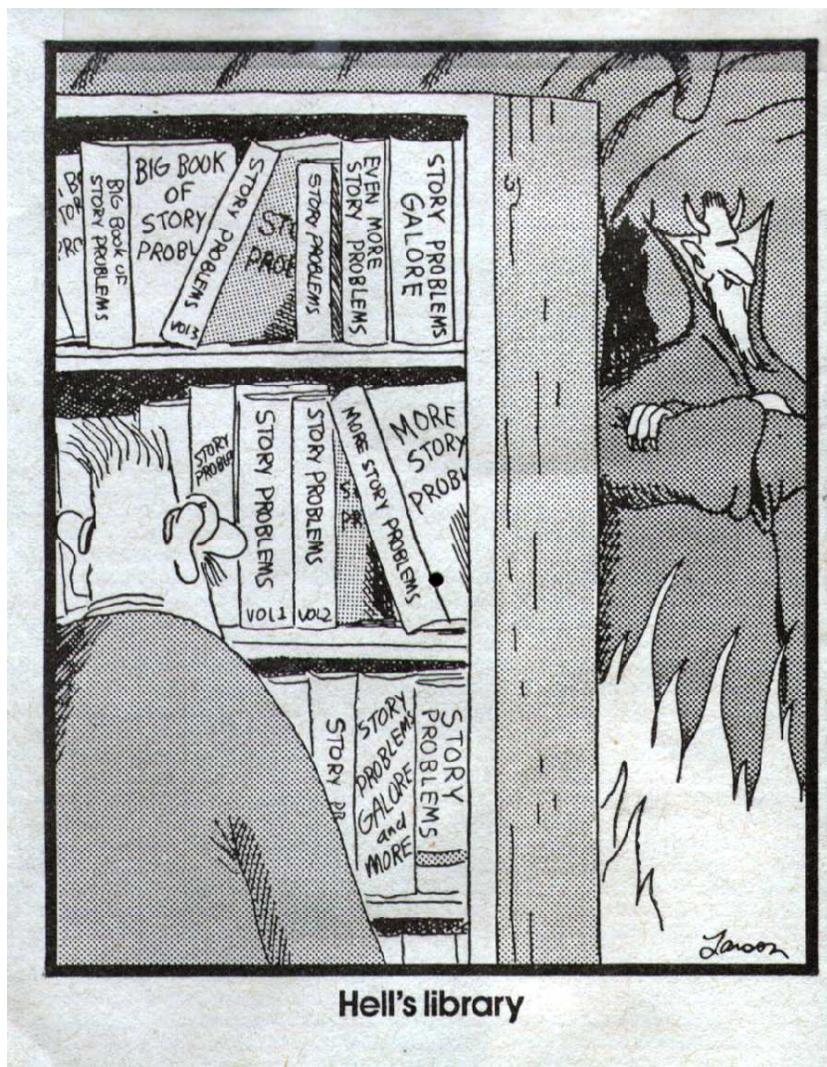
### 1. SUMMARY OF THE MAIN TOPICS

See the separate page about arithmetic you are expected to be able to do without a calculator. Also, remember that notation counts! See the separate notation page.

Here is a list of most of the topics of the course (not necessarily complete).

- (1) Understanding the meaning of the limit, both one sided and two sided, at finite values and at  $\pm\infty$ , and which have finite values, are  $\pm\infty$ , or do not exist and are not even  $\pm\infty$ . Understanding includes recognizing them from graphs and numerical interpretation, for example, how the values of  $f(2.1)$ ,  $f(2.01)$ ,  $f(2.001)$ , etc. and  $f(1.9)$ ,  $f(1.99)$ ,  $f(1.999)$  etc. relate to  $\lim_{x \rightarrow 2} f(x)$ .
- (2) Limit laws, and finding limits of all the kinds above. We have seen many methods for doing this. See Section 2 for more.
- (3) Continuity, in particular, its relation to limits and recognizing continuity from the graph.
- (4) Definition, meaning, and interpretation of the derivative:
  - (a)  $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$ , if the limit exists.
  - (b) Limit of the slopes of secant lines to the graph.
  - (c) Slope of the tangent line to the graph.
  - (d) Recognizing differentiability (existence of a tangent line) from the graph.
  - (e) Instantaneous rate of change. The most common example is velocity, but there are many other rates of change, some of which don't involve time. (Acceleration is included here: it is the rate of change of velocity.)
  - (f) Linear approximation.
- (5) Applications of derivatives. Some examples, not already in (4):
  - (a) Shape of a graph:
    - (i) Critical points.
    - (ii) Local minimums and maximums.
    - (iii) Relation between the sign of the derivative and whether the function is increasing or decreasing.
    - (iv) Relations between the second derivative and concavity.
    - (v) Inflection points.

- (vi) Second derivative test for a local minimum or maximum.
- (b) Finding the maximum or minimum of a function on a closed bounded interval.
- (c) Applied maximization and minimization problems (word problems). Gary Larson cartoon on word problems:



- (d) Related rates (also word problems).
- (e) Using the Mean Value Theorem to get information about the change in a function from information about its derivative.

## 2. ADDITIONAL COMMENTS ON LIMITS

**Warning:** using L'Hopital's Rule when its hypotheses don't apply will result in getting the entire problem wrong. This could cost **half or more of a letter grade** on the final exam.

**Warning:** you will be expected to know that  $\sin(0) = 0$ ,  $\cos(0) = 1$ ,  $\ln(e) = 1$ ,  $a^0 = 1$  when  $a \neq 0$ , etc. **without using a calculator.** Getting these wrong may result in a misidentification of an indeterminate form, and thus getting an entire limit problem wrong, costing **half or more of a letter grade** on the final exam.

We have seen at least seven methods for calculating limits at finite values of the variable.

- (1) Direct substitution: if  $f$  is continuous at  $a$ , then  $\lim_{x \rightarrow a} f(x) = f(a)$ .

**Always** check first if this case applies!

- (2) Cancellation of common factors in a fraction.  
 (3) Rationalization in a ratio containing roots.  
 (4) Identifying vertical asymptotes, for example in a limit of the form  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$  in which  $g(a) = 0$  but  $f(a) \neq 0$ . (Vertical asymptotes also occur elsewhere. For example,  $\lim_{x \rightarrow 0^+} \ln(x) = -\infty$ .)  
 (5) L'Hopital's Rule, but **only** for indeterminate forms!  
 (6) Squeeze Theorem.  
 (7) The limit does not exist because the function oscillates too much.  
 For example,  $\lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$  does not exist. (But  $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = 0$  by the Squeeze Theorem.)

As an example, specifically for fractions, consider  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ . For the purposes of illustration, assume that  $f$  and  $g$  are both continuous at  $a$ . Then there are three possibilities.

- (1)  $g(a) \neq 0$ . Then the function  $q(x) = \frac{f(x)}{g(x)}$  is continuous at  $a$ , so

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} q(x) = q(a) = \frac{f(a)}{g(a)}.$$

This is true even if  $f(a) = 0$ .

- (2)  $g(a) = 0$  but  $f(a) \neq 0$ . Then the function  $q(x) = \frac{f(x)}{g(x)}$  has a vertical asymptote at  $x = a$ . (The limit  $\lim_{x \rightarrow \pi/2} \tan(x)$  is of this type.) This is **not** an indeterminate form, since no finite limit is possible. You expect the one sided limits  $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)}$  and  $\lim_{x \rightarrow a^-} \frac{f(x)}{g(x)}$  to be  $\infty$  or  $-\infty$ .

If they aren't both the same, then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$  does not exist, even as  $\infty$  or  $-\infty$ .

- (3)  $g(a) = 0$  and  $f(a) = 0$ . This is an indeterminate form, and methods like cancelling common factors, rationalization, or L'Hopital's Rule might be useful. You can certainly try L'Hopital's Rule, but **only** in this case..

It may be that instead  $\lim_{x \rightarrow a} f(x) = \pm\infty$ ,  $\lim_{x \rightarrow a} g(x) = \pm\infty$ , or both. Among these cases, only  $\lim_{x \rightarrow a} f(x) = \pm\infty$  **and**  $\lim_{x \rightarrow a} g(x) = \pm\infty$  is an indeterminate form, namely " $\frac{\pm\infty}{\pm\infty}$ ". But some other combinations, for example  $\lim_{x \rightarrow a} f(x) = \pm\infty$  and  $\lim_{x \rightarrow a} g(x) = 0$ , that is, the form " $\frac{\pm\infty}{0}$ ", give vertical asymptotes.

We have seen a similar collection of methods for calculating limits at  $\pm\infty$ . First, here are examples of limits at infinity which we implicitly already knew before this course, for example just by looking at the graphs:

$$\begin{aligned} \lim_{x \rightarrow \infty} x &= \lim_{x \rightarrow \infty} x^2 = \lim_{x \rightarrow \infty} x^3 = \lim_{x \rightarrow \infty} \sqrt{x} = \infty, & \text{etc.} \\ \lim_{x \rightarrow \infty} x &= \lim_{x \rightarrow \infty} x^3 = \lim_{x \rightarrow \infty} x^5 = -\infty, & \text{etc.} \\ \lim_{x \rightarrow \infty} x^2 &= \lim_{x \rightarrow \infty} x^4 = \lim_{x \rightarrow \infty} x^6 = \infty, & \text{etc.} \\ \lim_{x \rightarrow \infty} e^x &= \infty, \quad \lim_{x \rightarrow -\infty} e^x = 0, \quad \lim_{x \rightarrow \infty} \ln(x) = \infty, \quad \lim_{x \rightarrow \infty} \arctan(x) = \frac{\pi}{2}, & \text{etc.} \\ \lim_{x \rightarrow \infty} \frac{1}{x} &= \lim_{x \rightarrow \infty} \frac{1}{x^2} = \lim_{x \rightarrow \infty} \frac{1}{x^3} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{x}} = 0, & \text{etc.} \\ \lim_{x \rightarrow -\infty} \frac{1}{x} &= \lim_{x \rightarrow -\infty} \frac{1}{x^2} = \lim_{x \rightarrow -\infty} \frac{1}{x^3} = 0, & \text{etc.} \end{aligned}$$

Here are some of the methods we have seen.

- (1) Limits of combinations of the functions appearing above gotten by using the limit laws and reasoning with  $\pm\infty$  in ways analogous to the limit laws. (These are the analogs of both direct substitution in continuous functions and some cases of identifying vertical asymptotes.)
- (2) Multiplying both numerator and denominator of a fraction by  $\frac{1}{x^n}$ , or similar things (for example,  $e^{-2x}$ ). This is the analog of cancellation of common factors in a fraction.
- (3) L'Hopital's Rule, but **only** for indeterminate forms!
- (4) Squeeze Theorem.
- (5) The limit does not exist because the function oscillates too much.

For example,  $\lim_{x \rightarrow \infty} \sin(x)$  does not exist. (But  $\lim_{x \rightarrow 0} \frac{\sin(x)}{x} = 0$  by the Squeeze Theorem.)