WORKSHEET SOLUTIONS: MEAN VALUE THEOREM; L'HOPITAL'S RULE 2

Names and student IDs: Solutions [$\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$]

The first problem is about the Mean Value Theorem. The second is on a case in which L'Hopital's Rule appears to apply, but for which it does not actually work. We already did an example like this one in the lecture of Monday 2 June, using methods like those on the worksheet of Tuesday 3 June. If we don't get to this problem in class today (likely), do it at home.

1. Let f be a differentiable function defined on the open interval $(0, 8)$. Suppose $f(1) = 10$ and $f'(x) \geq 5$ for all x in $(1, 4)$. What can you say about $f(4)$? Your answer should have one of the forms

$$f(4) \geq \underline{\hspace{2em}}, \qquad f(4) > \underline{\hspace{2em}}, \qquad f(4) \leq \underline{\hspace{2em}}, \qquad \text{or} \qquad f(4) < \underline{\hspace{2em}}.$$

Solution. When $f'(x) \geq M$ for all x in $(1, 4)$, we get $f(4) \geq f(1) + M(4 - 1)$. So

$$f(4) \geq f(1) + 5(4 - 1) = 10 + 5 \cdot 3 = 25.$$

In more detail, by the Mean Value Theorem, there is c in $(1, 4)$ such that

$$\frac{f(4) - f(1)}{4 - 1} = f'(c), \qquad \text{that is,} \qquad f(4) - f(1) = f'(c)(4 - 1).$$

But $f'(c) \geq 5$. □

Reminders on L'Hopital's Rule.

Do NOT confuse L'Hopital's Rule with the quotient rule!
Before using L'Hopital's Rule, you must check that its hypotheses are satisfied!

A special case: **if** $\lim_{x \to \infty} f(x) = \pm\infty$ **and** $\lim_{x \to \infty} g(x) = \pm\infty$**,** and if $\lim_{x \to \infty} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)}.$$

(Generally, one must have a fraction with an **indeterminate form**, such as "$\frac{0}{0}$" or "$\frac{\pm\infty}{\pm\infty}$".)

2. Consider $\lim_{x \to \infty} \frac{2x}{x + \sin(3x)}$. Set $f(x) = 2x$ and $g(x) = x + \sin(3x)$. Reminders for this problem: $\lim_{x \to \infty} \cos(3x)$ does not exist (the function oscillates between -1 and 1) and, by the Squeeze Theorem, $\lim_{x \to \infty} \frac{\sin(3x)}{x} = 0$.

(a) Find $\lim_{x \to \infty} f(x)$.

Solution. Since $\lim_{x \to \infty} x = \infty$, also

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} 2x = \infty.$$

□

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(b) Find $\lim_{x\to\infty} g(x)$.

Solution. Since $\lim_{x\to\infty} x = \infty$ and, for all x, we have $-1 \leq \sin(3x) \leq 1$, also

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \big(x + \sin(3x)\big) = \infty.$$

□

(c) Does $\lim_{x\to\infty} \frac{2x}{x+\sin(3x)}$ have an indeterminate form?

Solution. Yes, $\lim_{x\to\infty} \frac{2x}{x+\sin(3x)}$ has the indeterminate form "$\frac{\infty}{\infty}$". □

(d) Can we try L'Hopital's Rule on $\lim_{x\to\infty} \frac{2x}{x+\sin(3x)}$? Why?

Solution. Yes. It is the limit of a fraction, for which the limit has an indeterminate form. □

(e) What happens if we try L'Hopital's Rule on $\lim_{x\to\infty} \frac{2x}{x+\sin(3x)}$?

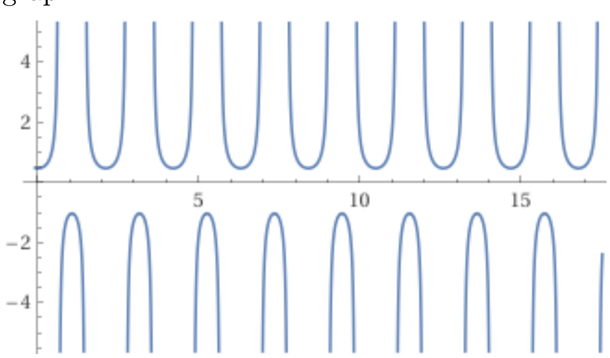
Solution. We have $f'(x) = 2$ and $g'(x) = 1 + 3\cos(3x)$. So we get

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{2}{1+3\cos(3x)}.$$

This limit doesn't exist, because $\cos(3x)$ oscillates between -1 and 1. So L'Hopital's Rule tells us nothing.

Things are actually worse than they appear at first: the function $q(x) = \frac{2}{1+3\cos(3x)}$ has infinitely many vertical asymptotes, namely whenever $\cos(3x) = -\frac{1}{3}$.

To illustrate, here is a graph.



Obviously the limit at ∞ does not exist, not even as ∞ or $-\infty$. □

(f) Use some other method to find $\lim_{x\to\infty} \frac{2x}{x+\sin(3x)}$. Hint: Multiply the numerator and denominator by $\frac{1}{x}$. We used this method on a similar problem in class Monday.

Solution. We have $\lim_{x\to\infty} \frac{\sin(3x)}{x} = 0$. (Squeeze Theorem.) So

$$\lim_{x\to\infty} \frac{2x}{x+\sin(3x)} = \lim_{x\to\infty} \frac{\frac{1}{x}(2x)}{\frac{1}{x}(x+\sin(3x))} = \lim_{x\to\infty} \frac{2}{1+\frac{\sin(3x)}{x}} = \frac{2}{1+\lim_{x\to\infty}\frac{\sin(3x)}{x}} = \frac{2}{1+0} = 2.$$

□