WORKSHEET SOLUTIONS: L'HOPITAL'S RULE

Names and student IDs: Solutions [$\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$]

Do NOT confuse L'Hopital's Rule with the quotient rule!

Before using L'Hopital's Rule, you must check that its hypotheses are satisfied!

A special case: **if** $\lim_{x\to a} f(x) = 0$ **and** $\lim_{x\to a} g(x) = 0$**,** and if $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

(Generally, one must have a fraction with an **indeterminate form**, such as "$\frac{0}{0}$" or "$\frac{\infty}{\infty}$".)

1. Consider $\lim_{x\to 0} \frac{e^{2x}-1}{\sin(3x)}$. In the above, we will take $f(x) = e^{2x} - 1$ and $g(x) = \sin(3x)$.

(a) What is $\lim_{x\to 0} (e^{2x} - 1)$?

Solution. $\lim_{x\to 0} (e^{2x} - 1) = e^{2\cdot 0} - 1 = 0$. □

(b) What is $\lim_{x\to 0} \sin(3x)$?

Solution. $\lim_{x\to 0} \sin(3x) = \sin(3 \cdot 0) = 0$. □

(c) Does $\lim_{x\to 0} \frac{e^{2x}-1}{\sin(3x)}$ have an indeterminate form?

Solution. Yes, $\lim_{x\to 0} \frac{e^{2x}-1}{\sin(3x)}$ has the indeterminate form "$\frac{0}{0}$". □

(d) Find $\lim_{x\to 0} \frac{e^{2x}-1}{\sin(3x)}$.

Solution. Since the limit has an indeterminate form we may try L'Hopital's Rule. We have

$$\frac{d}{dx}\left(e^{2x} - 1\right) = 2e^{2x}, \qquad \frac{d}{dx}\left(\sin(3x)\right) = 3\cos(3x), \qquad \text{and} \qquad \lim_{x\to 0} \frac{2e^{2x}}{3\cos(3x)} = \frac{2e^{2\cdot 0}}{3\cos(3\cdot 0)} = \frac{2}{3}.$$

Therefore, by L'Hopital's Rule,

$$\lim_{x\to 0} \frac{e^{2x}-1}{\sin(3x)} = \lim_{x\to 0} \frac{2e^{2x}}{3\cos(3x)} = \frac{2}{3}.$$

□

2. Differentiate the function $f(x) = \frac{e^{2x}-1}{\sin(3x)}$.

Solution. This problem has **nothing to do with L'Hopital's Rule!**. Use the quotient rule:

$$f'(x) = \frac{\frac{d}{dx}(e^{2x}-1)\sin(3x) - (e^{2x}-1)\frac{d}{dx}(\sin(3x))}{\sin^2(3x)} = \frac{2e^{2x}\sin(3x) - (e^{2x}-1)\cdot 3\cos(3x)}{\sin^2(3x)}.$$

□

3. Consider $\lim_{x\to 0} \frac{x}{x+2}$.

(a) What is $\lim_{x\to 0} x$?

Solution. $\lim_{x\to 0} x = 0$. □

(b) What is $\lim_{x\to 0} (x + 2)$?

Date: 4 June 2025.

Solution. $\lim_{x\to 0}(x+2) = 2$. This is neither 0 nor ∞, so you can tell that L'Hopital's Rule **does not apply**. ☐

(c) Does $\lim_{x\to 0} \frac{x}{x+2}$ have an indeterminate form?

Solution. No. You get $\frac{0}{2}$. ☐

(d) Find $\lim_{x\to 0} \frac{x}{x+2}$.

Solution. The function $f(x) = \frac{x}{x+2}$ is defined and continuous at 0, so

$$\lim_{x\to 0} \frac{x}{x+2} = \frac{0}{0+2} = 0.$$

L'Hopital's Rule would give **the wrong answer** (namely 1). ☐

A different special case: **if** $\lim_{x\to\infty} f(x) = \pm\infty$ **and** $\lim_{x\to\infty} g(x) = \pm\infty$**,** and if $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

(Generally, one must have a fraction with an **indeterminate form**, such as "$\frac{0}{0}$" or "$\frac{-\infty}{\infty}$".)

4. Consider $\lim_{x\to\infty} \frac{x}{e^{8x}+3x}$.

(a) What should $f(x)$ and $g(x)$ be?

Solution. $f(x) = x$ and $g(x) = e^{8x} + 3x$. ☐

(b) What is $\lim_{x\to\infty} f(x)$?

Solution. $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x = \infty$. ☐

(c) What is $\lim_{x\to\infty} g(x)$?

Solution. Since $\lim_{x\to\infty} e^{8x} = \infty$ and $\lim_{x\to\infty} 3x = \infty$, we have

$$\lim_{x\to\infty} g(x) = \lim_{x\to\infty} \left(e^{8x} + 3x\right) = \infty.$$

☐

(d) Does $\lim_{x\to\infty} \frac{x}{e^{8x}+3x}$ have an indeterminate form?

Solution. Yes, $\lim_{x\to\infty} \frac{x}{e^{8x}+3x}$ has the indeterminate form "$\frac{\infty}{\infty}$". ☐

(e) Find $\lim_{x\to\infty} \frac{x}{e^{8x}+3x}$.

Solution. Since the limit has an indeterminate form we may try L'Hopital's Rule. We have

$$\frac{d}{dx}(x) = 1, \qquad \frac{d}{dx}\left(e^{8x} + 3x\right) = 8^{8x} + 3, \qquad \lim_{x\to\infty} 1 = 1, \qquad \text{and} \qquad \lim_{x\to\infty}\left(e^{8x} + 3\right) = \infty.$$

Therefore $\lim_{x\to\infty} \frac{1}{e^{8x}+3} = 0$. (This limit has the form "$\frac{1}{\infty}$", which is **not** an indeterminate form!)

So, by L'Hopital's Rule, $\lim_{x\to\infty} \frac{x}{e^{8x}+3x} = \frac{1}{e^{8x}+3} = 0$. ☐