WORKSHEET SOLUTIONS: MORE ON LIMITS AT INFINITY

Names and student IDs: Solutions [$\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$]

See Section 4.6 of the book.

First, recall that

$$\lim_{x\to\infty}\frac{5}{x}, \qquad \lim_{x\to\infty}\left(-\frac{11}{x^2}\right), \qquad \lim_{x\to-\infty}\frac{3}{x^9},$$

etc. are all zero, while expressions like $\lim\limits_{x\to\pm\infty} 6x^2$ or $\lim\limits_{x\to\pm\infty}(-3x^5)$ are $\pm\infty$ (depending on the sign of the coefficient and, if the limit is at $-\infty$, whether the exponent is even or odd).

Consider $\lim\limits_{x\to\infty}\dfrac{x^3-1999x}{17x^3+1}$. This limit has the indeterminate form "$\frac{\infty}{\infty}$" (we will come back to this later), so work is needed. You expect the answer to be $\dfrac{1}{17}$. Here is one way to show work, following Monday's lecture. Multiply the numerator and denominator by $\dfrac{1}{x^3}$, and then use the limit laws:

$$\lim_{x\to\infty}\frac{x^3-1999x}{17x^3+1} = \lim_{x\to\infty}\frac{\left(\frac{1}{x^3}\right)\left(x^3-1999x\right)}{\left(\frac{1}{x^3}\right)\left(17x^3+1\right)} = \lim_{x\to\infty}\frac{1-\frac{1999}{x^2}}{17+\frac{1}{x^3}} = \frac{1-\lim_{x\to\infty}\frac{1999}{x^2}}{17+\lim_{x\to\infty}\frac{1}{x^3}} = \frac{1-0}{17+0} = \frac{1}{17}.$$

In an exam problem solution, it is enough to write

$$\lim_{x\to\infty}\frac{x^3-1999x}{17x^3+1} = \lim_{x\to\infty}\frac{\left(\frac{1}{x^3}\right)\left(x^3-1999x\right)}{\left(\frac{1}{x^3}\right)\left(17x^3+1\right)} = \lim_{x\to\infty}\frac{1-\frac{1999}{x^2}}{17+\frac{1}{x^3}} = \frac{1}{17}.$$

1. Why is it legitimate to multiply the numerator and denominator by $\dfrac{1}{x^3}$?

Solution. In the expression

$$\frac{\left(\frac{1}{x^3}\right)\left(x^3-1999x\right)}{\left(\frac{1}{x^3}\right)\left(17x^3+1\right)},$$

one can cancel the common factor $\dfrac{1}{x^3}$ in the numerator and denominator. □

2. Find the exact value of $\lim\limits_{x\to\infty}\dfrac{x^2-x+17}{7x^2+9x+19}$. (Be sure to show your work!)

Solution. This limit has the indeterminate form "$\frac{\infty}{\infty}$" (see below), so work is needed. So multiply the numerator and denominator by $\dfrac{1}{x^2}$, and then use the limit laws:

$$\lim_{x\to\infty}\frac{x^2-x+17}{7x^2+9x+19} = \lim_{x\to\infty}\frac{\left(\frac{1}{x^2}\right)\left(x^2-x+17\right)}{\left(\frac{1}{x^2}\right)\left(7x^2+9x+19\right)} = \lim_{x\to\infty}\frac{1-\frac{1}{x}+\frac{17}{x^2}}{7+\frac{9}{x}+\frac{19}{x^2}}$$

$$= \frac{1-\lim_{x\to\infty}\frac{1}{x}+\lim_{x\to\infty}\frac{17}{x^2}}{7+\lim_{x\to\infty}\frac{9}{x}+\lim_{x\to\infty}\frac{19}{x^2}} = \frac{1-0+0}{7+0+0} = \frac{1}{7}.$$

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Note the necessary parentheses:

$$\frac{\left(\frac{1}{x^2}\right) x^2 - x + 17}{\left(\frac{1}{x^2}\right) 7x^2 + 9x + 19}$$

is wrong, although

$$\frac{\frac{1}{x^2}\left(x^2 - x + 17\right)}{\frac{1}{x^2}\left(7x^2 + 9x + 19\right)}$$

is fine. □

3. Find the exact value of $\lim_{x\to\infty} \frac{3x+1}{5x^2-9}$. (Be sure to show your work!)

(Multiply the numerator and denominator by $\frac{1}{x^2}$.)

Solution. This limit has the indeterminate form "$\frac{\infty}{\infty}$", so work is needed. We multiply the numerator and denominator by $\frac{1}{x^2}$, and then use the limit laws:

$$\lim_{x\to\infty} \frac{3x+1}{5x^2-9} = \lim_{x\to\infty} \frac{\left(\frac{1}{x^2}\right)(3x+1)}{\left(\frac{1}{x^2}\right)(5x^2-9)} = \lim_{x\to\infty} \frac{\frac{3}{x} + \frac{1}{x^2}}{5 - \frac{9}{x^2}}$$

$$= \frac{\lim_{x\to\infty} \frac{3}{x} + \lim_{x\to\infty} \frac{1}{x^2}}{5 - \lim_{x\to\infty} \frac{9}{x^2}} = \frac{0+0}{5-0} = 0.$$

□

4. Find the exact value of $\lim_{x\to\infty} \frac{5x^2-9}{3x+1}$. (Be sure to show your work!)

(Multiply the numerator and denominator by $\frac{1}{x}$.)

Solution.

$$\lim_{x\to\infty} \frac{5x^2-9}{3x+1} = \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)\left(5x^2-9\right)}{\left(\frac{1}{x}\right)(3x+1)} = \lim_{x\to\infty} \frac{5x - \frac{9}{x}}{3 + \frac{1}{x}}.$$

Since

$$\lim_{x\to\infty} 5x = \infty \qquad \text{and} \qquad \lim_{x\to\infty} \frac{9}{x} = 0,$$

it follows that

$$\lim_{x\to\infty} \left(5x - \frac{9}{x}\right) = \infty.$$

Since also

$$\lim_{x\to\infty} \left(3 + \frac{1}{x}\right) = 3,$$

which is positive, not zero, and not ∞, it follows that

$$\lim_{x\to\infty} \frac{5x - \frac{9}{x}}{3 + \frac{1}{x}} = \infty.$$

□

General principle: choose the power of x to match the highest power which occurs in the denominator.