WORKSHEET: MORE ON LIMITS AT INFINITY

Names and student IDs: ____

See Section 4.6 of the book.

First, recall that

$$\lim_{x\to\infty} \frac{5}{x}, \qquad \lim_{x\to\infty} \left(-\frac{11}{x^2}\right), \qquad \lim_{x\to-\infty} \frac{3}{x^9},$$

etc. are all zero, while expressions like $\lim_{x\to\pm\infty} 6x^2$ or $\lim_{x\to\pm\infty} (-3x^5)$ are $\pm\infty$ (depending on the sign of the coefficient and, if the limit is at $-\infty$, whether the exponent is even or odd).

Consider $\lim_{x\to\infty} \frac{x^3 - 1999x}{17x^3+1}$. This limit has the indeterminate form "$\frac{\infty}{\infty}$" (we will come back to this later), so work is needed. You expect the answer to be $\frac{1}{17}$. Here is one way to show work, following Monday's lecture. Multiply the numerator and denominator by $\frac{1}{x^3}$, and then use the limit laws:

$$\lim_{x\to\infty} \frac{x^3 - 1999x}{17x^3+1} = \lim_{x\to\infty} \frac{\left(\frac{1}{x^3}\right)\left(x^3 - 1999x\right)}{\left(\frac{1}{x^3}\right)\left(17x^3+1\right)} = \lim_{x\to\infty} \frac{1 - \frac{1999}{x^2}}{17 + \frac{1}{x^3}} = \frac{1 - \lim_{x\to\infty} \frac{1999}{x^2}}{17 + \lim_{x\to\infty} \frac{1}{x^3}} = \frac{1-0}{17+0} = \frac{1}{17}.$$

In an exam problem solution, it is enough to write

$$\lim_{x\to\infty} \frac{x^3 - 1999x}{17x^3+1} = \lim_{x\to\infty} \frac{\left(\frac{1}{x^3}\right)\left(x^3 - 1999x\right)}{\left(\frac{1}{x^3}\right)\left(17x^3+1\right)} = \lim_{x\to\infty} \frac{1 - \frac{1999}{x^2}}{17 + \frac{1}{x^3}} = \frac{1}{17}.$$

1. Why is it legitimate to multiply the numerator and denominator by $\frac{1}{x^3}$?

2. Find the exact value of $\lim_{x\to\infty} \frac{x^2 - x + 17}{7x^2 + 9x + 19}$. (Be sure to show your work!)

Continued on back.

Date: 3 June 2025.

3. Find the exact value of $\lim_{x\to\infty} \frac{3x+1}{5x^2-9}$. (Be sure to show your work!)
(Multiply the numerator and denominator by $\frac{1}{x^2}$.)

4. Find the exact value of $\lim_{x\to\infty} \frac{5x^2-9}{3x+1}$. (Be sure to show your work!)
(Multiply the numerator and denominator by $\frac{1}{x}$.)

General principle: choose the power of x to match the highest power which occurs in the denominator.