WORKSHEET SOLUTIONS: DERIVATIVES AS RATES OF CHANGE

Names and student IDs: Solutions [$\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$]

See Section 3.4 of the book.

Recall: the derivative of something is its instantaneous rate of change. For example, if you are going 70 miles per hour along a long straight road, it means that, if you kept going at the same speed for an hour, you would be 70 miles farther along. If $P(x)$ is the profit in dollars Wang's Widgets Inc. makes when manufacturing x widgets, then saying $P'(300{,}000) = 6$ (in dollars per widget) means that, if the profit per additional widget is the same when the production level is between 300,000 and 300,001, then producing 300,001 widgets instead of 300,000 will generate an additional \$6 in profit. (This is really the linear approximation.)

1. A particle moves along a line. Time is measured in seconds, and position is measured in meters to the right of the origin. When $0 \leq t \leq 20$, the position is given by $p(t) = t^3 - 5t^2 - 2t$.

a. Where did the particle start?

Solution. At position $p(0) = 0$, so at the origin. □

b. What are the units of $p'(t)$?

Solution. Meters per second, or m/sec. (Units of $p(t)$ divided by units of t.) □

c. What was the velocity of the particle at $t = 2$ (including units)? Was it moving left or right?

Solution. We need $p'(t) = 3t^2 - 10t - 2$. So the velocity at $t = 2$ is $p'(2) = 12 - 20 - 2 = -10$, in m/sec. Since $p'(2) < 0$, the particle was moving left. □

d. What are the units of $p''(t)$ (acceleration)?

Solution. m/sec^2. (Units of $p'(t)$, which are m/sec, divided by units of t, which are seconds.) □

e. At time 5 seconds, was the particle speeding up or slowing down? (Be careful.)

Solution. We need $p''(t) = 6t - 10$. So $p''(5) = 30 - 10 = 20$, in m/sec^2. Since $p''(5) > 0$, the rightward velocity of the particle was increasing. Since $p'(5) = 23$, the particle was already moving rightwards, so it was speeding up.

Caution: we saw $p'(2) = -10$, so the particle was moving left at 2 seconds, but $p''(2) = 2 > 0$, so its **rightward** velocity was increasing. Here that means the particle is moving left more slowly, and conventionally one would say the particle is slowing down. **Watch the signs!**) □

2. Let $P(t)$ be the population of Megalopolis, in millions of people, at time t is years, counted conventionally.

a. What are the units of $P'(t)$?

Solution. Millions of people per year. (Units of $P(t)$ divided by units of t.) □

b. Suppose $P'(t) < 0$ when $2010 \leq t \leq 2020$. What does that say about the population?

Solution. The population was decreasing between 2010 and 2020, since the instantaneous rate of change was negative over the entire period. □

3. Let $C(x)$ be the cost, in dollars, to Wang's Widgets Inc. of producing x widgets.

a. What are the units of $C'(x)$?

Solution. Dollars per widget. (Units of $C(x)$ divided by units of x.) □

Date: 30 May 2025.

b. What does $C'(500{,}000)$ mean? (By the way, $C'(x)$ is called the *marginal cost.*)

Solution. The additional cost per widget produced when the production level is exactly 500,000 widgets. □

c. Do you expect $C'(500{,}000)$ to be positive or negative? Why?

Solution. Surely positive, because surely it costs more dollars to produce more widgets. □

4. Let $T(x)$ be the temperature, in degrees Fahrenheit (°F), along Interstate 5, x miles north of Eugene at 11:00 am on 25 July 2024.

a. What are the units of $T'(x)$?

Solution. °F per mile. (Units of $T(x)$ divided by units of x.) □

b. Do you think is is more likely that $T'(50) > 0$ or $T'(50) < 0$? Why?

Solution. Probably $T'(50) < 0$, since, in the northern hemisphere, usually it gets colder as you go north. (But features of the weather or terrain can easily override this.) □