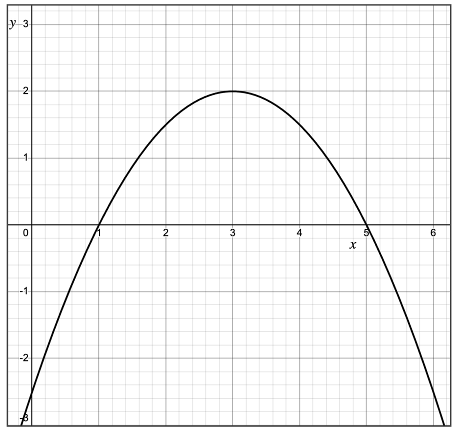
WORKSHEET SOLUTIONS: DERIVATIVES AND LOCAL EXTREMUMS

Names and student IDs: Solutions [$\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$]

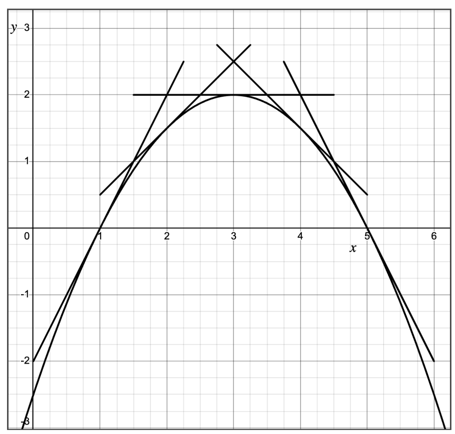
1. Shown below is the graph of $y = h(x)$ for some differentiable function h.



Recall that $h'(c)$ is the slope of the tangent line to the graph of $y = h(x)$ at $x = c$ (the point $(c, h(c))$ on the graph).

Estimate $h'(1), h'(2), \ldots, h'(5)$ (values of the derivative of h). (If you need to, draw tangent lines on the graph and estimate their slopes.)

Solution. Here is a graph showing the tangent lines.



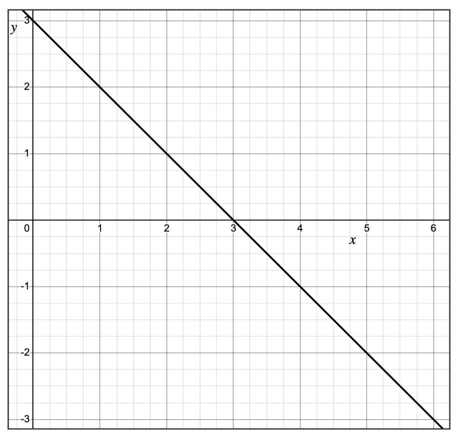
Using these, I got:

$$h'(1) \approx 2, \qquad h'(2) \approx 1, \qquad h'(3) \approx 0, \qquad h'(4) \approx -1, \qquad h'(5) \approx -2.$$

□

2. Using the values you found, draw a graph of $y = h'(x)$, the **derivative** of $h(x)$.

Solution. Graph:



Your graph should be at least somewhat close to this. □

Date: 21 May 2025.

3. Is the **derivative** $h'(x)$ of $h(x)$ is increasing on the interval $(0,6)$, decreasing on $(0,6)$, or increasing on parts of this interval and decreasing on other parts?

Solution. By the graph, decreasing on all of the interval $(0,6)$. □

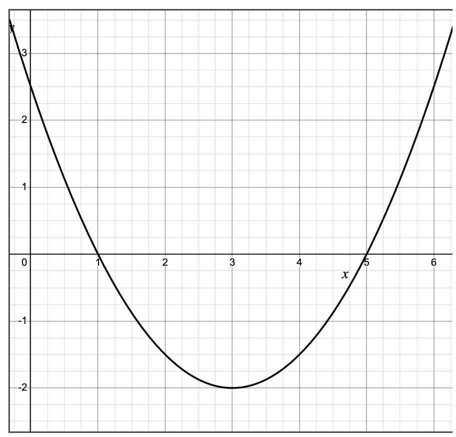
4. Recall that a differentiable function f is increasing on an open interval (a,b) exactly when $f'(x) \geq 0$ on (a,b), and that f is decreasing on (a,b) exactly when $f'(x) \leq 0$ on (a,b).

Apply this with $f = h'$, and decide whether the **second derivative** $h''(x)$ of $h(x)$ is positive on the interval $(0,6)$, negative on $(0,6)$, or positive on parts of this interval and negative on other parts.

Solution. Since h' is decreasing, $h''(x) \leq 0$ on all of the interval $(0,6)$. □

5. Draw the graph of some function $y = k(x)$ on the interval $(0,6)$ such that $h'(3) = 0$ and $h''(x) > 0$ on all of $(0,6)$. Does your function have a local minimum or a local maximum at $x = 3$? Is this function concave up or concave down on $(0,6)$?

Solution. Here is one such graph (out of many possibilities):



This function has a local minimum at $x = 3$, and is concave up on $(0,6)$. □

6. Let $g(x) = x^3 - 6x^2$. Then

$$g'(x) = 3x^2 - 12x = 3x(x-4) \qquad \text{and} \qquad g''(x) = 6x - 12 = 6(x-2).$$

By considering the signs of $g'(x)$ and $g''(x)$, find the critical points, intervals of increase and decrease, local minimums and maximums, and interval of concavity.

Solution. We have $g'(x) = 3x^2 - 12x = 3x(x-4)$, which is zero when $x = 0$ and $x = 4$. Therefore the critical points of g are at 0 and 4.

Since $g'(x) = 3x(x-4)$, we see that $g'(x) > 0$ on $(-\infty, 0)$ (since both x and $x-4$ are negative), $g'(x) < 0$ on $(0,4)$ (since $x > 0$ but $x - 4 < 0$), and $g'(x) > 0$ on $(4, \infty)$ (since both x and $x-4$ are positive). (You can also use test points. I think the best choices are -1, 1, and 5.)

Therefore g is increasing on $(-\infty, 0)$, decreasing on $(0,4)$, and increasing on $(4, \infty)$.

Use the second derivative test. We have $g''(x) = 6x - 12$. Therefore $g''(0) = -12 < 0$, so g has a local maximum at $x = 0$. Also, $g''(4) = 12 > 0$, so g has a local minimum at $x = 4$.

You can get the same result from the intervals of increase and decrease you found above.

We have $g''(x) = 6x - 12 = 6(x-2)$. So $g''(x) < 0$ on $(-\infty, 2)$, whence g is concave down there. Also $g''(x) > 0$ on $(2, \infty)$, whence g is concave up there. □