WORKSHEET SOLUTIONS: DERIVATIVES AND LOCAL EXTREMUMS 2

Names and student IDs: Solutions [$\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$]

Recall that a differentiable function f is increasing on (a, b) exactly when $f'(x) \geq 0$ on (a, b), and that f is decreasing on (a, b) exactly when $f'(x) \leq 0$ on (a, b).

Let $k(x) = -x^3 - 9x^2 + 21x + 2$. We are going to identify the open intervals on which k is increasing, those on which k is decreasing, and all critical points, local minimums, and local maximums.

Recall that x is a critical point if $k'(x) = 0$ or $k'(x)$ does not exist.

1. Find $k'(x)$.

Solution. We have

$$k'(x) = -3x^2 - 18x + 21.$$

□

2. Find all critical points of k. (You should find two of them, both integers.)

Solution. Clearly $k'(x)$ exists for all real numbers x. So we need to find all real numbers x such that $k'(x) = 0$. Factor the expression for $k'(x)$ above:

$$k'(x) = -3x^2 - 18x + 21 = -3(x^2 + 6x - 7) = -3(x - 1)(x + 7).$$

The solutions to $k'(x) = 0$ are therefore $x = 1$ and $x = -7$. These are the critical points. □

3. Find the possible open intervals of increase and decrease for k. (There should be three such intervals.)

Solution. Since k' is continuous and never zero on each of the intervals $(-\infty, -7)$, $(-7, 1)$, and $(1, \infty)$, the Intermediate Value Theorem tells us that k' must have the same sign everywhere on each of these intervals. Thus, k (**not** $\not{k'}$) is either increasing or decreasing on all of $(-\infty, -7)$, either increasing or decreasing on all of $(-7, 1)$ (but not necessarily the same choice as on $(-\infty, -7)$), and either increasing or decreasing on all of $(1, \infty)$ (again, not necessarily the same choice as on either of the previous two intervals). □

4. For each of the intervals above, determine whether k is increasing or decreasing on that interval.

Solution. For x in the interval $(-\infty, -7)$, we have $x < -7$, so $x - 1 < 0$ and $x + 7 < 0$, whence $k'(x) = -3(x - 1)(x + 7) < 0$. Therefore k is decreasing on $(-\infty, -7)$.

For x in the interval $(-7, 1)$, we have $x > -7$ and $x < 1$, so $x - 1 < 0$ and $x + 7 > 0$, whence $k'(x) = -3(x - 1)(x + 7) > 0$. Therefore k is increasing on $(-7, 1)$.

For x in the interval $(1, \infty)$, we have $x > 1$, so $x - 1 > 0$ and $x + 7 > 0$, whence $k'(x) = -3(x - 1)(x + 7) < 0$. Therefore k is decreasing on $(1, \infty)$.

Alternate method, using test points (valid because of what happened in part 3 above). I chose the test points -8 in $(-\infty, -7)$, 0 in $(-7, 1)$, and 2 in $(1, \infty)$. Then (using the easier choice of formula for $k'(x)$ at each point):

$$k'(-8) = -3(-8 - 1)(-8 + 7) = -3(-9)(-1) = -27 < 0,$$

$$k'(0) = -3 \cdot 0^2 - 18 \cdot 0 + 21 = 21 > 0,$$

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and

$$k'(2) = -3(2-1)(2+7) = -3(1)(9) = -27 < 0.$$

Therefore $k'(x) < 0$ on $(-\infty, -7)$, $k'(x) > 0$ on $(-7, 1)$, and $k'(x) < 0$ on $(1, \infty)$. So k is decreasing on $(-\infty, -7)$, increasing on $(-7, 1)$, and decreasing on $(1, \infty)$. □

5. Use the information above to determine, for each critical point, whether it is a local minimum or local maximum for k.

Solution. Since k is continuous, k is decreasing on $(-\infty, -7)$, and k is increasing on $(-7, 1)$, it follows that k has a local minimum at -7.

Since k is continuous, k is increasing on $(-7, 1)$, and k is decreasing on $(1, \infty)$, it follows that k has a local maximum at 1. □

6. Use the second derivative test to determine, for each critical point, whether it is a local minimum or local maximum for k.

Solution. We have $k''(x) = -6x - 18$. Therefore $k''(-7) = (-6)(-7) - 18 = 24 > 0$, so k has a local minimum at -7. Also, $k''(1) = (-6)(1) - 18 = -24 < 0$, so k has a local maximum at 1. □