WORKSHEET SOLUTIONS: RELATED RATES 2

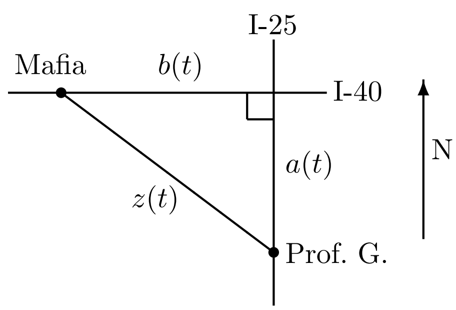
Names and student IDs: Solutions [$\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$]

Interstate 25 and Interstate 40 meet at a right angle in central Albuquerque (NM). At noon one day, Professor Greenbottle was driving north on Interstate 25 at 70 mph, and was 3 miles south of the intersection. At the same time, a Mafia hit squad was driving west on Interstate 40 at 50 mph, and was 4 miles west of the intersection. Were Professor Greenbottle and the Mafia hit squad getting closer together or further apart? At what rate? (Be sure to include the correct units.)

Steps:

(1) Understand the problem!
(2) Draw a picture if possible. (There is no picture in the solutions.) Name every quantity that varies with a **letter** (a function of time). (There are three of them in this problem.) Do **not** use a separate letter for any velocity, rate of change, etc. These are derivatives of quantities you should have already named with letters.
(3) State the information given, and what is to be found.
(4) Relate the variables, giving a relation that is valid for all values of time, not just the time for which the information in the problem is given. Choose a form of the relation that is easy to differentiate.
(5) Differentiate the relation in step (4) **with respect to time**, getting an equation involving quantities from step (2) and their derivatives. **Be sure to use the chain rule!** In this problem, if there are less than three derivatives, you made a mistake.
(6) Now, and only now, restrict to the particular time given in the problem. (In many solutions, this will be "put $t = t_0$", "put $t = 0$", or similar.) Put the known values in the equation above, and solve for the quantity asked for. (In this problem, you may need to find one distance at the time given from other distances at that time.)

Solution. Here is a picture, with appropriate quantities labelled.



In the picture, $a(t)$ is the distance (in miles) Professor Greenbottle's car is south of the intersection at time t (with time measured in hours), $b(t)$ is the distance (in miles) the hit squad is west of the intersection at time t, and $z(t)$ is the distance (also in miles) between the two cars at time t. Let t_0 represent noon on the day in question. Then the information given says that:

$$a(t_0) = 3, \qquad a'(t_0) = -70, \qquad b(t_0) = 4, \qquad \text{and} \qquad b'(t_0) = 50.$$

(Note that $a'(t_0)$ is negative, because Professor Greenbottle's car is getting closer to the intersection. There are other possible choices for measuring distance.) We want to find $z'(t_0)$.

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We know (from the Pythagorean Theorem) that $z(t)^2 = a(t)^2 + b(t)^2$. Differentiating, we get

$$2z(t)z'(t) = 2a(t)a'(t) + 2b(t)b'(t).$$

(Don't forget to use the chain rule!) Put $t = t_0$ and (for simplicity) divide by 2:

$$z(t_0)z'(t_0) = a(t_0)a'(t_0) + b(t_0)b'(t_0).$$

Next, substitute values. (Note that this can only be done *after* differentiating!) We need

$$z(t_0) = \sqrt{a(t_0)^2 + b(t_0)^2} = \sqrt{3^2 + 4^2} = 5,$$

and we then get

$$5 \cdot z'(t_0) = 3 \cdot (-70) + 4 \cdot 50 = -10.$$

Therefore $z'(t_0) = -2$, and Professor Greenbottle's car and the Mafia hit squad are getting closer to each other at 2 miles per hour. (The units are necessary!)

It is wrong to say that they are getting ~~closer at -2 mph~~. This statement means they are getting farther apart at 2 mph.

It is possible to differentiate the relation $z(t) = \sqrt{a(t)^2 + b(t)^2}$ instead of $z(t)^2 = a(t)^2 + b(t)^2$. The result is

$$\begin{aligned} z'(t) &= \frac{1}{2}\big(a(t)^2 + b(t)^2\big)^{-1/2} \frac{d}{dt}\big(a(t)^2 + b(t)^2\big) = \frac{1}{2}\big(a(t)^2 + b(t)^2\big)^{-1/2}\big(2a(t)a'(t) + 2b(t)b'(t)\big) \\ &= \frac{a(t)a'(t) + b(t)b'(t)}{\sqrt{a(t)^2 + b(t)^2}}. \end{aligned}$$

The differentiation was obviously messier. The advantage is that you do not need to solve for $z(t_0)$.

In physicists' notation, the equation relating the quantities is $z^2 = a^2 + b^2$. Differentiating (using the chain rule, because everything is a function of t!), we get

$$2z\frac{dz}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt},$$

so

$$z\frac{dz}{dt} = a\frac{da}{dt} + b\frac{db}{dt}.$$

Substituting values (implicitly putting $t = t_0$, and using $z = 5$ at $t = t_0$, as above):

$$5 \cdot \frac{dz}{dt} = 3 \cdot (-70) + 4 \cdot 50 = -10.$$

So $\dfrac{dz}{dt} = -2$. □