WORKSHEET SOLUTIONS: RELATED RATES

Names and student IDs: Solutions [$\pi\pi\pi$-$\pi\pi$-$\pi\pi\pi\pi$]

A spherical balloon is being inflated. At a particular time, the volume was $400\,\mathrm{m}^3$ and the inflation rate was $20\,\mathrm{m}^3/\mathrm{sec}$. At that time, was its radius increasing or decreasing? At what rate? (Be sure to include the correct units.) (The numbers will not work out nicely.)

Steps:

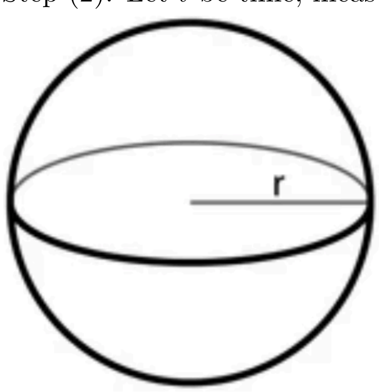
(1) Understand the problem!
(2) Draw a picture if possible. ~~(There is no picture in the solutions.)~~ Name every quantity that varies with a **letter** (a function of time). Do **not** use a separate letter for any velocity, rate of change, etc. These are derivatives of quantities you should have already named with letters.
(3) State the information given, and what is to be found.
(4) Relate the variables, giving a relation that is valid for all values of time, not just the time for which the information in the problem is given. Choose a form of the relation that is easy to differentiate.
(5) Differentiate the relation in step (4) **with respect to time**, getting an equation involving quantities from step (2) and their derivatives. **Be sure to use the chain rule!** In this problem, if there are less than two derivatives in the result, you made a mistake.
(6) ~~Put~~ Now, and only now, restrict to the particular time given in the problem. (In many solutions, this will be "put $t = t_0$", "put $t = 0$", or similar.) Put the known values in the equation above, and solve for the quantity asked for. (In this problem, you will need to find the radius at the time given from the volume at that time.)

You will need the volume V of a sphere of radius r. It is ~~$\frac{4}{3}\pi r^3$~~ $V = \frac{4}{3}\pi r^3$. (You will need to know this formula.)

1. Do steps (1) and (2) above.

Solution. Step (1): see the problem statement.

Step (2): Let t be time, measured in minutes.



Date: 12 May 2025.

The radius is r (or $r(t)$), as shown in the picture, measured in meters. We also need the volume. Call it V (or $V(t)$), measured in cubic meters. You **must** use a letter here, since the volume is changing with time! ☐

2. Do step (3) above.

Solution. Let t_0 be the time at which we are given the information. Then $V(t_0) = 400$ and $V'(t_0) = 20$. ☐

3. Do step (4) above.

Solution. $V(t) = \frac{4}{3}\pi r(t)^3$. In physicists' notation, $V = \frac{4}{3}\pi r^3$.

This must be done for arbitrary time, not just for the particular time at which the information is given. Otherwise, in the next step, you are just differentiating constants, and you will get zero. ☐

4. Do step (5) above.

Solution. Using the chain rule on the right,

$$V'(t) = \frac{4}{3}\pi \cdot 3r(t)^2 r'(t) = 4\pi r(t)^2 r'(t).$$

Getting ~~$4\pi r(t)^2$~~ on the right is a **catastrophic** error!

In physicists' notation,

$$\frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

Getting ~~$V \frac{dV}{dt}$~~ on the left, or ~~$4\pi r^2$~~ on the right, is a **catastrophic** error! ☐

5. Do step (6) above.

Solution. From step (5), $V'(t_0) = 4\pi r(t_0)^2 r'(t_0)$. We know $V'(t_0) = 20$. We still need $r(t_0)$. Since $V(t_0) = 400$ and $V(t_0) = \frac{4}{3}\pi r(t_0)^3$, we get

$$\frac{4}{3}\pi r(t_0)^3 = 400,$$

so

$$r(t_0) = \sqrt[3]{\frac{300}{\pi}}$$

Thus

$$20 = \left(\sqrt[3]{\frac{300}{\pi}}\right)^2 r'(t_0) = (300\pi^{-1})^{2/3} r'(t_0),$$

so

$$r'(t_0) = \frac{20}{(300\pi^{-1})^{2/3}} = 20(300\pi^{-1})^{-2/3}.$$

Thus, the radius is increasing at $20(300\pi^{-1})^{-2/3}$ m/sec. ☐