# WORKSHEET SOLUTIONS: LIMITS AT INFINITY 1

Names and student IDs: Solutions [πππ-ππ-ππππ]

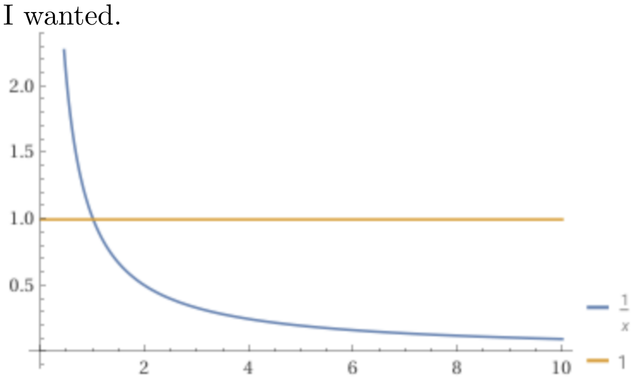
Recall: We say $\lim\limits_{x\to\infty} f(x) = L$ if one can force $f(x)$ to be as close to $L$ as one wants by requiring that $x$ be large enough. In particular, $f$ has a horizontal asymptote at $y = L$. Caution: the graph of $y = f(x)$ can cross the line $y = L$, even cross it infinitely often. See the third example below. We define $\lim\limits_{x\to-\infty} f(x) = L$ similarly. Combining this with the meaning of $\lim\limits_{x\to a} f(x) = \infty$ and related notions, one gets the meaning of $\lim\limits_{x\to\infty} f(x) = \infty$ etc. Caution: $\lim\limits_{x\to\infty} f(x) = \infty$ does not imply the existence of any asymptote. Examples:

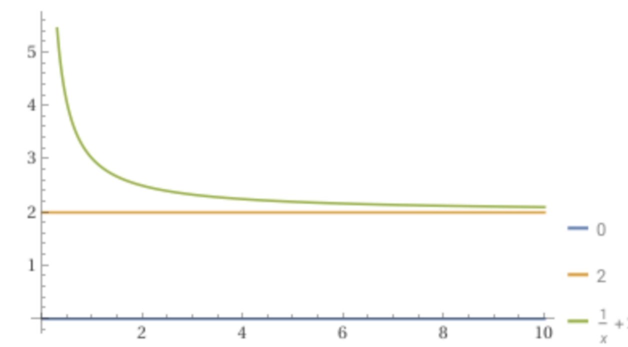
$$\lim_{x\to\infty} \frac{1}{x} = 0, \qquad \lim_{x\to\infty} \left(2 + \frac{1}{x}\right) = 2, \qquad \lim_{x\to\infty} \left(1 + \frac{\sin(8x)}{x}\right) = 1, \qquad \text{and} \qquad \lim_{x\to\infty} x^2 = \infty.$$

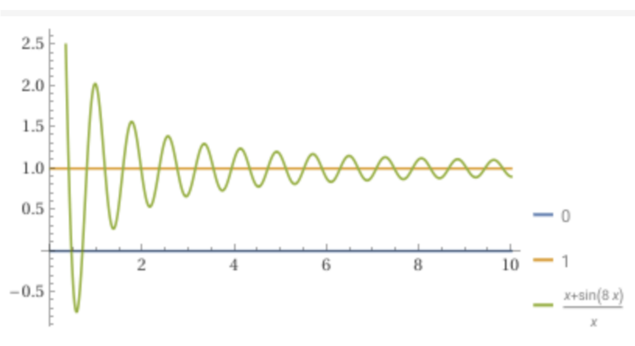
All the usual laws of limits at a real number $a$ also apply to limits at $\pm\infty$.

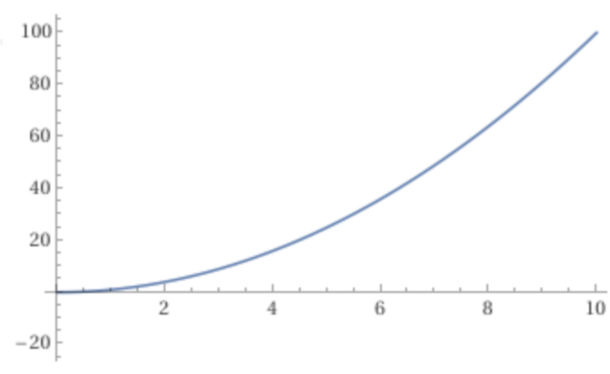
1. Plot the functions in the examples above, to see what the behavior looks like graphically. Draw the results on your paper. (Some of the extra coefficients were chosen to make the plots look better.)

*Solution.* Here are graphs, with different scales on the two axes (not the best quality, but fast to make). The extra horizontal lines in the second and third are the horizontal asymptotes. The extra horizontal line in the first was put in only to force the scale on the vertical axis to be close to what I wanted.



□

2. If $\lim\limits_{x\to\infty} f(x) = 8$ and $\lim\limits_{x\to\infty} g(x) = 11$, what are

$$\lim_{x\to\infty} (f(x) + g(x)), \qquad \lim_{x\to\infty} (21 - 3g(x)), \qquad \lim_{x\to\infty} f(x)g(x), \qquad \text{and} \qquad \lim_{x\to\infty} \frac{f(x)}{g(x)}?$$

*Date*: 9 May 2025.

*Solution.* Use the limit laws:

$$\lim_{x\to\infty}(f(x)+g(x)) = \lim_{x\to\infty} f(x) + \lim_{x\to\infty} g(x) = 8 + 11,$$

$$\lim_{x\to\infty}(21 - 3g(x)) = 21 - 3\lim_{x\to\infty} g(x) = 21 - 3\cdot 11 = -12,$$

$$\lim_{x\to\infty} f(x)g(x) = \left(\lim_{x\to\infty} f(x)\right)\left(\lim_{x\to\infty} g(x)\right) = 8\cdot 11 = 88,$$

and

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)} = \frac{8}{11}.$$

□

3. Recall that $\lim_{x\to\infty}\frac{x^2}{18} = \infty$. Given that $\lim_{x\to\infty}\frac{3}{x} = 0$ (you know this) and $\lim_{x\to\infty}\frac{290x^2 - x + 18}{7x^2 + 2x + 121} = \frac{290}{7}$ (we will see later how to find this limit), what are $\lim_{x\to\infty}\left(\frac{x^2}{18} + \frac{3}{x}\right)$ and $\lim_{x\to\infty}\left(\frac{x^2}{18} - \frac{290x^2 - x + 18}{7x^2 + 2x + 121}\right)$?

*Solution.* Since the sum of an extremely large positive number and a number close to zero is still an extremely large positive number, we still have $\lim_{x\to\infty}\left(\frac{x^2}{18} + \frac{3}{x}\right) = \infty$. Similarly, if you subtract $\frac{290}{7}$ from an extremely large positive number, the result is still an extremely large positive number. So also $\lim_{x\to\infty}\left(\frac{x^2}{18} - \frac{290x^2 - x + 18}{7x^2 + 2x + 121}\right) = \infty$. □

Since $\infty$ and $-\infty$ are not numbers, we don't write algebraic operations on them in equations. Thus, we do not write, for example, $\lim_{x\to\infty}\left(\frac{x^2}{18} + \frac{3}{x}\right) = \infty + 0 = \infty$. This is the convention in our textbook, and we will follow it. Other conventions are possible, but have other problems. (You will see a different convention in parts of Math 616.)

This does not prevent us from saying that a limit 'has the form "$\infty + 0$" ', just as we say that certain limits 'have the form "$\frac{\infty}{\infty}$" ', 'have the form "$\frac{0}{0}$" ', etc. These statements don't imply we are actually going to do an algebraic operation.

4. Recall that $\lim_{x\to\infty} x^2 = \infty$. What is $\lim_{x\to\infty}(x^2 + 5\sin(8x))$?

*Solution.* We always have $-5 \leq 5\sin(8x) \leq 5$. Since $\lim_{x\to\infty} x^2 = \infty$ and the sum of an extremely large positive number and a number between $-5$ and $5$ is still an extremely large positive number, we still have $\lim_{x\to\infty}\left(\frac{x^2}{18} + 5\sin(8x)\right) = \infty$. □

5. Some basic limits at infinity. (Answers may be $\pm\infty$, or that the limit does not exist and is not even $\infty$ or $-\infty$.)

a. What is $\lim_{x\to\infty} e^x$?

*Solution.* $\lim_{x\to\infty} e^x = \infty$. □

b. What is $\lim_{x\to\infty} e^{-x}$? (Remember that $e^{-x} = \frac{1}{e^x}$.)

*Solution.* Since $\lim_{x\to\infty} e^x = \infty$, we have

$$\lim_{x\to\infty} e^{-x} = \lim_{x\to\infty} \frac{1}{e^x} = 0.$$

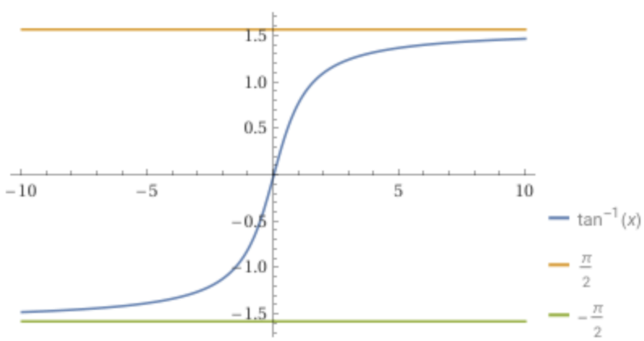
□

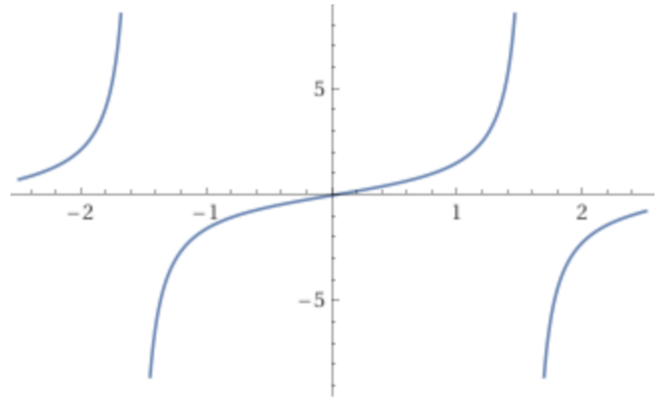
c. What is $\lim_{x\to\infty} \ln(x)$?

*Solution.* $\lim_{x\to\infty} \ln(x) = \infty$. To force $\ln(x) > M$, just take $x > e^M$. □

d. What is $\lim_{x\to\infty} \arctan(x)$? (Look at the graph of $y = \tan(x)$.)

*Solution.* $\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$. Graphs, $y = \arctan(x)$ first, with asymptotes, then $y = \tan(x)$:



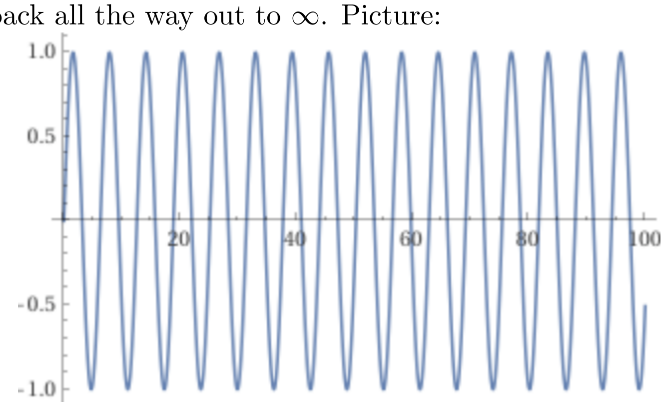




The graph of $y = \arctan(x)$ has a horizontal asymptote at $y = \frac{\pi}{2}$ because the graph of $y = \tan(x)$ has a vertical asymptote at $x = \frac{\pi}{2}$ as $x$ approaches $\frac{\pi}{2}$ from below (the side that is in the domain of the restricted version of $y = \tan(x)$).

□

e. What is $\lim_{x\to\infty} \sin(x)$?

*Solution.* $\lim_{x\to\infty} \sin(x)$ does not exist, not even as $\pm\infty$, because $\sin(x)$ oscillates from 1 to $-1$ and back all the way out to $\infty$. Picture:

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6. Find the following, after doing appropriate algebra:

$$\lim_{x\to\infty} \frac{x^2}{x^3}, \qquad \lim_{x\to\infty} \frac{x^3}{x^2}, \qquad \lim_{x\to\infty} \frac{3x^2}{8x^2}, \qquad \lim_{x\to\infty} \frac{290x^2}{2x^2}, \qquad \text{and} \qquad \lim_{x\to\infty} \frac{290x^2}{-2x^2}.$$

What does this tell you about the possibilities for $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ when $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$?

*Solution.*

$$\lim_{x\to\infty} \frac{x^2}{x^3} = \lim_{x\to\infty} \frac{1}{x} = 0,$$
$$\lim_{x\to\infty} \frac{x^3}{x^2} = \lim_{x\to\infty} x = \infty,$$
$$\lim_{x\to\infty} \frac{3x^2}{8x^2} = \lim_{x\to\infty} \frac{3}{8} = \frac{3}{8},$$
$$\lim_{x\to\infty} \frac{290x^2}{2x^2} = \lim_{x\to\infty} \frac{290}{2} = 145,$$

and

$$\lim_{x\to\infty} \frac{290x^2}{-2x^2} = \lim_{x\to\infty} \frac{290}{-2} = -145.$$

All of these are $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ for functions $f$ and $g$ such that $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$. This shows that, when $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$ but you know nothing more, nothing can be said about $\lim_{x\to\infty} \frac{f(x)}{g(x)}$. That is, "$\frac{\infty}{\infty}$" is an **indeterminate form**. □